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THE LINEAR ESCAPE LIMIT SET

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Abstract. If G is any Kleinian group, we show that the dimension of the limit set Λ is always equal to either the dimension of the bounded geodesics or the dimension of the geodesics that escape to infinity at linear speed.

Suppose G is a discrete group of isometries on hyperbolic space $\mathbb{B}^n$, $n \geq 2$. The limit set $\Lambda \subset S^{n-1}$ is defined to be the accumulation set of the G-orbit of $0 \in \mathbb{B}^n$. A point $x \in \Lambda$ can be associated to the radial segment that ends at x, which in turn projects to a geodesic ray γ (based at z_0, the projection of 0) in the quotient $M = \mathbb{B}^n/G$. We then write Λ as the disjoint union $\Lambda_c \cup \Lambda_e$, where Λ_c (the "conical limit set") corresponds to γ's that return to some compact set at arbitrarily large times and Λ_e (the "escaping limit set") corresponds to γ's that eventually leave every compact set. Obviously,

$$\dim(\Lambda) = \max(\dim(\Lambda_c), \dim(\Lambda_e))$$

(where dim denotes Hausdorff dimension). The purpose of this note is to show that this equality is still true if we replace both Λ_c and Λ_e by certain subsets.

Let Λ_b (the "bounded limit set") be the subset of Λ_c corresponding to γ's that remain bounded for all time. Parametrize geodesic rays by hyperbolic arclength and for $0 < \alpha < 1$, let Λ_α correspond to geodesic rays γ such that

$$\liminf_t \frac{\mathrm{dist}_M(\gamma(t), z_0)}{t} > \alpha,$$

and let $\Lambda_\ell = \bigcup_{0<\alpha<1} \Lambda_\alpha$ denote the "linear escape limit set". Related sets have been considered by Lundh in [2] ($\Lambda \setminus \Lambda_\alpha = \mathcal{L}(\frac{1}{1+\alpha})$ where $\mathcal{L}$ is as in Definition 3.14 of [2]).

Theorem 1. *For any discrete group G,* $\dim(\Lambda) = \max(\dim(\Lambda_b), \dim(\Lambda_\ell))$.

In other words, the dimension of Λ is determined either by the geodesic rays that stay bounded for all time or by those that escape to ∞ at the fastest possible speed. This is somewhat surprising since neither of these behaviors is "typical" in general. For example, if $n = 2$ and M is a finite area Riemann surface that is not compact, then Λ_c will have full Lebesgue measure but Λ_b will have measure zero (e.g., see [5]).

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If G is elementary, then $\dim(\Lambda) = 0$ and there is nothing to do. The Poincaré exponent δ of G is defined as a critical exponent of convergence of the Poincaré series, i.e.,

$$\delta = \inf\{s : \sum_{g\in G} e^{-s\rho(0,g(0))} < \infty\},$$

and in [1] it is shown that for non-elementary G, we have $\delta = \dim(\Lambda_c) = \dim(\Lambda_b)$. Theorem 1 follows from the following result (which is similar to Theorem 2.1.1 of [5]).

Lemma 2. *Suppose G is a discrete group of isometries on $\mathbb{B}^n$, $n \geq 2$ and $0 < \alpha < 1$ and assume that $\Lambda \setminus \Lambda_\alpha$ supports a positive measure μ such that*

$$\mu(B(x,r)) \leq \varphi(r) \tag{1}$$

for all balls and some increasing function φ that satisfies $\varphi(At) \leq B\varphi(t)$ for some $A > 1$, $B < \infty$. Then

$$\sum_{g\in G} \varphi((1-|g(0)|)^{(1+\beta)^{-1}}) = \infty, \tag{2}$$

for every $\beta > \alpha$.

We will prove this later. First we deduce a few consequences.

Corollary 3. *If G is a discrete group of isometries on $\mathbb{B}^n$, $n \geq 2$ and $0 < \alpha < 1$, then $\delta \geq \dim(\Lambda \setminus \Lambda_\alpha)/(1+\alpha)$.*

Proof. If $s = \dim(\Lambda \setminus \Lambda_\alpha)$, then Frostman's lemma (e.g., [3]) says it supports a measure satisfying (1) with $\varphi(t) = t^{s-\epsilon}$ for every $\epsilon > 0$. By (2), $\delta \geq (s-\epsilon)/(1+\beta)$. Taking $\epsilon \to 0$ gives the result. ☐

Corollary 4. *For any non-elementary group G we have $\delta = \dim(\Lambda \setminus \Lambda_\ell)$.*

Proof. Take $\alpha \to 0$ in the previous result to get "$\geq$". Since $\delta = \dim(\Lambda_c)$ and $\Lambda_c \subset \Lambda \setminus \Lambda_\ell$, the other direction is clear. ☐

Proof of Theorem 1. Our previous remarks imply that

$$\dim(\Lambda) = \max(\dim(\Lambda\setminus\Lambda_\ell), \dim(\Lambda_\ell)) = \max(\delta, \dim(\Lambda_\ell)) = \max(\dim(\Lambda_b), \dim(\Lambda_\ell)).$$

☐

Let $|E|_n$ denote the n-dimensional Lebesgue measure of the set E.

Corollary 5. *If G is a discrete group of isometries on $\mathbb{B}^{n+1}$ and $|\Lambda|_n > 0$, then $|\Lambda_\alpha|_n = |\Lambda|_n > 0$ for every $\alpha < \frac{n}{\delta} - 1$.*

Proof. Suppose α is such that $|\Lambda \setminus \Lambda_\alpha|_n > 0$. If we let μ be n-dimensional measure restricted to Λ_α, then it satisfies (1) with $\varphi(t) = t^n$; so by Lemma 2, the Poincaré series diverges at $n/(1+\beta)$ for all $\beta > \alpha$. Thus $\delta \geq n/(1+\alpha)$ and hence $\alpha \geq \frac{n}{\delta} - 1$, as desired. ☐

A Kleinian group ($n = 3$) is called analytically finite if $(S^2 \setminus \Lambda)/G$ is a finite union of finite area Riemann surfaces. By the Ahlfors finiteness theorem, all finitely generated Kleinian groups have this property.

Corollary 6. *If G is an analytically finite Kleinian group such that $\dim(\Lambda_b) \neq \dim(\Lambda)$, then $|\Lambda_\alpha|_2 > 0$ for all $\alpha < \frac{2}{\delta} - 1$.*

Proof. Corollary 1.4 of [1] implies that such a group has a positive area limit set. So this is a special case of the previous result. □

The Ahlfors conjecture claims there are no finitely generated Kleinian groups with $\Lambda \neq S^2$ and $|\Lambda|_2 > 0$. However, there are analytically finite examples.

Points of Λ_ℓ are closely related to McMullen's "deep points" ([4]). The convex hull of $M = \mathbb{B}^n/G$ is the quotient of the hyperbolic convex hull of Λ in $\mathbb{B}^n$. A point of Λ is called a deep point if the corresponding geodesic is such that $\mathrm{dist}(\gamma(t), \partial C(M))$ increases with an eventually linear lower bound. Clearly, all such points are in Λ_ℓ and the two sets coincide if $\partial C(M)$ is compact. In general, however, there can be points of Λ_ℓ that are not deep points (and the deep points can be empty even if Λ_ℓ is not, e.g., in some quasi-Fuchsian groups).

If $\partial C(M)$ is compact and $|\Lambda_\alpha|_2 > 0$, then Λ is a bit "thicker" than a general positive area set must be. In particular, if $w \in \Lambda_\alpha$, then the largest omitted disk in $\Lambda \cap A_n$, $A_n = \{w : 2^{-n} \leq |z-w| \leq 2^{-n+1}\}$, has diameter $\simeq 2^{-n} \cdot 2^{-\alpha n}$. On the other hand, Cantor sets of positive area can be easily constructed where the largest omitted ball is $\simeq 2^{-n} a_n$ for any series such that $\sum_n a_n < \infty$.

Proof of Lemma 2. Let $X_\alpha = \Lambda \setminus \Lambda_\alpha$. For any $1 > \beta > \alpha$, and $x \in X_\alpha$, the corresponding geodesic ray satisfies

$$\mathrm{dist}(\gamma(t_n), z_0) \leq \beta t_n,$$

for some sequence $t_n \nearrow \infty$ (the sequence may depend on the point x). Given a disk $D(x,r)$ on S^{n-1}, let z_D be the point on the radius from 0 to x at (Euclidean) distance r from S^{n-1}. By definition, every point of X_α is covered by arbitrarily small disks D so that z_D satisfies

$$\rho(z_D, G(0)) \leq \beta \rho(z_D, 0).$$

By the Vitali covering theorem, there is a disjoint subcollection of these disks which cover μ-almost every point of X_α. Let $\{D_n\}$ be an enumeration of this collection, $\{z_n\}$ the corresponding points, and choose $w_n \in G(0)$ so that $\rho(z_n, w_n) \leq \beta\rho(z_D, 0)$. Let $\mathcal{W} = \bigcup_n \{w_n\}$, i.e., is the collection of orbit points chosen.

To proceed further we need a simple lemma about hyperbolic geometry.

Lemma 7. *There is $M < \infty$ so that if $z, w \in \mathbb{B}^n$ and $\rho(z,w) \leq \beta\rho(0,z)$, then*

$$|z-w| \leq M(1-|w|)^{1/(1+\beta)}$$

and

$$1-|z| \leq M(1-|w|)^{1/(1+\beta)}.$$

Proof. Since $\rho(0,w) \leq \rho(0,z) + \rho(z,w) \leq (1+\beta)\rho(0,z)$, we have $\rho(0,z) \geq d \equiv \rho(0,w)/(1+\beta)$. This implies the second estimate since a point in the ball which is a hyperbolic distance d from 0 has Euclidean distance to the boundary $\simeq e^{-d} \simeq (1-|w|)^{1/(1+\beta)}$.

Let $t = \rho(0,w) - d = \beta d$ and $k = \rho(0,z) - d \geq 0$. If $|z-w| = M(1-|w|)^{1/(1+\beta)}$, with $M \gg 1$, then the part of the geodesic between z and w that lies inside $\{x : \rho(0,x) \leq d\}$ has hyperbolic length $\geq \log M - C_2$ for some absolute C_2. Thus

$$\rho(z,w) \geq \log M - C_2 + k + t = \log M + k + \beta d - C_2.$$

Since we also have $\rho(z,w) \leq \beta\rho(0,z) = \beta(d+k)$, we deduce that $\log M \leq C_2 + (\beta-1)k \leq C_2$ (recall $\beta \leq 1$), and the lemma is proven. □

We now continue with the proof of Lemma 2. For each $w \in \mathcal{W}$, let $r_w = M(1-|w|)^{1/(1+\beta)}$, with M as in Lemma 7. Thus $D_n \subset D(w^*, 4r_w)$ and

$$\sum_{D_n \in \mathcal{C}(w)} \mu(D_n) \quad \leq \quad \mu(D(w^*, 2r_w)) \leq C\varphi(r_w) \leq C\varphi((1-|w|)^{\frac{1}{1+\beta}}), \tag{3}$$

where $\mathcal{C}(w)$ is the set of all disks in $\{D_n\}$ associated to the point $w \in \mathcal{W}$. Since the disks $\{D_n\}$ cover full μ measure,

$$\|\mu\| \leq C \sum_{w \in \mathcal{W}} \varphi((1-|w|)^{\frac{1}{1+\beta}}).$$

Thus there is a finite subcollection $\mathcal{W}_1$ over which the sum is $\geq \frac{1}{2}\|\mu\|/C$. Repeating the argument starting with a covering of X_α by disks not in $\mathcal{W}_1$ we get a second, distinct, collection $\mathcal{W}_2$ with the same property. Continuing by induction we obtain an infinite family of such collections and this obviously implies that (2) diverges. $\square$

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