PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 5, Pages 1389–1392
S 0002-9939(03)07171-5
Article electronically published on December 18, 2003

LOCAL AUTOMORPHISMS AND DERIVATIONS ON $\mathbb{M}_n$

SANG OG KIM AND JU SEON KIM

(Communicated by Joseph A. Ball)

Abstract. The aim of this note is to give a short proof that 2-local derivations on M_n, the $n \times n$ matrix algebra over the complex numbers are derivations and to give a shorter proof that 2-local *-automorphisms on M_n are *-automorphisms.

A mapping ϕ of an algebra $\mathcal{A}$ into itself is called a local automorphism (respectively, local derivation) if for every $A \in \mathcal{A}$ there exists an automorphism (respectively, local derivation) ϕ_A of $\mathcal{A}$, depending on A, such that $\phi(A) = \phi_A(A)$. These notions were introduced by Kadison [Kad] and Larson and Sourour [LaSo]. In fact, their definitions were stronger. They have assumed that these mappings are also linear. Larson and Sourour proved that every local derivation on $B(X)$, the algebra of all bounded linear operators on a Banach space X, is a derivation, and provided that X is infinite dimensional, every surjective linear local automorphism of $B(X)$ is an automorphism. In [BrSe], they proved that the surjectivity assumption in the last result can be dropped if X is a separable Hilbert space.

It is easy to see that if we drop the assumption of linearity of the local maps, then the corresponding statements are no longer true. However, in [KoSl], they obtained the following result: If $\mathcal{A}$ is a unital Banach algebra and if $\phi : \mathcal{A} \to \mathbf{C}$ is a map (no linearity is assumed) having the property that $\phi(I) = 1$ and for every $A, B \in \mathcal{A}$, there exists a multiplicative linear functional $\phi_{A,B}$ on $\mathcal{A}$ such that $\phi(A) = \phi_{A,B}(A)$ and $\phi(B) = \phi_{A,B}(B)$, then ϕ is linear and multiplicative.

Motivated by the above considerations, Šemrl [Sem] introduced the following definition.

Definition. Let $\mathcal{A}$ be an algebra. A mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a 2-local automorphism (respectively, 2-local derivation) if for every $A, B \in \mathcal{A}$ there is an automorphism (respectively, derivation) $\phi_{A,B} : \mathcal{A} \to \mathcal{A}$, depending on A and B, such that $\phi(A) = \phi_{A,B}(A)$ and $\phi(B) = \phi_{A,B}(B)$.

Also, they showed the following result.

Theorem 1 ([Sem])**.** *Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Then every 2-local automorphism $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ (no linearity, surjectivity or continuity of ϕ is*

Received by the editors September 10, 2002 and, in revised form, November 20, 2002.
2000 *Mathematics Subject Classification.* Primary 47B49, 15A60.
Key words and phrases. 2-local *-automorphism, 2-local derivation.
This work was supported by the Research Grant from Hallym University, Korea.

assumed) is an automorphism and every 2*-local derivation* $\theta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *(no linearity or continuity of* θ *is assumed) is a derivation.*

In [Sem, Remark] they say that they get the same results in the case that $\mathcal{H}$ is finite dimensional by a long proof involving tedious computations. We found very recently that Molnár [Mol2] gave a short proof that every 2-local automorphism on $M_n(\mathbf{C})$ is an automorphism. It is the aim of this note to make a short proof that 2-local derivations on $M_n(\mathbf{C})$ are derivations and to make the proof shorter in the case of 2-local *-automorphisms (its definition is self-explanatory, i.e., the $\phi_{A,B}$ is a *-automorphism for every A and B). Note that the proof of the following Theorem 2 is similar to that in [Mol1].

Theorem 2. *Let* M_n *be the* $n \times n$ *matrix algebra over* $\mathbf{C}$ *and* $\phi : M_n \to M_n$ *a* 2*-local *-automorphism. Then* ϕ *is a *-automorphism.*

Proof. By the well-known result that every *-automorphism of M_n is of the form $A \mapsto UAU^*$ for some unitary $U \in M_n$, for every $A, B \in M_n$ there is a unitary U in M_n such that

$$\phi(A) = UAU^* \text{ and } \phi(B) = UBU^*.$$

Then if we let tr be the usual trace functional, we have

$$\operatorname{tr}(\phi(A)\phi(B)^*) = \operatorname{tr}(AB^*).$$

Then for any $C \in M_n$,

$$\operatorname{tr}[(\phi(A+B) - \phi(A) - \phi(B))\phi(C)^*] = \operatorname{tr}[((A+B) - A - B)C^*] = 0$$

by the linearity of tr, and then we obtain that

$$\operatorname{tr}[(\phi(A+B) - \phi(A) - \phi(B))(\phi(A+B) - \phi(A) - \phi(B))^*] = 0.$$

Consequently, it follows that ϕ is additive. Let A be an element of M_n and let λ be any scalar. If we use the 2-locality of ϕ to the elements A and λA, we have

$$\phi(\lambda A) = \phi_{A,\lambda A}(\lambda A) = \lambda\phi_{A,\lambda A}(A) = \lambda\phi(A).$$

Then ϕ is homogeneous, and hence ϕ is a linear map. Since the set of eigenvalues of $\phi(A)$ according to multiplicity is the same as that of $A \in M_n$ and $\phi(A^*) = \phi_{A,A^*}(A^*) = \phi_{A,A^*}(A)^* = \phi(A)^*$, there exists by [MaMo, Theorem 4] a unitary $U \in M_n$ such that ϕ is either of the form

$$\phi(A) = UAU^* \quad (A \in M_n)$$

or of the form

$$\phi(A) = UA^tU^* \quad (A \in M_n).$$

Suppose, on the contrary, that $\phi(A) = UA^tU^*$ for all $A \in M_n$. Take two matrices A and B in M_n such that $AB \neq 0$ and $BA = 0$. Then

$$0 \neq \phi_{A,B}(AB) = \phi(A)\phi(B) = UA^tU^*UB^tU^* = U(BA)^tU^* = 0.$$

This shows that $\phi(A) = UAU^*(A \in M_n)$, completing the proof. □

Now we consider 2-local derivations on M_n. We have to mention that the idea for the proof of the following Theorem 3 comes from that of [Sem].

Theorem 3. *Let* M_n *be the* $n \times n$ *matrix algebra over* $\mathbf{C}$ *and* $\phi : M_n \to M_n$ *a* 2*-local derivation. Then* ϕ *is a derivation.*

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbf{C}^n$. We define two matrices $A, N \in M_n$ by

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \ldots & 0 \\ 0 & \frac{1}{2^2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & \frac{1}{2^n} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

It is easy to see that $T \in M_n$ commutes with A if and only if it is diagonal, and if U commutes with N, then U is of the form

$$Ue_k = \mu_k e_1 + \mu_{k-1} e_2 + \cdots + \mu_1 e_k \quad (k = 1, 2, \ldots, n)$$

for some $\{\mu_1, \mu_2, \ldots, \mu_n \mid \mu_k \in \mathbf{C}\}$. That is, U is of the form

$$U = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 & \ldots & \mu_n \\ 0 & \mu_1 & \mu_2 & \ldots & \mu_{n-1} \\ 0 & 0 & \mu_1 & \ldots & \mu_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & \ldots & \ldots & \mu_1 & \mu_2 \\ 0 & 0 & \ldots & 0 & \mu_1 \end{pmatrix}.$$

Replacing ϕ by $\phi - \phi_{A,N}$, if necessary, we can assume that $\phi(A) = \phi(N) = 0$. Every derivation on M_n is inner. It follows that for every $T \in M_n$ there exist diagonal D and U of the above form, depending on T, such that

$$\phi(T) = TD - DT = TU - UT.$$

Let $\{E_{ij}\}_{i,j=1,\cdots,n}$ be the system of matrix units of M_n. Then for any fixed i and j, we have $\phi(E_{ij}) = E_{ij}D - DE_{ij} = E_{ij}U - UE_{ij}$ for some $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and U of the above form. Since $E_{ij}D - DE_{ij} = (\lambda_j - \lambda_i)E_{ij}$ and $E_{ij}U - UE_{ij}$ has 0 as (i,j)-entry, it follows that $\phi(E_{ij}) = 0$. Noting that E_{ij} is the rank one operator $e_i \otimes e_j$, we then have for any $T \in M_n$,

$$\begin{aligned} E_{ij}\phi(T)E_{ij} &= \phi_{E_{ij},T}(E_{ij}TE_{ij}) \\ &= \langle Te_i, e_j \rangle \phi_{E_{ij},T}(E_{ij}) \\ &= \langle Te_i, e_j \rangle \phi(E_{ij}) \\ &= 0. \end{aligned}$$

From this equation it follows that $\langle \phi(T)e_i, e_j \rangle E_{ij} = 0$ and hence $\phi(T) = 0$, completing the proof. □

References

[BrSe] M. Brešar and P. Šemrl, *On local automorphisms and mappings that preserve idempotents*, Studia Math. **113** (1995), 101–108. MR **96i:**47058

[Kad] R. V. Kadison, *Local derivations*, J. Algebra **130** (1990), 494–509. MR **91f:**46092

[KoSl] S. Kowalski and Z. Slodkowski, *A characterization of multiplicative linear functionals in Banach algebras*, Studia Math. **67** (1980), 215–223. MR **82d:**46070

[LaSo] D. R. Larson and A. R. Sourour, *Local derivations and local automorphisms of $B(X)$*, Proc. Sympos. Pure Math. 51, Part 2, American Mathematical Society, Providence, Rhode Island (1990), 187–194. MR **91k:**47106

[MaMo] M. Marcus and B. N. Moyls, *Linear transformations on algebras of matrices*, Canad. J. Math. **11** (1959), 61–66. MR **20:**6432

[Mol1] L. Molnár, *2-local isometries of some operator algebras*, Proc. Edinburgh Math. Soc. **45** (2002), 349–352. MR **2003e:**47067

[Mol2] L. Molnár, *Local automorphisms of operator algebras on Banach spaces*, arXiv:math.OA /0209059 (2002)

[Sem] P. Šemrl, *Local automorphisms and derivations on $B(H)$*, Proc. Amer. Math. Soc. **125** (1997), 2677–2680. MR **98e:**46082

Department of Mathematics, Hallym University, Chuncheon 200-702, Korea
E-mail address: sokim@hallym.ac.kr

Department of Mathematics Education, Seoul National University, Seoul, 151-742, Korea