LOCAL AUTOMORPHISMS AND DERIVATIONS ON $\mathbb{M}_n$

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(Communicated by Joseph A. Ball)

Abstract. The aim of this note is to give a short proof that 2-local derivations on $\mathbb{M}_n$, the $n \times n$ matrix algebra over the complex numbers are derivations and to give a shorter proof that 2-local *-automorphisms on $\mathbb{M}_n$ are *-automorphisms.

A mapping $\phi$ of an algebra $\mathcal{A}$ into itself is called a local automorphism (respectively, local derivation) if for every $A \in \mathcal{A}$ there exists an automorphism (respectively, local derivation) $\phi_A$ of $\mathcal{A}$, depending on $A$, such that $\phi(A) = \phi_A(A)$. These notions were introduced by Kadison [Kad] and Larson and Sourour [LaSo]. In fact, their definitions were stronger. They have assumed that these mappings are also linear. Larson and Sourour proved that every local derivation on $B(X)$, the algebra of all bounded linear operators on a Banach space $X$, is a derivation, and provided that $X$ is infinite dimensional, every surjective linear local automorphism of $B(X)$ is an automorphism. In [BrSe], they proved that the surjectivity assumption in the last result can be dropped if $X$ is a separable Hilbert space.

It is easy to see that if we drop the assumption of linearity of the local maps, then the corresponding statements are no longer true. However, in [KoSl], they obtained the following result: If $\mathcal{A}$ is a unital Banach algebra and if $\phi : \mathcal{A} \to \mathbb{C}$ is a map (no linearity is assumed) having the property that $\phi(I) = 1$ and for every $A, B \in \mathcal{A}$, there exists a multiplicative linear functional $\phi_{A,B}$ on $\mathcal{A}$ such that $\phi(A) = \phi_{A,B}(A)$ and $\phi(B) = \phi_{A,B}(B)$, then $\phi$ is linear and multiplicative.

Motivated by the above considerations, Šemrl [Sem] introduced the following definition.

Definition. Let $\mathcal{A}$ be an algebra. A mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a 2-local automorphism (respectively, 2-local derivation) if for every $A, B \in \mathcal{A}$ there is an automorphism (respectively, derivation) $\phi_{A,B} : \mathcal{A} \to \mathcal{A}$, depending on $A$ and $B$, such that $\phi(A) = \phi_{A,B}(A)$ and $\phi(B) = \phi_{A,B}(B)$.

Also, they showed the following result.

Theorem 1 ([Sem]). Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Then every 2-local automorphism $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ (no linearity, surjectivity or continuity of $\phi$ is
assumed) is an automorphism and every 2-local derivation \( \theta : B(\mathcal{H}) \to B(\mathcal{H}) \) (no linearity or continuity of \( \theta \) is assumed) is a derivation.

In [Sem, Remark] they say that they get the same results in the case that \( \mathcal{H} \) is finite dimensional by a long proof involving tedious computations. We found very recently that Molnár [Mol2] gave a short proof that every 2-local automorphism on \( M_n(\mathbb{C}) \) is an automorphism. It is the aim of this note to make a short proof that 2-local derivations on \( M_n(\mathbb{C}) \) are derivations and to make the proof shorter in the case of 2-local *-automorphisms (its definition is self-explanatory, i.e., the \( \phi_{A,B} \) is a *-automorphism for every \( A \) and \( B \)). Note that the proof of the following Theorem 2 is similar to that in [Mol1].

**Theorem 2.** Let \( M_n \) be the \( n \times n \) matrix algebra over \( \mathbb{C} \) and \( \phi : M_n \to M_n \) a 2-local *-automorphism. Then \( \phi \) is a *-automorphism.

**Proof.** By the well-known result that every *-automorphism of \( A \) can be any scalar. If we use the 2-locality of \( \phi \), then we obtain that
\[
\phi(A) = UAU^* \quad \text{and} \quad \phi(B) = UBU^*.
\]
Then if we let \( tr \) be the usual trace functional, we have
\[
tr(\phi(A)\phi(B)^*) = tr(AB^*).
\]
Then for any \( C \in M_n \),
\[
tr(((\phi(A + B) - \phi(A) - \phi(B))\phi(C)^*) = tr(((A + B) - A - B)C^*) = 0
\]
by the linearity of \( tr \), and then we obtain that
\[
tr(((\phi(A + B) - \phi(A) - \phi(B))\phi(A + B) - \phi(A) - \phi(B))^*) = 0.
\]
Consequently, it follows that \( \phi \) is additive. Let \( A \) be an element of \( M_n \) and let \( \lambda \) be any scalar. We use the 2-locality of \( \phi \) to the elements \( A \) and \( \lambda A \), we have
\[
\phi(\lambda A) = \phi_{\lambda,\lambda A}(\lambda A) = \lambda \phi_{A,\lambda A}(A) = \lambda \phi(A).
\]
Then \( \phi \) is homogeneous, and hence \( \phi \) is a linear map. Since the set of eigenvalues of \( \phi(A) \) according to multiplicity is the same as that of \( A \in M_n \) and \( \phi(A^*) = \phi_{A,A^*}(A^*) = \phi_{A,A^*}(A)^* = \phi(A)^* \), there exists by [MaMo] Theorem 4] a unitary \( U \in M_n \) such that \( \phi \) is either of the form
\[
\phi(A) = UAU^* \quad (A \in M_n)
\]
or of the form
\[
\phi(A) = UABU^* \quad (A \in M_n).
\]
Suppose, on the contrary, that \( \phi(A) = UABU^* \) for all \( A \in M_n \). Take two matrices \( A \) and \( B \) in \( M_n \) such that \( AB \neq 0 \) and \( BA = 0 \). Then
\[
0 \neq \phi_{A,B}(AB) = \phi(A)\phi(B) = UABU^*UBU^* = U(AB)U^* = 0.
\]
This shows that \( \phi(A) = UAU^*(A \in M_n) \), completing the proof.

Now we consider 2-local derivations on \( M_n \). We have to mention that the idea for the proof of the following Theorem 3 comes from that of [Sem].

**Theorem 3.** Let \( M_n \) be the \( n \times n \) matrix algebra over \( \mathbb{C} \) and \( \phi : M_n \to M_n \) a 2-local derivation. Then \( \phi \) is a derivation.
Proof. Let \( \{e_1, e_2, \ldots, e_n\} \) be the standard basis for \( \mathbb{C}^n \). We define two matrices \( A, N \in M_n \) by
\[
A = \begin{pmatrix}
\frac{1}{2} & 0 & \cdots & 0 \\
0 & \frac{1}{2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{2}
\end{pmatrix}, \quad N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

It is easy to see that \( T \in M_n \) commutes with \( A \) if and only if it is diagonal, and if \( U \) commutes with \( N \), then \( U \) is of the form
\[
U_{e_k} = \mu_k e_1 + \mu_{k-1} e_2 + \cdots + \mu_1 e_k \quad (k = 1, 2, \ldots, n)
\]
for some \( \{\mu_1, \mu_2, \ldots, \mu_n \mid \mu_k \in \mathbb{C}\} \). That is, \( U \) is of the form
\[
U = \begin{pmatrix}
\mu_1 & \mu_2 & \mu_3 & \cdots & \mu_n \\
0 & \mu_1 & \mu_2 & \cdots & \mu_{n-1} \\
0 & 0 & \mu_1 & \cdots & \mu_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \mu_1 \\
0 & 0 & \cdots & 0 & \mu_1
\end{pmatrix}.
\]
Replacing \( \phi \) by \( \phi - \phi_{A,N} \), if necessary, we can assume that \( \phi(A) = \phi(N) = 0 \). Every derivation on \( M_n \) is inner. It follows that for every \( T \in M_n \) there exist diagonal \( D \) and \( U \) of the above form, depending on \( T \), such that
\[
\phi(T) = TD - DT = TU - UT.
\]
Let \( \{E_{ij}\}_{i,j=1,\ldots,n} \) be the system of matrix units of \( M_n \). Then for any fixed \( i \) and \( j \), we have \( \phi(E_{ij}) = E_{ij}D - DE_{ij} = E_{ij}U - UE_{ij} \) for some \( D = \text{diag}(\lambda_1, \ldots, \lambda_n) \) and \( U \) of the above form. Since \( E_{ij}D - DE_{ij} = (\lambda_j - \lambda_i)E_{ij} \) and \( E_{ij}U - UE_{ij} \) has 0 as \((i, j)\)-entry, it follows that \( \phi(E_{ij}) = 0 \). Noting that \( E_{ij} \) is the rank one operator \( e_i \otimes e_j \), we then have for any \( T \in M_n \),
\[
E_{ij} \phi(T) E_{ij} = \phi(E_{ij}, T(E_{ij}) E_{ij})
\]
\[
= (Te_i, e_j) \phi(E_{ij}, T(E_{ij}))
\]
\[
= (Te_i, e_j) \phi(E_{ij})
\]
\[
= 0.
\]
From this equation it follows that \( \langle \phi(T) e_i, e_j \rangle E_{ij} = 0 \) and hence \( \phi(T) = 0 \), completing the proof. \( \square \)

References


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