A NOTE ON PERIODIC SOLUTIONS OF NONAUTONOMOUS SECOND-ORDER SYSTEMS

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Abstract. A multiplicity theorem is obtained for periodic solutions of nonautonomous second-order systems with partially periodic potentials by the minimax methods.

1. Introduction and main results

Consider the second-order systems

\[
\begin{cases}
\ddot{u}(t) + \nabla F(t, u(t)) = e(t) \\
u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0
\end{cases}
\]

where \( T > 0 \) and \( F : [0, T] \times \mathbb{R}^N \to \mathbb{R} \) satisfies the following assumption:

\[
F(t, 0) + |\nabla F(t, x)| \leq f(t)|x|^{\alpha} + g(t)
\]

for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \). Suppose that \( F(t, x) \) is \( T \)-periodic in \( x_i, 1 \leq i \leq r \), that is,

\[
F\left(t, x + \sum_{i=1}^{r} k_i T_i e_i\right) = F(t, x)
\]

for a.e. \( t \in [0, T] \), all \( x \in \mathbb{R}^N \) and all integers \( k_i, 1 \leq i \leq r \), where \( (e_i) \) \( 1 \leq i \leq N \) is the canonical basis of \( \mathbb{R}^N \).

With periodic potentials, that is, (3) holding with \( r = N \), the existence and multiplicity theorems are obtained for the nonautonomous second-order system (1) in \([1]\) and \([2]\) respectively. Note that (2) holds automatically with \( \alpha = 0 \) in this case. Under the condition (2) with \( \alpha = 0 \), \([3]\) and \([4]\) consider the nonautonomous second-order system (1) with partially periodic (that is, (3) holding with \( 0 \leq r \leq N \)) and partially uniformly coercive potentials \( (F(t, x) \to +\infty \text{ for every } (x_1, \ldots, x_r) \in \mathbb{R}^r) \) as \( (x_{r+1}, \ldots, x_N) \) tends to infinity in \( \mathbb{R}^{N-r} \). Recently \([5]\) obtains the same result...
as [4] by replacing the partially uniformly coercive condition with the partially semicoercive condition (that is, \( \int_0^T F(t,x)dt \to +\infty \) for every \((x_1,\ldots,x_r) \in \mathbb{R}^r\) as \((x_{r+1},\ldots,x_N)\) tends to infinity in \(\mathbb{R}^{N-r}\)).

In this paper we obtain the same result as [4] and [5] but under weaker coercivity conditions. In fact, we consider the nonautonomous second-order system (1) with the partially periodic potential and sublinear nonlinearity (that is, (2) holding with \(0 < \alpha < 1\)), which is motivated by [5] and [6]. Some results mentioned above are unified and generalized. The following main results are obtained by the minimax methods.

**Theorem.** Suppose that (3) holds and \(e \in L^1(0,T;\mathbb{R}^N)\) satisfying

\[
\int_0^T e(t)dt = 0.
\]

Assume that (2) holds and

\[
|x|^{-2\alpha} \int_0^T F(t,x)dt \to +\infty \quad (or \quad -\infty)
\]

as \(x\) tends to infinity in \(0 \times \mathbb{R}^{N-r}\). Then problem (1) has at least \(r+1\) geometrically distinct solutions in \(H^1_T\), where

\[
H^1_T = \left\{ u : [0,T] \to \mathbb{R}^N \mid u \text{ is absolutely continuous}, \quad u(0) = u(T) \text{ and } \dot{u} \in L^2(0,T;\mathbb{R}^N) \right\}
\]

is a Hilbert space with the norm given by

\[
\|u\| = \left( \int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{1}{2}}
\]

for \(u \in H^1_T\).

**Corollary.** Suppose that \(F(t,x)\) is measurable in \(t\) for every \(x \in \mathbb{R}^N\) and continuously differentiable in \(x\) for a.e. \(t \in [0,T]\), and there exists \(g \in L^1(0,T;\mathbb{R}^+)\) such that

\[
|F(t,0)| + |\nabla F(t,x)| \leq g(t)
\]

for all \(x \in \mathbb{R}^N\) and a.e. \(t \in [0,T]\) and

\[
\int_0^T F(t,x)dt \to +\infty \quad (or \quad -\infty)
\]

as \(x\) tends to infinity in \(0 \times \mathbb{R}^{N-r}\). Assume that (3) and (4) hold. Then problem (1) has at least \(r+1\) geometrically distinct solutions in \(H^1_T\).

**Remark.** Our Theorem unifies and generalizes Theorems 1-3 in [4] and Theorems 1-2 in [3], which are the special cases of our Theorem corresponding to \(\alpha = 0\) and \(r = 0\) respectively. Thus Theorem 2.1 in [3], Theorems 1.5, 1.6 and 4.8 in [2] and Theorem 0.3 in [1] all are the corollaries of our Theorem. There are functions \(F\) satisfying our Theorem and not satisfying the results in [1]–[10]. For example, let \(0 < \alpha < 1\) and

\[
F(t,x) = \beta(t) \left( r + 1 + \sin x_1 + \cdots + \sin x_r + \sum_{j=r+1}^{N} |x_j|^2 \right)^{\alpha + 1} + (e(t),x),
\]
where \( x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N, \beta \in L^1(0, T) \) with
\[
\int_0^T \beta(t)dt \neq 0
\]
and \( e \in L^1(0,T;\mathbb{R}^N) \) satisfying (4).

2. Proof of theorem
For \( u \in H_T^1 \), let \( \overline{u} = \frac{1}{T} \int_0^T u(t)dt \) and \( \tilde{u}(t) = u(t) - \overline{u} \). Then we have Sobolev’s inequality
\[
\| \tilde{u} \|_\infty^2 \leq \frac{T}{12} \| \dot{u} \|_{L^2}^2
\]
and Wirtinger’s inequality
\[
\| \tilde{u} \|_{L^2} \leq \frac{T}{2\pi} \| \dot{u} \|_{L^2}
\]
for all \( u \in H_T^1 \) (see Proposition 1.3 in [7]). Put \( \hat{u}(t) = P\overline{u} + Q\tilde{u} + \tilde{u}(t) \), where
\[
P\overline{u} = \sum_{i=r+1}^N (\overline{u}, e_i)e_i,
\]
\[
Q\overline{u} = \sum_{i=1}^r [(\overline{u}, e_i) - l_i T_i] e_i,
\]
and \( l_i \) (1 \( \leq i \leq r \)) is the unique integer such that
\[
0 \leq (\overline{u}, e_i) - l_i T_i < T_i.
\]
Define the functional \( \varphi \) on \( H_T^1 \) by
\[
\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t, u(t))dt + \int_0^T (e(t), u(t))dt.
\]
Then \( \varphi \) is continuously differentiable by Theorem 1.4 in [7], because (2) implies that
\[
|F(t,x)| \leq |F(t,0)| + \int_0^1 (|\nabla F(t, sx), x|)ds \leq f(t)|x|^{\alpha+1} + (|x| + 1)g(t)
\]
for a.e. \( t \in [0,T] \) and all \( x \in \mathbb{R}^N \). Moreover, one has
\[
\langle \varphi'(u), v \rangle = \int_0^T (\dot{u}(t), \dot{v}(t))dt - \int_0^T (\nabla F(t, u(t)), v(t))dt + \int_0^T (e(t), v(t))dt
\]
for all \( u, v \in H_T^1 \).

Let
\[
G = \left\{ \sum_{i=1}^r k_i T_i e_i \mid k_i \text{ is integer, } 1 \leq i \leq r \right\}
\]
be a discrete subgroup of \( H_T^1 \), and let \( \pi : H_T^1 \rightarrow H_T^1/G \) be the canonical surjection. It is obvious that \( H_T^1/G = X \times V \), where \( X = Y + Z, Y = H_T^1 = \{ u \in H_T^1 \mid \overline{u} = 0 \}, Z = \text{span}\{e_{r+1}, \ldots, e_N\} \) and \( V = \text{span}\{e_1, \ldots, e_r\}/G \) is isomorphic to the torus \( T^r \). Define \( \psi : X \times V \rightarrow R \) by
\[
\psi(\pi(u)) = \varphi(u).
\]
It follows from (3) that \( \psi \) is well-defined. Moreover, \( \psi \) is continuously differentiable.

Now we begin to prove our main result.
Proof of Theorem. In the case that (5\textsuperscript{\textast}) holds, the proof relies on the generalized saddle point theorem due to Liu \cite{Liu}. First assume that (5\textsuperscript{\textast}) holds, that is, \(\psi(\pi(u_n))\) is bounded and \(\psi'(\pi(u_n)) \to 0\). Then \(\varphi(u_n)\) is bounded and \(\varphi'(u_n) \to 0\). It follows from (2) and Sobolev’s inequality (6) that

\[
\left| \int_0^T (\nabla F(t, \bar{u}(t)), \bar{u}(t))dt \right| = \left| \int_0^T (\nabla F(t, Q\bar{u} + P\bar{u} + \bar{u}(t)), \bar{u}(t))dt \right|
\]

\[
\leq \int_0^T f(t) |Q\bar{u} + P\bar{u} + \bar{u}(t)| |\bar{u}(t)|dt + \int_0^T g(t) |\bar{u}(t)|dt
\]

\[
\leq \int_0^T 2f(t) (|Q\bar{u} + P\bar{u}|^\alpha + |\bar{u}(t)|^\alpha) |\bar{u}(t)|dt + \int_0^T g(t) |\bar{u}(t)|dt
\]

\[
\leq 2(|Q\bar{u} + P\bar{u}|^\alpha + \|\bar{u}\|_\infty^\alpha) \|\bar{u}\|_\infty \int_0^T f(t)dt + \|\bar{u}\|_\infty \int_0^T g(t)dt
\]

\[
\leq \frac{3}{T} \|\bar{u}\|_\infty^2 + \frac{T}{3} (Q\bar{u} + P\bar{u})^{2\alpha} \left( \int_0^T f(t)dt \right)^2
\]

\[
+ 2 \|\bar{u}\|_\infty^{\alpha+1} \int_0^T f(t)dt + \|\bar{u}\|_\infty \int_0^T g(t)dt
\]

\[
\leq \frac{1}{4} \int_0^T |\dot{u}(t)|^2 dt + C_1(|P\bar{u}|^{2\alpha} + 1)
\]

\[
+ C_2 \left( \int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{\alpha+1}{2}} + C_3 \left( \int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{1}{2}}
\]

for all \(u \in H^1_T\) and some positive constants \(C_1, C_2\) and \(C_3\), where we have made use of the fact that \(|Q\bar{u}|\) is bounded. Hence one has

\[
\|\bar{u}_n\| \geq |\langle \varphi'(u_n), \bar{u}_n \rangle|
\]

\[
= |\langle \varphi'(\bar{u}_n), \bar{u}_n \rangle|
\]

\[
= \left| \int_0^T |\ddot{u}_n(t)|^2 dt - \int_0^T (\nabla F(t, \dot{u}_n(t)), \bar{u}_n(t))dt + \int_0^T (e(t), \bar{u}_n(t))dt \right|
\]

\[
\geq \frac{3}{4} \int_0^T |\ddot{u}_n(t)|^2 dt - C_1(|P\bar{u}_n|^{2\alpha} + 1) - C_2 \left( \int_0^T |\ddot{u}_n(t)|^2 dt \right)^{\frac{\alpha+1}{2}}
\]

\[
- C_3 \left( \int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{1}{2}} - \|\bar{u}_n\|_\infty \int_0^T |e(t)|dt
\]

\[
\geq \frac{3}{4} \int_0^T |\ddot{u}_n(t)|^2 dt - C_1(|P\bar{u}_n|^{2\alpha} + 1) - C_2 \left( \int_0^T |\ddot{u}_n(t)|^2 dt \right)^{\frac{\alpha+1}{2}}
\]

\[
- C_4 \left( \int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{1}{2}}
\]
for large $n$ and positive constant $C_4$ by the fact that $\varphi'(u_n) \to 0$, (4) and Sobolev’s inequality. It follows from Wirtinger’s inequality (7) that

$$
\left(1 + \frac{T^2}{4\pi^2}\right)\left(\int_0^T |\dddot{u}_n(t)|^2\,dt\right)\frac{1}{2} \geq \|\dddot{u}_n\|
$$

$$
\geq \frac{3}{4} \int_0^T |\dddot{u}_n(t)|^2\,dt - C_1(|\overline{P\pi}|^{2\alpha} + 1)
$$

$$
- C_2 \left(\int_0^T |\dddot{u}_n(t)|^2\,dt\right)\frac{\alpha + 1}{2} - C_4 \left(\int_0^T |\ddot{u}_n(t)|^2\,dt\right)^{\frac{1}{2}}
$$

for large $n$, which implies that

$$
(8) \quad C|\overline{P\pi}|^\alpha \geq \left(\int_0^T |\ddot{u}_n(t)|^2\,dt\right)^{\frac{1}{2}} - C_5
$$

for some $C > 0$, $C_5 > 0$ and all large $n$. By (2) and Sobolev’s inequality (6) we have

$$
\left|\int_0^T [F(t, \ddot{u}(t)) - F(t, \overline{P\pi})]\,dt\right|
$$

$$
= \left|\int_0^T \int_0^1 \langle \nabla F(t, \overline{P\pi} + s(\overline{Q\pi} + \ddot{u}(t))), Q\pi + \ddot{u}(t)\rangle ds\,dt\right|
$$

$$
\leq \int_0^T \int_0^1 f(t)|\overline{P\pi} + s(\overline{Q\pi} + \ddot{u}(t))|^\alpha|Q\pi + \ddot{u}(t)|ds\,dt
$$

$$
+ \int_0^T g(t)|Q\pi + \ddot{u}(t)|\,dt
$$

$$
\leq \int_0^T 2f(t)(|\overline{P\pi}|^\alpha + |\overline{Q\pi} + \ddot{u}(t)|^\alpha)|Q\pi + \ddot{u}(t)|\,dt
$$

$$
+ \int_0^T g(t)|Q\pi + \ddot{u}(t)|\,dt
$$

$$
\leq 2 \left(|\overline{P\pi}|^\alpha + (|\overline{Q\pi}| + \|\ddot{u}\|_\infty)^\alpha\right) \left(|\overline{Q\pi}| + \|\ddot{u}\|_\infty\right) \int_0^T f(t)\,dt
$$

$$
+ (|\overline{Q\pi}| + \|\ddot{u}\|_\infty) \int_0^T g(t)\,dt
$$

$$
\leq \frac{3}{2}(|\overline{Q\pi}| + \|\ddot{u}\|_\infty)^2 + \frac{T^3}{3} |\overline{P\pi}|^{2\alpha} \left(\int_0^T f(t)\,dt\right)^2
$$

$$
+ 2(|\overline{Q\pi}| + \|\ddot{u}\|_\infty)^{\alpha + 1} \int_0^T f(t)\,dt + (|\overline{Q\pi}| + \|\ddot{u}\|_\infty) \int_0^T g(t)\,dt
$$

$$
\leq \frac{1}{4} \int_0^T |\ddot{u}(t)|^2\,dt + C_6(|\overline{P\pi}|^{2\alpha} + 1)
$$

$$
+ C_7 \left(\int_0^T |\ddot{u}(t)|^2\,dt\right)^\frac{\alpha + 1}{2} + C_8 \left(\int_0^T |\ddot{u}(t)|^2\,dt\right)^{\frac{1}{2}}
$$
for all \( u \in H^1_0 \) and some positive constants \( C_6, C_7 \) and \( C_8 \). It follows from the boundedness of \( \{ \varphi(u_n) \} \), (8), (4), (9) and Sobolev’s inequality (6) that

\[
C_9 \leq \varphi(u_n) = \varphi(\tilde{u}_n) = \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 dt - \int_0^T [F(t, \dot{u}_n(t)) - F(t, P\overline{u}_n)] dt \\
- \int_0^T F(t, P\overline{u}_n) dt + \int_0^T (\tilde{u}_n(t)) \, dt \\
\leq \frac{3}{4} \int_0^T |\dot{u}_n(t)|^2 dt + C_6(\|\overline{u}_n\|^{2\alpha} + 1) + C_7 \left( \int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{\alpha + 1}{2}} \\
+ C_8 \left( \int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{1}{2}} - \int_0^T F(t, P\overline{u}_n) dt + \|\overline{u}_n\| \int_0^T |\tilde{u}_n(t)| \, dt \\
\leq \frac{3}{4} \int_0^T |\dot{u}_n(t)|^2 dt + C_6(\|\overline{u}_n\|^{2\alpha} + 1) + C_7 \left( \int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{\alpha + 1}{2}} \\
+ C_10 \left( \int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{1}{2}} - \int_0^T F(t, P\overline{u}_n) dt
\]

(10)

for all large \( n \) and some constants \( C_9, C_{10} \). Then (10), (8) and (5) imply that \( (|P\overline{u}_n|) \) is bounded. In fact, if not, without loss of generality we may assume that \( |P\overline{u}_n| \to \infty \) as \( n \to \infty \). Then from (8) and (10) one obtains

\[
\lim \inf_{n \to \infty} |P\overline{u}_n|^{-2\alpha} \int_0^T F(t, P\overline{u}_n) dt > -\infty,
\]

which contradicts (5). Hence \( (|P\overline{u}_n|) \) is bounded. Furthermore, \( (\tilde{u}_n) \) is bounded by (8), which implies that \( \dot{\tilde{u}}_n \) is bounded. Arguing then as in Proposition 4.1 in \[7\], we conclude that the \((PS)\) condition is satisfied.

Now we check the link condition that

(a) \( \inf \{ \psi(\pi(u)) \mid \pi(u) \in Y \times V \} > -\infty \), and

(b) \( \psi(\pi(x)) \to -\infty \) uniformly for \( \pi(Qx) \in V \) as \( |Px| \to \infty \), where \( x \in \mathbb{R}^N \).

For \( \pi(u) \in Y \times V, u = Q\overline{u} + \tilde{u} \). It follows from (9) that

\[
\left| \int_0^T [F(t, u(t)) - F(t, 0)] dt \right| \leq \frac{1}{4} \int_0^T |\dot{u}(t)|^2 dt + C_6 + C_7 \left( \int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{\alpha + 1}{2}} \\
+ C_8 \left( \int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{1}{2}}
\]

(11)
for all \( \pi(u) \in Y \times V \). Hence, by (4), Sobolev’s inequality (6) and (11), we have

\[
\psi(\pi(u)) = \psi(\pi(\tilde{u} + Q\pi)) = \varphi(\tilde{u} + Q\pi)
\]

\[
= \frac{1}{2} \int_0^T |\dot{\tilde{u}}(t)|^2 dt - \int_0^T F(t, 0) dt - \int_0^T [F(t, \tilde{u}(t) + Q\pi) - F(t, 0)] dt
\]

\[
+ \int_0^T (e(t), \tilde{u}(t) + Q\pi) dt
\]

\[
\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t, 0) dt - (||Q\pi||_{\infty} + ||\tilde{u}||_{\infty}) \int_0^T |e(t)| dt
\]

\[
- C_6 - C_7 \left( \int_0^T |\dot{u}(t)|^2 \right)^{\frac{2+1}{2}} - C_8 \left( \int_0^T |\dot{u}(t)|^2 \right)^{\frac{1}{2}}
\]

\[
\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - C_{11} - C_7 \left( \int_0^T |\dot{u}(t)|^2 \right)^{\frac{2+1}{2}} - C_{12} \left( \int_0^T |\dot{u}(t)|^2 \right)^{\frac{1}{2}}
\]

for all \( \pi(u) \in Y \times V \), which implies (a). It follows from (4) and (2) that

\[
\psi(\pi(x)) = \varphi(x)
\]

\[
= \varphi(\ddot{x})
\]

\[
= - \int_0^T F(t, Px + Qx) dt
\]

\[
= - \int_0^T F(t, Px) dt - \int_0^T \int_0^1 (\nabla F(t, Px + sQx), Qx) ds dt
\]

\[
\leq - \int_0^T F(t, Px) dt + \int_0^T f(t)|Px + Qx|^\alpha |Qx| dt + \int_0^T g(t)|Qx| dt
\]

\[
\leq - \int_0^T F(t, Px) dt + C_{13} |Px|^\alpha + C_{14}
\]

for all \( x \in R^N \) and some constants \( C_{13} \) and \( C_{14} \). Hence we have, by (5+),

\[
\limsup_{|Px| \to \infty} |Px|^{-2\alpha} \psi(\pi(x)) \leq -\infty
\]

uniformly for all \( Qx \in R^r \), which implies (b). It follows from the generalized saddle point theorem (Theorem 1.7 in [4]) that \( \psi \) has at least \( r + 1 \) critical points. Hence \( \varphi \) has at least \( r + 1 \) geometrically distinct critical points. Therefore, in this case, problem (1) has at least \( r + 1 \) geometrically distinct solutions in \( H^1_T \).
In the case that (5−) holds, the proof relies on Theorem 4.12 in [7]. By (4), (9) and Sobolev’s inequality (6), we have

\[ \varphi(u) = \varphi(\bar{u}) \]
\[ = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t, P\bar{u}) dt + \int_0^T (e(t), Q\bar{u} + \bar{u}(t)) dt 
- \int_0^T \int_0^1 (\nabla F(t, P\bar{u} + s(Q\bar{u} + \bar{u}(t)), Q\bar{u} + \bar{u}(t)) ds dt 
\ge \frac{1}{4} \int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t, P\bar{u}) dt - (|Q\bar{u}| + \|\bar{u}\|_\infty) \int_0^T |e(t)| dt 
- C_6(|P\bar{u}|^{2\alpha} + 1) - C_7 \left( \int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{\alpha+1}{2}} - C_8 \left( \int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{1}{2}} 
\ge \frac{1}{4} \int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t, P\bar{u}) dt - C_{15}(|P\bar{u}|^{2\alpha} + 1) 
- C_7 \left( \int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{\alpha+1}{2}} - C_{16} \left( \int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{1}{2}} \]

for all \( u \in H^1 \), which implies that \( \varphi \) is bounded from below. Moreover, the functional \( \varphi \) satisfies the \((PS)_C\) condition; that is, for every sequence \((u_n)\) in \( H^1 \) such that \( \varphi(u_n) \) is bounded and \( \varphi'(u_n) \to 0 \), the sequence \( \pi(u_n) \) has a convergent subsequence (see Definition 4.2 in [7]). In fact, the boundedness of \( \varphi(u_n) \), (5−) and (12) imply that \( (\bar{u}_n) \) and \( (P\bar{u}_n) \) are bounded. Hence \( (\bar{u}_n) \) is bounded. As in the proof of Proposition 4.1 in [7], \( (\bar{u}_n) \) has a convergent subsequence; so does \( \pi(u_n) = \pi(\bar{u}_n) \). Now the Theorem in this case follows from Theorem 4.12 in [7].

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