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A NOTE ON PERIODIC SOLUTIONS OF NONAUTONOMOUS SECOND-ORDER SYSTEMS

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ABSTRACT. A multiplicity theorem is obtained for periodic solutions of nonautonomous second-order systems with partially periodic potentials by the minimax methods.

1. INTRODUCTION AND MAIN RESULTS

Consider the second-order systems

$$\begin{cases} \ddot{u}(t) + \bigtriangledown F(t, u(t)) = e(t) & \text{a.e. } t \in [0, T] \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases} \tag{1}$$

where $T > 0$ and $F : [0, T] \times R^N \to R$ satisfies the following assumption:

$F(t, x)$ is measurable in t for every $x \in R^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $f, g \in L^1(0, T; R^+)$ and $\alpha \in [0, 1[$ such that

$$|F(t, 0)| + |\nabla F(t, x)| \leq f(t)|x|^\alpha + g(t) \tag{2}$$

for all $x \in R^N$ and a.e. $t \in [0, T]$. Suppose that $F(t, x)$ is T_i-periodic in x_i, $1 \leq i \leq r$, that is,

$$F\left(t, x + \sum_{i=1}^{r} k_i T_i e_i\right) = F(t, x) \tag{3}$$

for a.e. $t \in [0, T]$, all $x \in R^N$ and all integers $k_i, 1 \leq i \leq r$, where (e_i) $(1 \leq i \leq N)$ is the canonical basis of R^N.

With periodic potentials, that is, (3) holding with $r = N$, the existence and multiplicity theorems are obtained for the nonautonomous second-order system (1) in [1] and [2] respectively. Note that (2) holds automatically with $\alpha = 0$ in this case. Under the condition (2) with $\alpha = 0$, [3] and [4] consider the nonautonomous second-order system (1) with partially periodic (that is, (3) holding with $0 \leq r \leq N$) and partially uniformly coercive potentials ($F(t, x) \to +\infty$ for every $(x_1, \ldots, x_r) \in R^r$ as $(x_{r+1}, \ldots, x_N)$ tends to infinity in R^{N-r}). Recently [5] obtains the same result

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as [4] by replacing the partially uniformly coercive condition with the partially semicoercive condition (that is, $\int_0^T F(t,x)dt \to +\infty$ for every $(x_1,\dots,x_r) \in R^r$ as $(x_{r+1},\dots,x_N)$ tends to infinity in R^{N-r}).

In this paper we obtain the same result as [4] and [5] but under weaker coercivity conditions. In fact, we consider the nonautonomous second-order system (1) with the partially periodic potential and sublinear nonlinearity (that is, (2) holding with $0 \le \alpha < 1$), which is motivated by [5] and [6]. Some results mentioned above are unified and generalized. The following main results are obtained by the minimax methods.

Theorem. *Suppose that* (3) *holds and* $e \in L^1(0,T;R^N)$ *satisfying*

$$\int_0^T e(t)dt = 0. \tag{4}$$

Assume that (2) *holds and*

$$|x|^{-2\alpha}\int_0^T F(t,x)dt \to +\infty \ (or\ -\infty) \tag{$5^{\pm}$}$$

as x *tends to infinity in* $0 \times R^{N-r}$*. Then problem* (1) *has at least* $r+1$ *geometrically distinct solutions in* H_T^1*, where*

$$H_T^1 = \left\{ u:[0,T]\to R^N \;\middle|\; \begin{array}{l} u \text{ is absolutely continuous,} \\ u(0)=u(T) \text{ and } \dot u \in L^2(0,T;R^N) \end{array} \right\}$$

is a Hilbert space with the norm given by

$$\|u\| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot u(t)|^2 dt\right)^{\frac{1}{2}}$$

for $u \in H_T^1$.

Corollary. *Suppose that* $F(t,x)$ *is measurable in* t *for every* $x \in R^N$ *and continuously differentiable in* x *for a.e.* $t \in [0,T]$*, and there exists* $g \in L^1(0,T;R^+)$ *such that*

$$|F(t,0)| + |\nabla F(t,x)| \le g(t)$$

for all $x \in R^N$ *and a.e.* $t \in [0,T]$ *and*

$$\int_0^T F(t,x)dt \to +\infty \ (or\ -\infty)$$

as x *tends to infinity in* $0 \times R^{N-r}$*. Assume that* (3) *and* (4) *hold. Then problem* (1) *has at least* $r+1$ *geometrically distinct solutions in* H_T^1*.*

Remark. Our Theorem unifies and generalizes Theorems 1-3 in [5] and Theorems 1-2 in [6], which are the special cases of our Theorem corresponding to $\alpha = 0$ and $r = 0$ respectively. Thus Theorem 2.1 in [4], Theorems 1.5, 1.6 and 4.8 in [7] and Theorem 0.3 in [2] all are the corollaries of our Theorem. There are functions F satisfying our Theorem and not satisfying the results in [1]–[10]. For example, let $0 < \alpha < 1$ and

$$F(t,x) = \beta(t)\left(r+1+\sin x_1 + \cdots + \sin x_r + \sum_{j=r+1}^{N} |x_j|^2\right)^{\frac{\alpha+1}{2}} + (e(t),x),$$

where $x = (x_1, x_2, \dots, x_N) \in R^N$, $\beta \in L^1(0,T)$ with

$$\int_0^T \beta(t)dt \neq 0$$

and $e \in L^1(0,T;R^N)$ satisfying (4).

2. Proof of theorem

For $u \in H_T^1$, let $\overline{u} = \frac{1}{T}\int_0^T u(t)dt$ and $\widetilde{u}(t) = u(t) - \overline{u}$. Then we have Sobolev's inequality

$$\|\widetilde{u}\|_\infty^2 \leq \frac{T}{12}\|\dot{u}\|_{L^2}^2 \tag{6}$$

and Wirtinger's inequality

$$\|\widetilde{u}\|_{L^2} \leq \frac{T}{2\pi}\|\dot{u}\|_{L^2} \tag{7}$$

for all $u \in H_T^1$ (see Proposition 1.3 in [7]). Put $\hat{u}(t) = P\overline{u} + Q\overline{u} + \widetilde{u}(t)$, where

$$P\overline{u} = \sum_{i=r+1}^{N} (\overline{u}, e_i)e_i, \qquad Q\overline{u} = \sum_{i=1}^{r} \left[(\overline{u}, e_i) - l_iT_i\right] e_i,$$

and l_i $(1 \leq i \leq r)$ is the unique integer such that

$$0 \leq (\overline{u}, e_i) - l_iT_i < T_i.$$

Define the functional φ on H_T^1 by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t,u(t))dt + \int_0^T (e(t), u(t))dt.$$

Then φ is continuously differentiable by Theorem 1.4 in [7], because (2) implies that

$$|F(t,x)| \leq |F(t,0)| + \left|\int_0^1 (\nabla F(t,sx), x)ds\right| \leq f(t)|x|^{\alpha+1} + (|x|+1)g(t)$$

for a.e. $t \in [0,T]$ and all $x \in R^N$. Moreover, one has

$$\langle \varphi'(u), v\rangle = \int_0^T (\dot{u}(t), \dot{v}(t))dt - \int_0^T (\nabla F(t,u(t)), v(t))dt + \int_0^T (e(t), v(t))dt$$

for all $u, v \in H_T^1$.

Let

$$G = \left\{ \sum_{i=1}^{r} k_iT_ie_i \,\middle|\, k_i \text{ is integer, } 1 \leq i \leq r \right\}$$

be a discrete subgroup of H_T^1 and let $\pi : H_T^1 \to H_T^1/G$ be the canonical surjection. It is obvious that $H_T^1/G = X \times V$, where $X = Y + Z, Y = \widetilde{H}_T^1 = \{u \in H_T^1 \mid \overline{u} = 0\}, Z = \operatorname{span}\{e_{r+1}, \dots, e_N\}$ and $V = \operatorname{span}\{e_1, \dots, e_r\}/G$ is isomorphic to the torus T^r. Define $\psi : X \times V \to R$ by

$$\psi(\pi(u)) = \varphi(u).$$

It follows from (3) that ψ is well-defined. Moreover, ψ is continuously differentiable.

Now we begin to prove our main result.

Proof of Theorem. In the case that (5^+) holds, the proof relies on the generalized saddle point theorem due to Liu [4]. First assume that $(\pi(u_n))$ is a (PS) sequence for ψ, that is, $\psi(\pi(u_n))$ is bounded and $\psi'(\pi(u_n)) \to 0$. Then $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$. It follows from (2) and Sobolev's inequality (6) that

$$\begin{aligned}
&\left|\int_0^T (\nabla F(t, \hat{u}(t)), \widetilde{u}(t))dt\right| = \left|\int_0^T (\nabla F(t, Q\overline{u} + P\overline{u} + \widetilde{u}(t)), \widetilde{u}(t))dt\right| \\
&\leq \int_0^T f(t)|Q\overline{u} + P\overline{u} + \widetilde{u}(t)|^\alpha |\widetilde{u}(t)|dt + \int_0^T g(t)|\widetilde{u}(t)|dt \\
&\leq \int_0^T 2f(t)(|Q\overline{u} + P\overline{u}|^\alpha + |\widetilde{u}(t)|^\alpha)|\widetilde{u}(t)|dt + \int_0^T g(t)|\widetilde{u}(t)|dt \\
&\leq 2(|Q\overline{u} + P\overline{u}|^\alpha + \|\widetilde{u}\|_\infty^\alpha)\|\widetilde{u}\|_\infty \int_0^T f(t)dt + \|\widetilde{u}\|_\infty \int_0^T g(t)dt \\
&\leq \frac{3}{T}\|\widetilde{u}\|_\infty^2 + \frac{T}{3}|Q\overline{u} + P\overline{u}|^{2\alpha}\left(\int_0^T f(t)dt\right)^2 \\
&\quad + 2\|\widetilde{u}\|_\infty^{\alpha+1}\int_0^T f(t)dt + \|\widetilde{u}\|_\infty \int_0^T g(t)dt \\
&\leq \frac{1}{4}\int_0^T |\dot{u}(t)|^2 dt + C_1(|P\overline{u}|^{2\alpha} + 1) \\
&\quad + C_2\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{\alpha+1}{2}} + C_3\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}
\end{aligned}$$

for all $u \in H_T^1$ and some positive constants C_1, C_2 and C_3, where we have made use of the fact that $|Q\overline{u}|$ is bounded. Hence one has

$$\begin{aligned}
\|\widetilde{u}_n\| &\geq |\langle \varphi'(u_n), \widetilde{u}_n\rangle| \\
&= |\langle \varphi'(\hat{u}_n), \widetilde{u}_n\rangle| \\
&= \left|\int_0^T |\dot{u}_n(t)|^2 dt - \int_0^T (\nabla F(t, \hat{u}_n(t)), \widetilde{u}_n(t))dt + \int_0^T (e(t), \widetilde{u}_n(t))dt\right| \\
&\geq \frac{3}{4}\int_0^T |\dot{u}_n(t)|^2 dt - C_1(|P\overline{u}_n|^{2\alpha} + 1) - C_2\left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{\alpha+1}{2}} \\
&\quad - C_3\left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{1}{2}} - \|\widetilde{u}_n\|_\infty \int_0^T |e(t)|dt \\
&\geq \frac{3}{4}\int_0^T |\dot{u}_n(t)|^2 dt - C_1(|P\overline{u}_n|^{2\alpha} + 1) - C_2\left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{\alpha+1}{2}} \\
&\quad - C_4\left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{1}{2}}
\end{aligned}$$

for large n and positive constant C_4 by the fact that $\varphi'(u_n) \to 0$, (4) and Sobolev's inequality. It follows from Wirtinger's inequality (7) that

$$\begin{aligned}\left(1+\frac{T^2}{4\pi^2}\right)^{\frac{1}{2}}\left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{1}{2}} &\geq \|\widetilde{u}_n\| \\ &\geq \frac{3}{4}\int_0^T |\dot{u}_n(t)|^2 dt - C_1(|P\overline{u}_n|^{2\alpha}+1) \\ &\quad - C_2\left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{\alpha+1}{2}} - C_4\left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{1}{2}}\end{aligned}$$

for large n, which implies that

$$C|P\overline{u}_n|^{\alpha} \geq \left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{1}{2}} - C_5 \tag{8}$$

for some $C > 0$, $C_5 > 0$ and all large n. By (2) and Sobolev's inequality (6) we have

$$\begin{aligned}&\left|\int_0^T [F(t,\hat{u}(t)) - F(t,P\overline{u})]dt\right| \\ &\quad = \left|\int_0^T \int_0^1 (\nabla F(t, P\overline{u} + s(Q\overline{u}+\widetilde{u}(t))), Q\overline{u}+\widetilde{u}(t))dsdt\right| \\ &\quad \leq \int_0^T \int_0^1 f(t)|P\overline{u} + s(Q\overline{u}+\widetilde{u}(t))|^{\alpha}|Q\overline{u}+\widetilde{u}(t)|dsdt \\ &\quad\quad + \int_0^T g(t)|Q\overline{u}+\widetilde{u}(t)|dt \\ &\quad \leq \int_0^T 2f(t)(|P\overline{u}|^{\alpha} + |Q\overline{u}+\widetilde{u}(t)|^{\alpha})|Q\overline{u}+\widetilde{u}(t)|dt \\ &\quad\quad + \int_0^T g(t)|Q\overline{u}+\widetilde{u}(t)|dt \\ &\quad \leq 2\left(|P\overline{u}|^{\alpha} + (|Q\overline{u}|+\|\widetilde{u}\|_{\infty})^{\alpha}\right)\left(|Q\overline{u}|+\|\widetilde{u}\|_{\infty}\right)\int_0^T f(t)dt \\ &\quad\quad + \left(|Q\overline{u}+\|\widetilde{u}\|_{\infty}\right)\int_0^T g(t)dt \\ &\quad \leq \frac{3}{T}(|Q\overline{u}|+\|\widetilde{u}\|_{\infty})^2 + \frac{T}{3}|P\overline{u}|^{2\alpha}\left(\int_0^T f(t)dt\right)^2 \\ &\quad\quad + 2(|Q\overline{u}|+\|\widetilde{u}\|_{\infty})^{\alpha+1}\int_0^T f(t)dt + (|Q\overline{u}|+\|\widetilde{u}\|_{\infty})\int_0^T g(t)dt \\ &\quad \leq \frac{1}{4}\int_0^T |\dot{u}(t)|^2 dt + C_6(|P\overline{u}|^{2\alpha}+1) \\ &\quad\quad + C_7\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{\alpha+1}{2}} + C_8\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}\end{aligned} \tag{9}$$

for all $u \in H_T^1$ and some positive constants C_6, C_7 and C_8. It follows from the boundedness of $\{\varphi(u_n)\}$, (8), (4), (9) and Sobolev's inequality (6) that

$$\begin{aligned} C_9 &\leq \varphi(u_n) \\ &= \varphi(\hat{u}_n) \\ &= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2 dt - \int_0^T [F(t,\hat{u}_n(t)) - F(t,P\overline{u}_n)]dt \\ &\quad - \int_0^T F(t,P\overline{u}_n)dt + \int_0^T (e(t),\widetilde{u}_n(t))dt \\ &\leq \frac{3}{4}\int_0^T |\dot{u}_n(t)|^2 dt + C_6(|\overline{u}_n|^{2\alpha}+1) + C_7\left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{\alpha+1}{2}} \\ &\quad + C_8\left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{1}{2}} - \int_0^T F(t,P\overline{u}_n)dt + \|\widetilde{u}_n\|_\infty \int_0^T |e(t)|dt \\ &\leq \frac{3}{4}\int_0^T |\dot{u}_n(t)|^2 dt + C_6(|\overline{u}_n|^{2\alpha}+1) + C_7\left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{\alpha+1}{2}} \\ &\quad + C_{10}\left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{1}{2}} - \int_0^T F(t,P\overline{u}_n)dt \end{aligned} \tag{10}$$

for all large n and some constants C_9, C_{10}. Then (10), (8) and (5^+) imply that $(|P\overline{u}_n|)$ is bounded. In fact, if not, without loss of generality we may assume that $|P\overline{u}_n| \to \infty$ as $n \to \infty$. Then from (8) and (10) one obtains

$$\liminf_{n\to\infty} |P\overline{u}_n|^{-2\alpha} \int_0^T F(t,P\overline{u}_n)dt > -\infty,$$

which contradicts (5^+). Hence $(|P\overline{u}_n|)$ is bounded. Furthermore, $(\widetilde{u}_n)$ is bounded by (8), which implies that $\hat{u}_n$ is bounded. Arguing then as in Proposition 4.1 in [7], we conclude that the (PS) condition is satisfied.

Now we check the link condition that

(a) $\inf\{\psi(\pi(u)) \mid \pi(u) \in Y \times V\} > -\infty$, and

(b) $\psi(\pi(x)) \to -\infty$ uniformly for $\pi(Qx) \in V$ as $|Px| \to \infty$, where $x \in R^N$.

For $\pi(u) \in Y \times V$, $u = Q\overline{u} + \widetilde{u}$. It follows from (9) that

$$\begin{aligned} \left|\int_0^T [F(t,u(t)) - F(t,0)]dt\right| &\leq \frac{1}{4}\int_0^T |\dot{u}(t)|^2 dt + C_6 + C_7\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{\alpha+1}{2}} \\ &\quad + C_8\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}} \end{aligned} \tag{11}$$

for all $\pi(u) \in Y \times V$. Hence, by (4), Sobolev's inequality (6) and (11), we have

$$\begin{aligned}
\psi(\pi(u)) &= \psi(\pi(\widetilde{u} + Q\overline{u})) \\
&= \varphi(\widetilde{u} + Q\overline{u}) \\
&= \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t,0)dt - \int_0^T [F(t,\widetilde{u}(t) + Q\overline{u}) - F(t,0)]dt \\
&\quad + \int_0^T (e(t), \widetilde{u}(t) + Q\overline{u})dt \\
&\geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t,0)dt - (|Q\overline{u}|_\infty + \|\widetilde{u}\|_\infty)\int_0^T |e(t)|dt \\
&\quad - C_6 - C_7\left(\int_0^T |\dot{u}(t)|^2\right)^{\frac{\alpha+1}{2}} - C_8\left(\int_0^T |\dot{u}(t)|^2\right)^{\frac{1}{2}} \\
&\geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - C_{11} - C_7\left(\int_0^T |\dot{u}(t)|^2\right)^{\frac{\alpha+1}{2}} - C_{12}\left(\int_0^T |\dot{u}(t)|^2\right)^{\frac{1}{2}}
\end{aligned}$$

for all $\pi(u) \in Y \times V$, which implies (a). It follows from (4) and (2) that

$$\begin{aligned}
\psi(\pi(x)) &= \varphi(x) \\
&= \varphi(\hat{x}) \\
&= -\int_0^T F(t, Px + Qx)dt \\
&= -\int_0^T F(t, Px)dt - \int_0^T \int_0^1 (\nabla F(t, Px + sQx), Qx)dsdt \\
&\leq -\int_0^T F(t, Px)dt + \int_0^T f(t)|Px + Qx|^\alpha |Qx|dt + \int_0^T g(t)|Qx|dt \\
&\leq -\int_0^T F(t, Px)dt + C_{13}|Px|^\alpha + C_{14}
\end{aligned}$$

for all $x \in R^N$ and some constants C_{13} and C_{14}. Hence we have, by (5^+),

$$\limsup_{|Px|\to\infty} |Px|^{-2\alpha}\psi(\pi(x)) \leq -\infty$$

uniformly for all $Qx \in R^r$, which implies (b). It follows from the generalized saddle point theorem (Theorem 1.7 in [4]) that ψ has at least $r+1$ critical points. Hence φ has at least $r+1$ geometrically distinct critical points. Therefore, in this case, problem (1) has at least $r+1$ geometrically distinct solutions in H_T^1.

In the case that (5^-) holds, the proof relies on Theorem 4.12 in [7]. By (4), (9) and Sobolev's inequality (6), we have

$$\begin{aligned}
\varphi(u) &= \varphi(\hat{u}) \\
&= \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t, P\overline{u})dt + \int_0^T (e(t), Q\overline{u} + \widetilde{u}(t))dt \\
&\quad - \int_0^T \int_0^1 (\nabla F(t, P\overline{u} + s(Q\overline{u} + \widetilde{u}(t))), Q\overline{u} + \widetilde{u}(t))ds dt \\
&\geq \frac{1}{4}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t, P\overline{u})dt - (|Q\overline{u}| + \|\widetilde{u}\|_\infty)\int_0^T |e(t)|dt \\
&\quad - C_6(|P\overline{u}|^{2\alpha} + 1) - C_7\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{\alpha+1}{2}} - C_8\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}} \\
&\geq \frac{1}{4}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t, P\overline{u})dt - C_{15}(|P\overline{u}|^{2\alpha} + 1) \\
&\quad - C_7\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{\alpha+1}{2}} - C_{16}\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}
\end{aligned}$$

for all $u \in H_T^1$, which implies that φ is bounded from below. Moreover, the functional φ satisfies the $(PS)_G$ condition; that is, for every sequence (u_n) in H_T^1 such that $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$, the sequence $\pi(u_n)$ has a convergent subsequence (see Definition 4.2 in [7]). In fact, the boundedness of $\varphi(u_n)$, (5^-) and (12) imply that $(\widetilde{u}_n)$ and $(P\overline{u}_n)$ are bounded. Hence $(\hat{u}_n)$ is bounded. As in the proof of Proposition 4.1 in [7], $(\hat{u}_n)$ has a convergent subsequence; so does $\pi(u_n) = \pi(\hat{u}_n)$. Now the Theorem in this case follows from Theorem 4.12 in [7].

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