

ON BOHMAN'S CONJECTURE RELATED TO A SUM PACKING PROBLEM OF ERDŐS

R. AHLWEDE, H. AYDINIAN, AND L. H. KHACHATRIAN

(Communicated by John R. Stembridge)

ABSTRACT. Motivated by a sum packing problem of Erdős, Bohman discussed an extremal geometric problem which seems to have an independent interest. Let H be a hyperplane in \mathbb{R}^n such that $H \cap \{0, \pm 1\}^n = \{0^n\}$. The problem is to determine

$$f(n) \triangleq \max_H |H \cap \{0, \pm 1, \pm 2\}^n|.$$

Bohman (1996) conjectured that

$$f(n) = \frac{1}{2}(1 + \sqrt{2})^n + \frac{1}{2}(1 - \sqrt{2})^n.$$

We show that for some constants c_1, c_2 we have $c_1(2, 538)^n < f(n) < c_2(2, 723)^n$ —disproving the conjecture. We also consider a more general question of the estimation of $|H \cap \{0, \pm 1, \dots, \pm m\}^n|$, when $H \cap \{0, \pm 1, \dots, \pm k\}^n = \{0^n\}$, $m > k > 1$.

1. INTRODUCTION AND STATEMENT OF THE RESULT

Let H be a hyperplane in \mathbb{R}^n so that $H \cap \{0, \pm 1\}^n = \{0^n\}$. Let

$$f(n) = \max_H |H \cap \{0, \pm 1, \pm 2\}^n|.$$

The problem (of determination of $f(n)$) was raised by Bohman [1] in connection with a subset sum problem of Erdős [2].

A set S of positive integers $b_1 < b_2 < \dots < b_n$ has distinct subset sums if all sums of subsets are distinct. Erdős [2] asked for the value of

$$g(n) \triangleq \min\{a_n : S \text{ has distinct subset sums, } |S| = n\}.$$

A long-standing conjecture of Erdős claims that $g(n) \geq c2^n$ for some constant c .

In [1] Bohman explained the relationship between functions $f(n)$ and $g(n)$, and noticed that studying the function $f(n)$ might be helpful for further investigation of the problem of Erdős.

Suppose a hyperplane H defined by the equation

$$(1.1) \quad \sum_{i=0}^{n-1} a_i x_i = 0; \quad a_0, \dots, a_{n-1} \in \mathbb{N}$$

Received by the editors October 22, 2001 and, in revised form, August 22, 2002 and January 15, 2003.

2000 *Mathematics Subject Classification*. Primary 11P99; Secondary 05D05.

satisfies $H \cap \{0, \pm 1\}^n = \{0^n\}$. This clearly means that $\{a_0, \dots, a_{n-1}\}$ has distinct subset sums. A simple example of such a set with $a_{n-1} \leq 2^{n-1}$ is $\{1, 2, 2^2, \dots, 2^{n-1}\}$. For more complicated examples see [1], [3].

For $f(n)$, Bohman [1] conjectured that

$$f(n) = \frac{1}{2}(1 + \sqrt{2})^n + \frac{1}{2}(1 - \sqrt{2})^n,$$

showing that this number can be achieved, taking $a_i = 2^i$ ($i = 0, \dots, n - 1$) in (1.1).

Let us consider now the hyperplanes defined by

$$(1.2) \quad \sum_{i=0}^{n-1} 2^i \lambda_i x_i = 0,$$

where $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ are odd integers.

One can easily see that the set $\{\lambda_0, 2\lambda_1, \dots, 2^{n-1}\lambda_{n-1}\}$ has distinct subset sums.

Let $f^*(n)$ denote the maximum possible number of solutions $x^n \in \{0, \pm 1, \pm 2\}^n$ of equation (1.2) over all choices of odd integers $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$.

Theorem 1. *For some constants c', c''*

$$c'(2, 538)^n < f^*(n) < c''(2, 547)^n.$$

Clearly this means that $f(n) > c_1(2, 538)^n$ and the conjecture of Bohman fails.

Our next goal is to give an upper bound for $f(n)$. A simple upper bound is

$$(1.3) \quad f(n) \leq 3^n.$$

Indeed, let X be the set of solutions $x^n \in \{0, \pm 1, \pm 2\}^n$ of equation (1.1). Then observe that for any $u^n, v^n \in \{0, 1\}^n$, $u^n \neq v^n$, we have $(X + u^n) \cap (X + v^n) = \emptyset$. This implies that $|X + \{0, 1\}^n| = |X| |\{0, 1\}^n| = |X| 2^n$. On the other hand, $\{X + \{0, 1\}^n\} \subset \{0, \pm 1, \pm 2, 3\}^n$. Hence $|X| 2^n \leq 6^n$ and thus (1.3). The next result improves bound (1.3).

Theorem 2. *For some constant c ,*

$$f(n) < c(2, 723)^n.$$

Conjecture 1. For some constant c ,

$$f(n) \sim c\beta^n,$$

where β is the largest real root of the equation $z^8 - 8z^6 + 10z^4 + 1 = 0$ ($\beta = 2, 5386 \dots$). The construction attaining this number is given in section 2.

We also consider a more general problem. Let $Q \subset \mathbb{Z}$ be finite and $F = \{0, \pm 1, \dots, \pm k\}$. Then

$$f(n, Q, F) \triangleq \max\{|H \cap Q^n| : H \text{ is a hyperplane and } H \cap F^n = \{0^n\}\}.$$

In some cases we succeed in giving the exact answer.

Theorem 3. (i) *Let $Q = \{0, \pm 1, \dots, \pm m\}$, $F = \{0, \pm 1, \dots, \pm k\}$ and $k + 1 | 2m + 1$. Then*

$$f(n, Q, F) = \left(\frac{2m + 1}{k + 1}\right)^{n-1}.$$

(ii) *Let $Q = \{0, \pm 1, \dots, \pm(m - 1), m\}$, $F = \{0, \pm 1, \dots, \pm k\}$ and $k + 1 | 2m$. Then*

$$f(n, Q, F) = \left(\frac{2m}{k + 1}\right)^{n-1}.$$

An interesting case is

$$Q = \{0, \pm 1, \dots, \pm(k+1)\}, \quad F = \{0, \pm 1, \dots, \pm k\}, \quad k \geq 1.$$

Note that for $k = 1$ we have Bohman's problem. It can be shown that

$$(1 + \sqrt{2})^n \leq f(n, Q, F) \leq 3^n.$$

The upper bound is derived exactly as above for $k = 1$. For the lower bound consider the equation

$$(1.4) \quad x_0 + (k+1)x_1 + \dots + (k+1)^{n-1}x_{n-1} = 0.$$

Let $X \subset Q^n$ denote the set of solutions of (1.4). Clearly, $X \cap F^n = \{0^n\}$. On the other hand, one can show that $|X| = \frac{1}{2}(1 + \sqrt{2})^n + \frac{1}{2}(1 - \sqrt{2})^n$ (as for $k = 1$). We believe that Bohman's conjecture is true for $k \geq 2$, that is,

Conjecture 2. For $Q = \{0, \pm 1, \dots, \pm(k+1)\}$, $F = \{0, \pm 1, \dots, \pm k\}$ and $k \geq 2$ (or a weaker condition for $k > k_0$) one has

$$f(n, Q, F) = \frac{1}{2}(1 + \sqrt{2})^n + \frac{1}{2}(1 - \sqrt{2})^n.$$

2. PROOF OF THEOREM 1

We start with an auxiliary statement. Let $f_\lambda^*(n)$ denote the maximum number of solutions $x^n = (x_0, \dots, x_{n-1}) \in \{0, \pm 1, \pm 2\}^n$ of the equation

$$(2.1) \quad \sum_{i=0}^{n-1} 2^i \lambda_i x_i = \lambda$$

over all choices of odd integers $\lambda_0, \dots, \lambda_{n-1}$ and a given integer λ . Remember that $f_0^*(n) = f^*(n)$.

Lemma 1.

$$f^*(n) \geq \frac{1}{25} f_\lambda^*(n).$$

Proof. Suppose we have an optimal equation (2.1). That is, for the solutions of (2.1), $X \subset \{0, \pm 1, \pm 2\}^n$, we get $|X| = f_\lambda^*(n)$.

For an integer μ consider the equation

$$(2\mu + 1)y + 2z + 4\lambda_0 x_0 + \dots + 2^{n+1} \lambda_{n-1} x_{n-1} = 0.$$

Then taking $y = -2$, $z = 1$, we come to equation $\sum_{i=0}^{n-1} 2^i \lambda_i x_i = \mu$, which implies that $f^*(n+2) \geq \max_{\mu} f_\mu(n)$. On the other hand, clearly

$$\max_{\mu} f_\mu(n) \geq \frac{1}{25} f_\lambda^*(n+2).$$

□

Consider the equation

$$(2.2) \quad x_0 + 2x_1 + \dots + 2^{n-1}x_{n-1} = \lambda.$$

Let $X(\lambda)$ be the set of all solutions (from $\{0, \pm 1, \pm 2\}^n$) of (2.2). With the help of this lemma we can get a lower bound using an average argument. There are 5^n vectors $(x_0, \dots, x_{n-1}) \in \{0, \pm 1, \pm 2\}^n$. On the other hand, there are $4(2^n - 1) + 1$

possible values for λ for which equation (2.2) has solutions. Hence there exists a λ such that

$$|X(\lambda)| \geq \frac{5^n}{4(2^n - 1) + 1}.$$

This together with Lemma 1 implies that $f(n) \geq c(2, 5)^n$ for some constant c , which actually disproves the conjecture of Bohman. However, we can improve this bound constructively.

Lower bound. As above let $X(\lambda) = H \cap \{0, \pm 1, \pm 2\}^n$, where H is the hyperplane defined by (2.2).

Also, let $h_\lambda(n)$ denote the number of solutions of (2.2), that is, $h_\lambda(n) = |X(\lambda)|$. Suppose that $\lambda = 2s$, where s is an integer. Then observe that

$$(2.3) \quad h_{2s}(n) = h_{s-1}(n-1) + h_s(n-1) + h_{s+1}(n-1).$$

Correspondingly, if $\lambda = 2s + 1$, then

$$(2.4) \quad h_{2s+1}(n) = h_s(n-1) + h_{s+1}(n-1).$$

For a positive integer n , define

$$(2.5) \quad S_n = \begin{cases} 2^{n-1} + 2^{n-3} + \cdots + 2^3 + 2, & \text{if } 2 \mid n; \\ 2^{n-1} + 2^{n-3} + \cdots + 2^2 + 1, & \text{if } 2 \nmid n. \end{cases}$$

Claim. For $2 \mid n$ and some constant c ,

$$(2.6) \quad h_{S_n}(n) > c(2, 538)^n.$$

Proof. In view of (2.3) we have

$$(2.7) \quad h_{S_n}(n) = h_{S_{n-1}-1}(n-1) + h_{S_{n-1}}(n-1) + h_{S_{n-1}+1}(n-1).$$

Correspondingly,

$$(2.8) \quad \begin{aligned} h_{S_{n-1}-1}(n-1) &= h_{S_{n-2}-1}(n-2) + h_{S_{n-2}}(n-2) + h_{S_{n-2}+1}(n-2), \\ h_{S_{n-1}}(n-1) &= h_{S_{n-2}}(n-2) + h_{S_{n-2}+1}(n-2), \\ h_{S_{n-1}+1}(n-1) &= h_{S_{n-2}}(n-2) + h_{S_{n-2}+1}(n-2) + h_{S_{n-2}+2}(n-2). \end{aligned}$$

It is easy to see that $h_{S_n}(n)$ can be represented by linear combinations of the functions $h_{S_{n-i}-1}(n-i)$, $h_{S_{n-i}}(n-i)$, $h_{S_{n-i}+1}(n-i)$, $h_{S_{n-i}+2}(n-i)$.

In view of (2.7) and (2.8) we can write

$$\begin{aligned}
 h_{s_n}(n) &= h_{S_{n-1}+1}(n-1) + h_{S_{n-1}}(n-1) + h_{S_{n-1}}(n-1) \\
 &= h_{S_{n-2}-1}(n-2) + 3h_{S_{n-2}}(n-2) + 3h_{S_{n-2}+1}(n-2) \\
 &\quad + h_{S_{n-2}+2}(n-2) \\
 &= 4h_{S_{n-3}-1}(n-3) + 8h_{S_{n-3}}(n-3) + 7h_{S_{n-3}+1}(n-3) \\
 &\quad + h_{S_{n-3}+2}(n-3) \\
 &= 4h_{S_{n-4}-1}(n-4) + 19h_{S_{n-4}}(n-4) + 20h_{S_{n-4}+1}(n-4) \\
 &\quad + 8h_{S_{n-4}+2}(n-4) \\
 &\quad \dots \\
 &= a_i h_{S_{n-i}-1}(n-i) + b_i h_{S_{n-i}}(n-i) + c_i h_{S_{n-i}+1}(n-i) \\
 &\quad + d_i h_{S_{n-i}+2}(n-i) \\
 &\quad \dots \\
 &= a_{n-1} h_{S_1-1}(1) + b_{n-1} h_{S_1}(1) + c_{n-1} h_{S_1+1}(1) + d_{n-1} h_{S_1+2}(1) \\
 &= a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}.
 \end{aligned}
 \tag{2.9}$$

From (2.7), (2.8) and (2.9) we obtain the following recurrences for the coefficients a_i, b_i, c_i, d_i in (2.9):

$$\begin{aligned}
 a_{2i} &= a_{2i-1}, \\
 b_{2i} &= a_{2i-1} + b_{2i-1} + c_{2i-1}, \\
 c_{2i} &= a_{2i-1} + b_{2i-1} + c_{2i-1} + d_{2i-1}, \\
 d_{2i} &= c_{2i-1} + d_{2i-1};
 \end{aligned}
 \tag{2.10}$$

$$\begin{aligned}
 a_{2i+1} &= a_{2i} + b_{2i}, \\
 b_{2i+1} &= a_{2i} + b_{2i} + c_{2i} + d_{2i}, \\
 c_{2i+1} &= b_{2i} + c_{2i} + d_{2i}, \\
 d_{2i+1} &= d_{2i} \quad (i = 1, 2, \dots).
 \end{aligned}
 \tag{2.11}$$

Here are the first ten values of a_i, b_i, c_i, d_i .

$$\begin{array}{rcccccccccc}
 a_i: & 1 & 1 & 4 & 4 & 23 & 23 & 144 & 144 & 921 & 921, \\
 b_i: & 1 & 3 & 8 & 19 & 51 & 121 & 328 & 777 & 2113 & 5003, \\
 c_i: & 1 & 3 & 7 & 20 & 47 & 129 & 305 & 832 & 1969 & 5363, \\
 d_i: & 0 & 1 & 1 & 8 & 8 & 55 & 55 & 360 & 360 & 2329.
 \end{array}$$

From (2.10) and (2.11) we obtain by elementary algebraic transformations the following recurrences:

$$t_{i+8} = 8t_{i+6} - 10t_{i+4} - t_i \text{ for } t_i \in \{a_i, b_i, c_i, d_i\}, \quad i = 1, 2, \dots .$$

In particular, we have

$$c_{2i+8} = 8c_{2i+6} - 10c_{2i+4} - c_{2i} \tag{2.12}$$

with initial values $c_2 = 3, c_4 = 20, c_6 = 129, c_8 = 832$.

The characteristic equation of (2.12),

$$z^8 - 8z^6 + 10z^4 + 1 = 0, \tag{2.13}$$

has a largest real root $\beta = 2, 5386\dots$

Thus c_{2i} can be estimated from below by $c_{2i} \geq c\beta^{2i} > c(2.538)^{2i}$, for some constant c definable from the initial values of c_{2i} .

Furthermore, in view of (2.9) and (2.10) for $n = 2k$ we have

$$h_{S_n}(n) = a_{2k-1} + b_{2k-1} + c_{2k-1} + d_{2k-1} = c_n,$$

which implies that $h_{S_n}(n) > c(2, 538)^n$.

Thus we have proved that $f_{S_n}^*(n) > (2, 538)^n$. This with Lemma 1 completes the proof of the lower bound.

Upper bound. Consider the equation

$$(2.14) \quad \lambda_0 x_0 + 2\lambda_1 x_1 + \dots + 2^{n-1} \lambda_{n-1} x_{n-1} = \lambda.$$

We distinguish the three cases:

- (α) $\lambda \equiv 2 \pmod{4}$: Then denote by $h_\alpha(n)$ the maximum possible number of solutions (from $\{0, \pm 1, \pm 2\}^n$ of equation (2.14)).
- (β) $\lambda \equiv 0 \pmod{4}$: The corresponding notation for this case is $h_\beta(n)$.
- (γ) $\lambda \equiv 1$ or $3 \pmod{4}$: The corresponding notation for this case is $h_\gamma(n)$.

Then one can easily observe that the following recurrence relations hold:

$$(2.15) \quad \begin{aligned} h_\alpha(n) &\leq h_\alpha(n-1) + h_\beta(n-1) + h_\gamma(n-1), \\ h_\beta(n) &\leq \max\{h_\alpha(n-1), h_\beta(n-1)\} + 2h_\gamma(n-1), \\ h_\gamma(n) &\leq \max\{h_\alpha(n-1), h_\beta(n-1)\} + h_\gamma(n-1). \end{aligned}$$

We also have that $h_\alpha(1) = h_\beta(1) = h_\gamma(1) = 1$.

Now introduce the functions $g_\alpha(n)$, $g_\beta(n)$, and $g_\gamma(n)$, so that $g_\alpha(1) = g_\beta(1) = g_\gamma(1) = 1$, and

$$\begin{aligned} g_\alpha(n) &= g_\alpha(n-1) + g_\beta(n-1) + g_\gamma(n-1), \\ g_\beta(n) &= \max\{g_\alpha(n-1), g_\beta(n-1)\} + 2g_\gamma(n-1), \\ g_\gamma(n) &= \max\{g_\alpha(n-1), g_\beta(n-1)\} + g_\gamma(n-1). \end{aligned}$$

Clearly, we have that $g_\alpha(n) \geq h_\alpha(n)$, $g_\beta(n) \geq h_\beta(n)$, $g_\gamma(n) \geq h_\gamma(n)$.

Observe also that for $n \geq 3$ we have $g_\alpha(n) > g_\beta(n) > g_\gamma(n)$.

Hence, finally, we come to the recurrences

$$(2.16) \quad \begin{aligned} g_\alpha(n) &= g_\alpha(n-1) + g_\beta(n-1) + g_\gamma(n-1), \\ g_\beta(n) &= g_\alpha(n-1) + 2g_\gamma(n-1), \\ g_\gamma(n) &= g_\alpha(n-1) + g_\gamma(n-1). \end{aligned}$$

From (2.16) we obtain the recurrence

$$(2.17) \quad g_\alpha(n) = 2g_\alpha(n-1) + g_\alpha(n-2) + g_\alpha(n-3)$$

with initial values $g_\alpha(1) = 1, g_\alpha(2) = 3, g_\alpha(3) = 8$.

Now to estimate the function $f^*(n)$ it remains to solve recurrence (2.17), since $f^*(n) \leq g_\alpha(n)$. The latter gives the estimation

$$g_\alpha(n) \leq c''(2, 547)^n$$

for some constant c'' definable from the initial values. This completes the proof of Theorem 1. □

3. PROOF OF THEOREM 2

Suppose that $\{a_1, \dots, a_n\} \subset \mathbb{N}$ has distinct subset sums. Let X denote the set of all solutions $x^n \in \{0, \pm 1, \pm 2\}^n$ of the equation $\sum_{i=1}^n a_i x_i = \lambda$.

Consider two mappings φ_0 and φ_1 from $\{0, \pm 1, \pm 2\}$ to $\{0, \pm 1\}$ $\varphi_0(-2) = \varphi_1(-2) = -1$, $\varphi_0(2) = \varphi_1(2) = 1$, $\varphi_0(\pm 1) = \varphi_1(\pm 1) = 0$, and $\varphi_0(0) = -1$, $\varphi_1(0) = 1$.

Next for $x^n \in X$ define

$$\varphi(x^n) = \{(\varphi_{\varepsilon_1}(x_1), \dots, \varphi_{\varepsilon_n}(x_n)) : \varepsilon_i \in \{0, 1\}, i = 1, \dots, n\}.$$

Claim 1. For $x^n, y^n \in X$, $x^n \neq y^n$,

$$\varphi(x^n) \cap \varphi(y^n) = \emptyset.$$

Proof. Suppose the opposite. Then it is not hard to verify that $x^n - y^n \in \{0, \pm 2\}^n \setminus \{0^n\}$, a contradiction. \square

Let us define

$$\alpha(x^n) = \text{the number of zero coordinates in } x^n.$$

Claim 2. For any $x^n \in X$,

$$|\varphi(x^n)| = 2^{\alpha(x^n)}.$$

Proof. This immediately follows from the definition of $\varphi(x^n)$. \square

Combining Claims 1 and 2, we conclude that

$$(3.1) \quad \sum_{x^n \in X} 2^{\alpha(x^n)} \leq 3^n.$$

Now consider the mapping $\Psi : X \rightarrow \{0, \pm 1\}^n$, defined by $\Psi(x^n) = (\Psi_0(x_1), \dots, \Psi_0(x_n))$, where

$$\Psi_0(x_i) = \begin{cases} -1, & \text{if } x_i = -2, -1, \\ 1, & \text{if } x_i = 2, 1, \\ 0, & \text{if } x_i = 0; i = 1, \dots, n. \end{cases}$$

Claim 3. For $x^n, z^n \in X$, $x^n \neq z^n$ holds for $\Psi(x^n) \neq \Psi(z^n)$.

Proof. Assuming the opposite, we will get $x^n - z^n \in \{0, \pm 1\}^n \setminus \{0^n\}$, a contradiction. \square

Note (and this is important for us) that Ψ leaves the zero coordinates fixed. This with (3.1) implies that

$$\sum_{y^n \in \Psi(X)} 2^{\alpha(y^n)} \leq 3^n.$$

Since $|X| = |\Psi(X)|$, we can bound $|X|$ by the maximum cardinality of a set $Y \subset \{0, \pm 1\}^n$ satisfying

$$(3.2) \quad \sum_{y^n \in Y} 2^{\alpha(y^n)} \leq 3^n.$$

Define

$$Y_i = \{y^n \in Y : \alpha(y^n) = i\}, i = 0, 1, \dots, n.$$

Note that $|Y_i| \leq 2^{n-i} \binom{n}{i}$.

Now (3.2) can be rewritten in the form

$$(3.3) \quad \sum_{i=0}^n |Y_i| 2^i \leq 3^n.$$

Observe that to maximize $|Y| = \sum_{i=0}^n |Y_i|$ we have to take

$$|Y_i| = \begin{cases} \binom{n}{i} 2^{n-i}, & \text{if } i \leq \ell(n); \\ 0, & \text{if } i > \ell(n) \end{cases}$$

where $\ell(n)$ is the maximal index for which we get

$$\sum_{i=0}^{\ell(n)} 2^{n-i} \binom{n}{i} 2^i \leq 3^n.$$

This gives (using a standard technique) that $\ell(n) \geq \lfloor 0,1402 n \rfloor$. Correspondingly, we get an estimation for $|Y|$ and consequently for $|X|$,

$$|X| \leq |Y| < c \frac{3^n}{2^{0,14n}} < c(2,723)^n,$$

for some constant c . □

4. PROOF OF THEOREM 3

Let $Q = \{0, \pm 1, \dots, \pm m\}$, $F = \{0, \pm 1, \dots, \pm k\}$ with $\alpha = (2m + 1)/(k + 1)$.

(a) First we will show that $f(n, Q, F) \leq \alpha^{n-1}$. Let H be defined by

$$(4.1) \quad \sum_{i=1}^n a_i x_i = 0.$$

Also, let $H \cap F^n = \{0^n\}$ and $H \cap Q^n = X$ with $|X| = f(n, Q, F)$.

Define $Q_j = \{a \in Q : a \equiv j \pmod{\alpha}\}$, $j = 0, 1, \dots, \alpha - 1$.

Then consider the mapping $\varphi : X \rightarrow \mathbb{Z}_\alpha^n$, defined by the transformation of coordinates, $\varphi(x_1, \dots, x_n) = (\varphi_0(x_1), \dots, \varphi_0(x_n))$, where $\varphi_0(x_i) = j$ ($i = 1, \dots, n$) if $x_i \in Q_j$; $j \in \{0, \dots, \alpha - 1\}$. Observe that φ is an injection. Hence $|X| = |\varphi(X)|$.

Note that now

$$\dim(\text{span}\varphi(X)) \leq \dim(\text{span}(X)) = n - 1.$$

This implies that

$$(4.2) \quad |X| = |\varphi(X)| \leq \alpha^{n-1}.$$

(b) Next, we will show that bound (4.2) can be achieved by taking the hyperplane H defined by

$$(4.3) \quad x_0 + (k + 1)x_1 + \dots + (k + 1)^{n-1}x_{n-1} = 0.$$

In fact, $H \cap F^n = \{0^n\}$. Moreover, we claim that for any $-m \leq \lambda \leq m$, the equation

$$(4.4) \quad \sum_{i=0}^{n-1} x_i (k + 1)^i = \lambda$$

has exactly α^{n-1} solutions $x^n \in Q^n$. This can be shown using induction on n .

The case $n = 1$ is trivial.

Induction step from $n - 1$ to n : Clearly, $x_0 \in \{a : -m \leq a \leq m, a \equiv \lambda \pmod{(k+1)}\}$. Thus x_0 can take α many values $x_0 \in [-m, m]$. For each x_0 we come to the equation

$$x_1 + (k+1)x_2 + \cdots + (k+1)^{n-2}x_{n-1} = \frac{\lambda - x_0}{k+1}$$

with $\left| \frac{\lambda - x_0}{k+1} \right| \leq \frac{2m}{k+1} \leq m$. Hence we get the result by the induction hypothesis. This completes the proof of Theorem 3 in the case (i). The case (ii) can be proved similarly. \square

ACKNOWLEDGEMENT

The authors are indebted to the referee for his comments and suggestions which have substantially helped to improve the style of the paper.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BIELEFELD, POB 100131, 33501 BIELEFELD, GERMANY

E-mail address: ahlswede@mathematik.uni-bielefeld.de

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BIELEFELD, POB 100131, 33501 BIELEFELD, GERMANY

E-mail address: ayd@mathematik.uni-bielefeld.de

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BIELEFELD, POB 100131, 33501 BIELEFELD, GERMANY

E-mail address: lk@mathematik.uni-bielefeld.de