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A FORMULA FOR THE JOINT LOCAL SPECTRAL RADIUS

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ABSTRACT. We give a formula for the joint local spectral radius of a bounded subset of bounded linear operators on a Banach space X in terms of the dual of X.

Let X be a Banach space and $\mathcal{L}(X)$ the algebra of all bounded linear operators in X. The joint spectral radius $\rho(M)$ of a bounded subset M of $\mathcal{L}(X)$ was introduced by G.-C. Rota and W. G. Strang [5] as

$$\rho(M) = \limsup_{n\to\infty} \|M^n\|^{1/n},$$

where M^n is the set of all products $T_1 \circ \ldots \circ T_n$ $(T_i \in M)$ and $\|M^n\| = \sup_{T\in M^n} \|T\|$. Recently the notion of the joint local spectral radius $\rho_x(M)$ at a point $x \in X$ was introduced by R. Drnovšek [2] for a finite subset M of $\mathcal{L}(X)$ and by V. S. Shulman and Yu. V. Turovskii [6] for a bounded $M \subseteq \mathcal{L}(X)$ as

$$\rho_x(M) = \limsup_{n\to\infty} \|M^n x\|^{1/n},$$

where $\|M^n x\| = \sup_{T\in M^n} \|Tx\|$. In this note we present the following formula for the joint local spectral radius.

Theorem 1. *For any bounded $M \subseteq \mathcal{L}(X)$ and for any $x \in X$ the following holds:*

$$\rho_x(M) = \sup_{f\in X^*} \limsup_{n\to\infty} |f \circ M^n(x)|^{1/n}, \tag{1}$$

where $|f \circ M^n(x)| = \sup\{|f \circ T(x)| : T \in M^n\}$. In particular, $\rho_x(M) = 0$ if and only if $\lim_{n\to\infty} |f \circ M^n(x)|^{1/n} = 0$ for all $f \in X^*$.

The proof of the theorem is based on a lemma which generalizes the recent result [3, Thm. 3] of S. Onal and the second author in the following way.

Lemma 2. *Let X be a Banach space. Then for any sequence $(x_n)_n$ in X and for any nonnegative sequence $(\epsilon_n)_n$ with $\epsilon_n \to 0$ the following holds:*

$$\limsup_{n\to\infty} \|x_n\|^{\epsilon_n} = \sup_{f\in X^*} \limsup_{n\to\infty} |f(x_n)|^{\epsilon_n}.$$

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Proof. We may assume that the sequence $(\epsilon_n)_n$ is decreasing. Denote by l and r the left and the right sides of the formula. Obviously,

$$r = \sup_{f \in X^*,\ \|f\| \le 1} \limsup_{n \to \infty} |f(x_n)|^{\epsilon_n}.$$

Thus $l \ge r$ is trivial, because $\|x_n\| \ge \|f\|\|x_n\| \ge |f(x_n)|$ for all $f \in X^*$, $\|f\| \le 1$.

Suppose $l > r$. There exists an $\alpha > r$ and a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $k\epsilon_{n_k} \to 0$ and

$$\|x_{n_k}\|^{\epsilon_{n_k}} \ge \alpha$$

for all k. Put $y_k = \alpha^{-\epsilon_{n_k}^{-1}} x_{n_k}$. Note that $\|y_{n_k}\| \ge 1$ for all k and

$$|f(y_k)| = |f(x_{n_k})| \ /\alpha^{1/\epsilon_{n_k}} \to 0 \quad (\forall f \in X^*),$$

since

$$\limsup_{k \to \infty} |f(x_{n_k})|^{\epsilon_{n_k}} \le r < \alpha \quad (\forall f \in X^*).$$

Applying the Bessaga-Pelczynski selection principle (see, for example, [1, p. 42]), take a subsequence $(y_{k_i})_i$ that is a basis in the closure of the linear span Y of $\{y_{k_i}\}_{i=1}^{\infty}$. Take the linear functionals u_i on Y satisfying

$$u_i(y_{k_j}) = \delta_{ij}, \quad (\forall i, j).$$

Then the sequence $(\|u_i\|)_{i=1}^{\infty}$ is bounded by the basis constant of $(y_{k_i})_i$. Set

$$u := \sum_{k=1}^{\infty} 2^{-k} u_k,$$

and extend u to a functional $\hat{u} \in X^*$. Then $|\hat{u}(y_{k_i})| = 2^{-i}$ and $|\hat{u}(x_{n_{k_i}})| = \alpha^{1/\epsilon_{n_{k_i}}} |\hat{u}(y_{k_i})| = 2^{-i}\alpha^{1/\epsilon_{n_{k_i}}}$ for each i. On the other hand,

$$\limsup_{i \to \infty} |\hat{u}(x_{n_{k_i}})|^{\epsilon_{n_{k_i}}} \le r$$

implies that

$$\limsup_{i \to \infty} 2^{-i\epsilon_{n_{k_i}}} \le r/\alpha < 1,$$

which contradicts $i\epsilon_{n_{k_i}} \to 0$. □

Proof of Theorem 1. Denote by l and r the left and the right sides of (1). The inequality $l \ge r$ is trivial. Suppose that $l > r$. Then there exists a sequence $(T_{n_k} \in M^{n_k})_k$ such that

$$\|T_{n_k} x\|^{1/n_k} \ge \alpha > r$$

for some α and for all k. Applying Lemma 2 to the sequences $(x_k)_k$, $x_k = T_{n_k} x$ and $(\epsilon_k)_k$, $\epsilon_k = 1/n_k$ we find an $f \in X^*$ such that

$$\limsup_{k \to \infty} |f \circ T_{n_k}(x)|^{1/n_k} > r,$$

which contradicts

$$r = \sup_{f \in X^*} \limsup_{n \to \infty} |f \circ M^n(x)|^{1/n}.$$

Consequently, $l = r$ and the proof of the theorem is complete. □

As an application of Theorem 1 we give the following formula for the joint spectral radius of bounded subsets of a Banach algebra. As usual, for a bounded subset M of a Banach algebra A, by $\rho(M)$ is denoted the joint spectral radius $\rho(M) = \limsup_{n\to\infty}\|M^n\|^{1/n}$ of M (see, for example, [4]). The relation between the joint spectral radius and the geometric spectral radius of noncommuting Banach algebra elements is investigated by P. Rosenthal and A. Soltysiak in [4].

Corollary 3. *Let A be a Banach algebra and M a bounded subset of A. Then*

$$\rho(M) = \sup_{f\in A^*} \limsup_{n\to\infty} |f(M^n)|^{1/n},$$

where $|f(M^n)| = \sup\{|f(a)| : a \in M^n\}$. In particular, $\rho(M) = 0$ if and only if $\lim_{n\to\infty} |f(M^n)|^{1/n} = 0$ for all $f \in A^*$.

Proof. We may assume that the algebra A has a unit, say e. Consider the bounded subset $\mathcal{M} = \{T_a : a \in M\}$ of $\mathcal{L}(A)$, where T_a is defined as the left multiplication $T_a(x) = ax$. Then $\rho(M) = \rho_e(\mathcal{M})$ and $|f(M^n)| = |f \circ \mathcal{M}^n(e)|$ for any f and n. To complete the proof it is enough to apply Theorem 1 to $\mathcal{M} \subseteq \mathcal{L}(A)$ and e. □

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