AN ENDPOINT ESTIMATE
FOR THE DISCRETE SPHERICAL MAXIMAL FUNCTION

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Abstract. We prove that the discrete spherical maximal function extends to a bounded operator from $L^d\left(\mathbb{Z}^d\right)$ to $L^d\left(\mathbb{Z}^d\right)$ in dimensions $d \geq 5$. This is an endpoint estimate for a recent theorem of Magyar, Stein and Wainger.

1. Introduction

The discrete spherical maximal function is defined as the operator

$$A_* (f)(n) = \sup_{r \in \Lambda} \frac{1}{N_d(r)} \sum_{|m|=r} |f(n-m)|,$$

where $f : \mathbb{Z}^d \to \mathbb{C}$ is a function, $d \geq 2$, $N_d(r)$ denotes the number of lattice points on the sphere $\{x \in \mathbb{R}^d : |x| = r\}$ and $\Lambda = \{r \geq 0 : N_d(r) \neq 0\}$. The sum in the definition of the operator $A_*$ is taken over the lattice points $m$ on the sphere of radius $r$. This operator is the discrete analogue of the classical Euclidean spherical maximal function

$$A_* (f)(x) = \sup_{r \in (0, \infty)} |f| * d\sigma_r(x)$$

where $d\sigma_r$ denotes the normalized invariant measure on the sphere $|x| = r$ and $f : \mathbb{R}^d \to \mathbb{C}$ is a suitable function. It is well known that the operator $A_*$ extends to a bounded operator on $L^p(\mathbb{R}^d)$ for $d \geq 2$ and $p > d/(d-1)$ (see Stein [7] in the case $d \geq 3$ and Bourgain [2] in the case $d = 2$).

The question of boundedness on $L^p(\mathbb{Z}^d)$ of the operator $A_*$ was considered by Magyar [4] and Magyar, Stein and Wainger [5]. The main theorem in [5] is the following.

Theorem. (Magyar, Stein and Wainger [5]). The maximal operator $A_*$ extends to a bounded operator on $L^p(\mathbb{Z}^d)$ if and only if $d \geq 5$ and $p > d/(d-2)$ or $d \leq 4$ and $p = \infty$.

The distinction between the cases $d \geq 5$ and $d \leq 4$ is related to the behavior of the function $N_d(r)$. If $d \leq 4$, this function is irregular. On the other hand, it is well known that if $d \geq 5$, then there is a constant $C_d \geq 1$ such that $C_d^{-1}r^{d-2} \leq $
In this note we prove an endpoint estimate for the theorem of Magyar, Stein and Wainger \cite{4}. First, we show how to use Lemma 2 to prove the theorem due to Magyar \cite{4}. Instead, we decompose our operator into an abstract version of this argument fails in dimension $d = 2$ (see \cite{6} Proposition 1.5). Our proof of Theorem 1 follows the line of the proof of the theorem of Magyar, Stein and Wainger \cite{5}. The main ingredients are the circle method of Hardy, Littlewood and Ramanujan, the Poisson summation formula, and a transference principle. Our simplification is that we will not need the dyadic version of the theorem due to Magyar \cite{4}. Instead, we decompose our operator into an $L^1$ part and an $L^2$ part depending on a parameter $\alpha$. We use the discrete Hardy-Littlewood maximal function to establish the $L^1$ bounds and the error analysis in \cite{5} together with a lemma of Bourgain \cite{1} for the $L^2$ bounds.

I would like to thank S. Wainger for several useful discussions on the subject.

2. Proof of Theorem 1

We first replace the operator $A_\ast$ with the operator

$$
\widetilde{A}_\ast(f)(n) = \sup_{r \in \Lambda} \left| \frac{1}{r^{d-2}} \sum_{|m| = r} f(n-m) \right|
$$

where $\Lambda = \{ r \in [1, \infty) : r^2 \in \mathbb{Z} \}$ and $f : \mathbb{Z}^d \to \mathbb{C}$ is compactly supported. This is possible since $d \geq 5$ and $N_d(r) \approx r^{d-2}$. It remains to prove that $\widetilde{A}_\ast$ extends to a bounded operator from $L^{p_{d,1}}(\mathbb{Z}^d)$ to $L^{p_{d,\infty}}(\mathbb{Z}^d)$. This is an easy consequence of the following lemma.

Lemma 2. For any $\alpha \in (0, 1]$ there are two subadditive operators $A_\alpha^1$ and $A_\alpha^2$ with the property that $\widetilde{A}_\ast(f)(n) \leq |A_\alpha^1(f)(n)| + |A_\alpha^2(f)(n)|$ for any $n \in \mathbb{Z}^d$,

\begin{equation}
\|A_\alpha^1(f)\|_{L^1_{\infty}} \leq C\alpha^{-2}\|f\|_{L^1},
\end{equation}

and

\begin{equation}
\|A_\alpha^2(f)\|_{L^2} \leq C\alpha^{(d-4)/2}\|f\|_{L^2}
\end{equation}

for any compactly supported function $f : \mathbb{Z}^d \to \mathbb{C}$.

The method of proving restricted weak type inequalities by decomposing the operator as in Lemma 2 is due to Bourgain \cite{1}. An abstract version of this argument may be found in the appendix of \cite{5}. First, we show how to use Lemma 2 to prove the theorem. By the general theory of Lorentz spaces, the $L^{p_{d,1}} \to L^{p_{d,\infty}}$ boundedness of the operator $A_\ast$ is equivalent to

\begin{equation}
\|\widetilde{A}_\ast(f)\|_{L^{p_{d,\infty}}} \leq C|F|^{1/p_d}
\end{equation}
for any finite set \( F \), where \( \chi_F \) denotes the characteristic function of the set \( F \) and |\( F \)| denotes its cardinality. Clearly, \( \widetilde{A}_r(\chi_F)(n) \leq C_d \) for any \( n \), where \( C_d \) is the constant with the property that \( N_d(r) \leq C_d r^{d-2} \) for any \( r \geq 1 \). Thus (2.3) is equivalent to proving that for any \( \lambda \in (0, 1] \),

\[
\lambda^{p_d} \| \{ n : \widetilde{A}_r(\chi_F)(n) > \lambda \} \| \leq C|F|.
\]

By Lemma 2 we have

\[
\| \{ n : \widetilde{A}_r(\chi_F)(n) > \lambda \} \| \leq \| \{ n : A^2_3(\chi_F)(n) > \lambda/2 \} \| + \| \{ n : A^3_3(\chi_F)(n) > \lambda/2 \} \| \\
\leq \frac{2}{\lambda} \| A^3_3(\chi_F) \|_{L^1} + \frac{4}{\lambda^2} \| A^3_3(\chi_F) \|_{L^2}^2 \\
\leq C \lambda^{-1} \alpha^{-2} |F| + C \lambda^{-2} \alpha^{-d-4} |F|.
\]

Since \( p_d = d/(d-2) \), the estimate (2.4) follows by taking \( \alpha = \lambda^{1/(d-2)} \).

It remains to prove Lemma 2. Fix \( \alpha \in (0, 1] \). We will use some of the notation in \( 5 \). Let

\[
A_r(f)(n) = \frac{1}{r^{(d-2)}} \sum_{|m|=r} f(n-m)
\]

and

\[
M_r(f) = \sum_{q=1}^\infty \sum_{1 \leq a \leq q, (a,q)=1} e^{-2\pi i a/q} M^{\alpha/q}_r(f),
\]

where, as in \( 5 \), \( M^{\alpha/q}_r \) is the convolution operator whose multiplier is

\[
\sum_{q \in \mathbb{Z}^d} G(a/q, \ell) \psi_q(\xi - \ell/q) \widehat{d\sigma}(\xi - \ell/q).
\]

Here \( G(a/q, \ell) \) is the normalized Gauss sum

\[
G(a/q, \ell) = (q^d \sum_{n \in \mathbb{Z}^d/(q^d)\mathbb{Z}} e^{2\pi i (|n|^2 a/q + n \cdot \ell/q)}).
\]

\( \psi \) is a smooth cutoff function supported in the cube \( Q/2 = \{ \xi : |\xi_j| \leq 1/4, j = 1, \ldots, d \} \) and identically equal to 1 in the cube \( Q/4 \), \( \psi_q(q) = \psi(q) \), \( \widehat{d\sigma} \) is the Fourier transform of the invariant measure on the sphere \( |x| = 1 \) normalized with total measure 1, and \( \widehat{d\sigma}(q) = \widehat{d\sigma}(r \eta) \). The reason for considering the operators \( M_r \) is that they are good approximations (in \( L^2 \)) of the operators \( A_r \). In what follows we assume that \( r \) is restricted so that \( r \geq 1 \) and \( r^2 \in \mathbb{Z} \). Let \( N = 1/\alpha \geq 1 \). We have

\[
\widetilde{A}_r(f)(n) = \sup_r |A_r(f)(n)| \leq \sup_{r \leq 10N} |A_r(f)(n)| + \sup_{r \geq 10N} |A_r(f)(n)| \\
\leq \sup_{r \leq 10N} |A_r(f)(n)| + \sup_{r \geq 10N} |(A_r - M_r)(f)(n)| + \sup_{r \geq 10N} |M_r(f)(n)| \\
= A^{1,1}_\alpha(f)(n) + A^{2,1}_\alpha(f)(n) + \sup_{r \geq 10N} |M_r(f)(n)|.
\]

Let

\[
M(f)(n) = \sup_r \frac{1}{r^d} \sum_{|m| \leq r} |f(n-m)|
\]
denote the discrete Hardy-Littlewood maximal function. By the same argument as in Euclidean spaces we have
\begin{equation}
(2.5) \quad ||M(f)||_{L^{1,\infty}(\mathbb{Z}^d)} \leq C||f||_{L^1(\mathbb{Z}^d)}.
\end{equation}
We use the operator $M$ to bound the operator $A_{1,1}^a$. We have
\begin{equation}
A_{1,1}^a(f)(n) \leq \sup_{r \leq 10N} \frac{1}{r^{d-2}} \sum_{|m| = r} |f(n-m)| \leq 100N^2M(f)(n).
\end{equation}
The desired bound
\begin{equation}
(2.6) \quad ||A_{1,1}^a(f)||_{L^{1,\infty}} \leq C\alpha^{-2}||f||_{L^1}
\end{equation}
follows from (2.5) and the fact that $N^2 = \alpha^{-2}$.

For the operator $A_{2,1}^a$ we use Proposition 4.1 in [5], which can be written in the form
\begin{equation}
|| \sup_{r \in [R,2R]} |A_r(f) - M_r(f)||_{L^2} \leq CR^{-(d-4)/2}||f||_{L^2}
\end{equation}
for $R \geq 1$ and $d \geq 5$. Since the supremum in the definition of the operator $A_{2,1}^a$ is taken over $r \geq 10N$, it follows that
\begin{equation}
(2.7) \quad ||A_{2,1}^a(f)||_{L^2} \leq C\alpha^{(d-4)/2}||f||_{L^2}
\end{equation}
as desired.

It remains to decompose the operator
\begin{equation}
f \rightarrow \sup_{r \geq 10N} |M_r(f)(n)|.
\end{equation}
For this we write first
\begin{equation}
\sup_{r \geq 10N} |M_r(f)(n)| \leq C \sup_{r \geq 10N} \sum_{q=1}^{N/10} \sum_{1 \leq a \leq q} \sum_{(a,q) = 1} |M_{r/q}^a(f)(n)| + CA_{2,2}^a(f)(n)
\end{equation}
where
\begin{equation}
A_{2,2}^a(f)(n) = \sum_{q \geq N/10} \sum_{1 \leq a \leq q} \sup_r |M_{r/q}^a(f)(n)|.
\end{equation}
To bound the operator $A_{2,2}^a$ we use Proposition 3.1(a) in [5] for $p = 2$:
\begin{equation}
|| \sup_r M_{r/q}^a(f)||_{L^2} \leq Cq^{-d/2}||f||_{L^2}.
\end{equation}
We can sum this bound over $q \geq N/10$ and $a \in [1,q] \cap \mathbb{Z}$ to obtain
\begin{equation}
(2.8) \quad ||A_{2,2}^a(f)||_{L^2} \leq C\alpha^{(d-4)/2}||f||_{L^2}
\end{equation}
as desired.

It remains to decompose the operators $M_{r/q}^a$ for integers $q \in [1,N/10]$. For this let $M_{r,a}^{q,1}$ denote the convolution operator given by the multiplier
\begin{equation}
\sum_{\ell \in \mathbb{Z}^d} G(a/q,\ell)\Psi_q(\xi - \ell/q)d\sigma_r(\xi - \ell/q)\Psi_{rq/N}(\xi - \ell/q),
\end{equation}
and let $M_{r,a}^{q,2}$ denote the convolution operator given by the multiplier
\begin{equation}
\sum_{\ell \in \mathbb{Z}^d} G(a/q,\ell)\Psi_q(\xi - \ell/q)d\sigma_r(\xi - \ell/q)(1 - \Psi)_{rq/N}(\xi - \ell/q).
\end{equation}
The notation is, as before, \( F_\lambda (\eta ) = F(\lambda \eta ) \). Clearly, \( M^\alpha_{r,q} = M^\alpha_{r,q,1} + M^\alpha_{r,q,2} \). Let

\[
M^1_\alpha (f)(n) = \sup_{r \geq 10N} \sum_{q=1}^{N/10} \sum_{1 \leq a \leq q, (a,q) = 1} |M^\alpha_{r,q,1}(f)(n)|
\]

and

\[
M^2_\alpha (f)(n) = \sum_{q=1}^{N/10} \sum_{1 \leq a \leq q, (a,q) = 1} \sup_{r \geq 10N} |M^\alpha_{r,q,2}(f)(n)|.
\]

This is the decomposition of the remaining operator into an \( L^1 \) part and an \( L^2 \) part.

For the \( L^1 \) estimate we will prove that

\[
(2.9) \quad \|M^1_\alpha (f)\|_{L^1,\infty} \leq C\alpha^{-2} \|f\|_{L^1}.
\]

For this we need an estimate on the kernel of the operator \( M^\alpha_{r,q,1} \). Let \( K_{r,q,1} \) denote this kernel. Notice that \( \phi \xi(\xi - \ell / q) \phi_{r,q}/N(\xi - \ell / q) = \phi_{r,q}/N(\xi - \ell / q) \) since \( r \geq 10N \).

Let \( Q \) denote the standard cube \( Q = \{ \xi = (\xi_1, \ldots, \xi_d) : -1/2 < \xi_j \leq 1/2 \} \). Then by letting \( \ell = \ell' + qs, \ell' \in \{0, 1, \ldots, q - 1\}^d \), \( s \in \mathbb{Z}^d \) we have

\[
K_{r,q,1}(m) = \int_Q e^{2\pi i m \cdot \xi} \sum_{\ell' \in \mathbb{Z}^d/(q\mathbb{Z})^d} G(a/q, \ell') \widehat{\sigma}_r(\xi - \ell / q) \phi_{r,q}/N(\xi - \ell / q) d\xi
\]

\[
= \sum_{\ell' \in \mathbb{Z}^d/(q\mathbb{Z})^d} G(a/q, \ell') \int_{\mathbb{R}^d} e^{2\pi i m \cdot \eta} \phi_{r,q}/N(\eta) d\eta = e^{2\pi i m \phi(q/q)/r} \int_{\mathbb{R}^d} e^{2\pi i m \phi(q/q)/r} \phi_{r,q}/N(\eta) d\eta = e^{2\pi i m \phi(q/q)/r} \phi_{r,q}/N(m/r).
\]

Here \( \phi \) is the inverse (Euclidean) Fourier transform of \( \phi \), the convolution denotes the Euclidean convolution and \( \phi_{r,q}/N(x) = (N/q)^d \phi(xN/q) \). Since \( \phi \) is a Schwartz function, it is easy to see that

\[
|d\sigma \ast \phi_{r,q}/N(x)| \leq C(1 + |x|)^{-(d+1)} N/q
\]

for any \( x \in \mathbb{R}^d \). Thus,

\[
|K_{r,q,1}(m)| \leq C r^{-d} (1 + |m|/r)^{-(d+1)} N/q.
\]

By summing this bound over \( a \in [1, q] \cap \mathbb{Z} \) and \( q \in [1, N/10] \cap \mathbb{Z} \) we have

\[
\sum_{q=1}^{N/10} \sum_{1 \leq a \leq q, (a,q) = 1} |M^\alpha_{r,q,1}(f)(n)| \leq C|f| \ast K_{r,a}(n)
\]

where

\[
K_{r,a}(m) = N^2 r^{-d} (1 + |m|/r)^{-(d+1)}.
\]

Thus,

\[
M^1_\alpha (f)(n) \leq CN^2 M(f)(n)
\]

and the estimate (2.9) follows from (2.5).

For the \( L^2 \) estimate we will prove that

\[
(2.10) \quad \|M^2_\alpha (f)\|_{L^2} \leq C\alpha^{(d-4)/2} \|f\|_{L^2}.
\]
By the formula of $M^2_{\alpha}$, it suffices to prove that
\begin{equation}
\|M^0_{r,\alpha}q^2(f)\|_{L^2_{\infty}(\Lambda)} \leq C q^{-1} N^{-(d-2)/2} \|f\|_{L^2},
\end{equation}
where $L^2_{\infty}(\Lambda)$ denotes the space of $L^2$ functions on $\mathbb{Z}^d$ with values in the Banach space $L^\infty(\Lambda)$. We first argue as in \[4\]. Let $\Psi$ be a smooth function supported in $Q$ with the property that $\Psi(\xi) \equiv 1$ in $Q/2$. The operator $M^0_{r,\alpha}$ can be written as the composition of two operators with multipliers
\[ \sum_{\ell \in \mathbb{Z}^d} G(a/q, \ell) \Psi_q(\xi - \ell/q) \]
and
\[ \sum_{\ell \in \mathbb{Z}^d} \Psi_q(\xi - \ell/q) \widehat{\sigma}_r(\xi - \ell/q)(1 - \Psi)q/N(\xi - \ell/q), \]
respectively. Let $S^{a,q}$ and $T^q_{r,\alpha}$ denote the two operators. Since $|G(a/q, \ell)| \leq C q^{-d/2}$, we have
\begin{equation}
\|S^{a,q}\|_{L^2 \to L^2} \leq C q^{-d/2}.
\end{equation}
Let
\[ m^{q,a}_{r,\alpha}(\eta) = \Psi_q(\eta) \widehat{\sigma}_r(\eta)(1 - \Psi)q/N(\eta). \]
The multiplier $m^{q,a}_{r,\alpha}$ is supported in $Q/q$. By the transference principle of Magyar, Stein and Wainger (Corollary 2.1 in \[5\]), we have
\begin{equation}
\|T^q_{r,\alpha}\|_{L^2(\mathbb{Z}^d) \to L^2_{\infty}(\mathbb{Z}^d)} \leq C \|\widehat{T^q_{r,\alpha}}\|_{L^2(\mathbb{R}^d) \to L^2_{\infty}(\mathbb{R}^d)},
\end{equation}
where $\widehat{T^q_{r,\alpha}}$ denotes the operator with multiplier $m^{q,a}_{r,\alpha}$ acting on functions in $L^2(\mathbb{R}^d)$. It remains to prove that
\begin{equation}
\|\widehat{T^q_{r,\alpha}}\|_{L^2(\mathbb{R}^d) \to L^2_{\infty}(\mathbb{R}^d)} \leq C(q/N)^{(d-2)/2}.
\end{equation}
The operator $\widehat{T^q_{r,\alpha}}$ can be written as the composition of two operators with multipliers $\eta \to \Psi_q(\eta)$ and $\eta \to \widehat{m}_r(\eta)$, respectively, where
\[ \widehat{m}(\eta) = \widehat{\sigma}(\eta)(1 - \Psi)(q\eta/N) \]
and $\widehat{m}_r(\eta) = \widehat{m}(r\eta)$. The operator defined by the multiplier $\eta \to \Psi_q(\eta)$ is bounded on $L^2(\mathbb{R}^d)$ uniformly in $q$. Let $\widehat{U}_r$ denote the operator with multiplier $\widehat{m}_r$ acting on functions in $L^2(\mathbb{R}^d)$. We will use the following lemma of Bourgain (Proposition 2 in \[4\]):

**Lemma** (Bourgain \[1\]). Assume that $m : \mathbb{R}^d \to \mathbb{C}$ is a smooth function and $U_r$ is the operator defined by the multiplier $\eta \to m_r(\eta) = m(r\eta)$. Then
\[ \|\sup_{r > 0} \|U_r f\|_{L^2} \leq C T(m) \|f\|_{L^2} \]
for any Schwartz function $f$ where
\[ \Gamma(m) = \sum_{j \in \mathbb{Z}} \alpha_j^{1/2} (\alpha_j^{1/2} + \beta_j^{1/2}) \]
with
\[ \alpha_j = \sup_{|\eta| \in [2^j, 2^{j+1}]} |m(\eta)| \text{ and } \beta_j = \sup_{|\eta| \in [2^j, 2^{j+1}]} |\nabla m(\eta) \cdot \eta|. \]

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In our case, the multiplier $\tilde{m}$ is supported in the set $|\eta| \geq N/(8q)$ since $\Psi(\eta) = 1$ is $|\eta| \leq 1/8$. In addition, $|\tilde{d}(\eta)| \leq C(1 + |\eta|)^{-(d-1)/2}$ and $|\nabla \tilde{d}(\eta)| \leq C(1 + |\eta|)^{-(d-1)/2}$. Thus in our case, $\alpha_j \leq C2^{-j(d-1)/2}$ and $\beta_j \leq C2^{-j(d-3)/2}$ if $2^j \geq N/(16q)$ and $\alpha_j = \beta_j = 0$ if $2^j < N/(16q)$. Thus $\Gamma(\tilde{m}) \leq C(q/N)^{(d-2)/2}$. By Bourgain’s lemma,

$$||\tilde{U}_r||_{L^2(R^d) \rightarrow L^2_{L^\infty(R^d)}(R^d)} \leq C(q/N)^{(d-2)/2}.$$ 

This proves (2.14). The estimate (2.11) follows from (2.12), (2.13) and (2.14), and the estimate (2.10) follows by summing over $q$ and $a$.

We can now finish the proof of Lemma 2. Let $A_0(f) = A_{\alpha}^2(f) + M_1(f)$ and $A_0^2(f) = A_{\alpha}^2(f) + A_{\alpha}^2(f) + M_2(f)$. The estimate (2.1) follows from (2.6) and (2.8), and the estimate (2.2) follows from (2.7), (2.9) and (2.10).

References


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