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AN ENDPOINT ESTIMATE FOR THE DISCRETE SPHERICAL MAXIMAL FUNCTION

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Abstract. We prove that the discrete spherical maximal function extends to a bounded operator from $L^{d/(d-2),1}(\mathbb{Z}^d)$ to $L^{d/(d-2),\infty}(\mathbb{Z}^d)$ in dimensions $d \geq 5$. This is an endpoint estimate for a recent theorem of Magyar, Stein and Wainger.

1. Introduction

The discrete spherical maximal function is defined as the operator

$$A_*(f)(n) = \sup_{r\in\Lambda} \frac{1}{N_d(r)} \sum_{|m|=r} |f(n-m)|,$$

where $f : \mathbb{Z}^d \to \mathbb{C}$ is a function, $d \geq 2$, $N_d(r)$ denotes the number of lattice points on the sphere $\{x \in \mathbb{R}^d : |x| = r\}$ and $\Lambda = \{r \geq 0 : N_d(r) \neq 0\}$. The sum in the definition of the operator A_* is taken over the lattice points m on the sphere of radius r. This operator is the discrete analogue of the classical Euclidean spherical maximal function

$$\mathcal{A}_*(f)(x) = \sup_{r\in(0,\infty)} |f| * d\sigma_r(x)$$

where $d\sigma_r$ denotes the normalized invariant measure on the sphere $|x| = r$ and $f : \mathbb{R}^d \to \mathbb{C}$ is a suitable function. It is well known that the operator $\mathcal{A}_*$ extends to a bounded operator on $L^p(\mathbb{R}^d)$ for $d \geq 2$ and $p > d/(d-1)$ (see Stein [7] in the case $d \geq 3$ and Bourgain [2] in the case $d = 2$).

The question of boundedness on $L^p(\mathbb{Z}^d)$ of the operator A_* was considered by Magyar [4] and Magyar, Stein and Wainger [5]. The main theorem in [5] is the following.

Theorem. (Magyar, Stein and Wainger [5]). *The maximal operator A_* extends to a bounded operator on $L^p(\mathbb{Z}^d)$ if and only if $d \geq 5$ and $p > d/(d-2)$ or $d \leq 4$ and $p = \infty$.*

The distinction between the cases $d \geq 5$ and $d \leq 4$ is related to the behavior of the function $N_d(r)$. If $d \leq 4$, this function is irregular. On the other hand, it is well known that if $d \geq 5$, then there is a constant $C_d \geq 1$ such that $C_d^{-1} r^{d-2} \leq$

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$N_d(r) \leq C_d r^{d-2}$ for any $r \geq 1$ with the property that $r^2 \in \mathbb{Z}$. In particular, the set Λ in the definition of the operator A_* is equal to $\{r \geq 0 : r^2 \in \mathbb{Z}\}$ if $d \geq 5$.

In this note we prove an endpoint estimate for the theorem of Magyar, Stein and Wainger. For $p, q \in [1, \infty]$ let $L^{p,q}(\mathbb{Z}^d)$ denote the usual Lorentz space of functions on $\mathbb{Z}^d$. We have the following restricted weak type estimate.

Theorem 1. *Assume that $d \geq 5$ and let $p_d = d/(d-2)$. The discrete spherical maximal function A_* extends to a bounded operator from $L^{p_d,1}(\mathbb{Z}^d)$ to $L^{p_d,\infty}(\mathbb{Z}^d)$.*

The Euclidean analogue of this theorem was proved by Bourgain [1]: the Euclidean spherical maximal function $\mathcal{A}_*$ extends to a bounded operator from $L^{d/(d-1),1}(\mathbb{R}^d)$ to $L^{d/(d-1),\infty}(\mathbb{R}^d)$ if $d \geq 3$. This restricted weak type estimate fails in dimension $d = 2$ (see [6, Proposition 1.5]).

Our proof of Theorem 1 follows the line of the proof of the theorem of Magyar, Stein and Wainger [5]. The main ingredients are the circle method of Hardy, Littlewood and Ramanujan, the Poisson summation formula, and a transference principle. Our simplification is that we will not need the dyadic version of the theorem due to Magyar [4]. Instead, we decompose our operator into an L^1 part and an L^2 part depending on a parameter α. We use the discrete Hardy-Littlewood maximal function to establish the L^1 bounds and the error analysis in [5] together with a lemma of Bourgain [1] for the L^2 bounds.

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2. Proof of Theorem 1

We first replace the operator A_* with the operator

$$\widetilde{A_*}(f)(n) = \sup_{r \in \Lambda} \left| \frac{1}{r^{(d-2)}} \sum_{|m|=r} f(n-m) \right|$$

where $\Lambda = \{r \in [1, \infty) : r^2 \in \mathbb{Z}\}$ and $f : \mathbb{Z}^d \to \mathbb{C}$ is compactly supported. This is possible since $d \geq 5$ and $N_d(r) \approx r^{d-2}$. It remains to prove that $\widetilde{A_*}$ extends to a bounded operator from $L^{p_d,1}(\mathbb{Z}^d)$ to $L^{p_d,\infty}(\mathbb{Z}^d)$. This is an easy consequence of the following lemma.

Lemma 2. *For any $\alpha \in (0, 1]$ there are two subadditive operators A^1_α and A^2_α with the property that $\widetilde{A_*}(f)(n) \leq \left|A^1_\alpha(f)(n)\right| + \left|A^2_\alpha(f)(n)\right|$ for any $n \in \mathbb{Z}^d$,*

$$||A^1_\alpha(f)||_{L^{1,\infty}} \leq C\alpha^{-2}||f||_{L^1} \tag{2.1}$$

and

$$||A^2_\alpha(f)||_{L^2} \leq C\alpha^{(d-4)/2}||f||_{L^2} \tag{2.2}$$

for any compactly supported function $f : \mathbb{Z}^d \to \mathbb{C}$.

The method of proving restricted weak type inequalities by decomposing the operator as in Lemma 2 is due to Bourgain [1]. An abstract version of this argument may be found in the appendix of [3]. First, we show how to use Lemma 2 to prove the theorem. By the general theory of Lorentz spaces, the $L^{p_d,1} \to L^{p_d,\infty}$ boundedness of the operator $\widetilde{A_*}$ is equivalent to

$$||\widetilde{A_*}(\chi_F)||_{L^{p_d,\infty}} \leq C|F|^{1/p_d} \tag{2.3}$$

for any finite set F, where χ_F denotes the characteristic function of the set F and $|F|$ denotes its cardinality. Clearly, $\widetilde{A_*}(\chi_F)(n) \leq C_d$ for any n, where C_d is the constant with the property that $N_d(r) \leq C_d r^{d-2}$ for any $r \geq 1$. Thus (2.3) is equivalent to proving that for any $\lambda \in (0,1]$,

$$\lambda^{p_d}|\{n : \widetilde{A_*}(\chi_F)(n) > \lambda\}| \leq C|F|. \tag{2.4}$$

By Lemma 2 we have

$$\begin{aligned}|\{n : \widetilde{A_*}(\chi_F)(n) > \lambda\}| &\leq |\{n : A^1_\alpha(\chi_F)(n) > \lambda/2\}| + |\{n : A^2_\alpha(\chi_F)(n) > \lambda/2\}| \\ &\leq \frac{2}{\lambda}||A^1_\alpha(\chi_F)||_{L^{1,\infty}} + \frac{4}{\lambda^2}||A^2_\alpha(\chi_F)||^2_{L^2} \\ &\leq C\lambda^{-1}\alpha^{-2}|F| + C\lambda^{-2}\alpha^{d-4}|F|.\end{aligned}$$

Since $p_d = d/(d-2)$, the estimate (2.4) follows by taking $\alpha = \lambda^{1/(d-2)}$.

It remains to prove Lemma 2. Fix $\alpha \in (0,1]$. We will use some of the notation in [5]. Let

$$A_r(f)(n) = \frac{1}{r^{(d-2)}} \sum_{|m|=r} f(n-m)$$

and

$$M_r(f) = \sum_{q=1}^{\infty} \sum_{1\leq a\leq q,\, (a,q)=1} e^{-2\pi i r^2 a/q} M_r^{a/q}(f),$$

where, as in [5], $M_r^{a/q}$ is the convolution operator whose multiplier is

$$\sum_{\ell\in\mathbb{Z}^d} G(a/q,\ell)\Psi_q(\xi-\ell/q)\widehat{d\sigma}_r(\xi-\ell/q).$$

Here $G(a/q,\ell)$ is the normalized Gauss sum

$$G(a/q,\ell) = q^{-d} \sum_{n\in\mathbb{Z}^d/(q\mathbb{Z})^d} e^{2\pi i(|n|^2 a/q + n\cdot\ell/q)},$$

Ψ is a smooth cutoff function supported in the cube $Q/2 = \{\xi : |\xi_j| \leq 1/4, j = 1,\dots,d\}$ and identically equal to 1 in the cube $Q/4$, $\Psi_q(\eta) = \Psi(q\eta)$, $\widehat{d\sigma}$ is the Fourier transform of the invariant measure on the sphere $|x| = 1$ normalized with total measure 1, and $\widehat{d\sigma}_r(\eta) = \widehat{d\sigma}(r\eta)$. The reason for considering the operators M_r is that they are good approximations (in L^2) of the operators A_r. In what follows we assume that r is restricted so that $r \geq 1$ and $r^2 \in \mathbb{Z}$. Let $N = 1/\alpha \geq 1$. We have

$$\begin{aligned}\widetilde{A}_*(f)(n) = \sup_r |A_r(f)(n)| &\leq \sup_{r\leq 10N} |A_r(f)(n)| + \sup_{r\geq 10N} |A_r(f)(n)| \\ &\leq \sup_{r\leq 10N} |A_r(f)(n)| + \sup_{r\geq 10N} |(A_r - M_r)(f)(n)| + \sup_{r\geq 10N} |M_r(f)(n)| \\ &= A^{1,1}_\alpha(f)(n) + A^{2,1}_\alpha(f)(n) + \sup_{r\geq 10N} |M_r(f)(n)|.\end{aligned}$$

Let

$$\mathcal{M}(f)(n) = \sup_r \frac{1}{r^d} \sum_{|m|\leq r} |f(n-m)|$$

denote the discrete Hardy-Littlewood maximal function. By the same argument as in Euclidean spaces we have

$$||\mathcal{M}(f)||_{L^{1,\infty}(\mathbb{Z}^d)} \leq C||f||_{L^1(\mathbb{Z}^d)}. \tag{2.5}$$

We use the operator $\mathcal{M}$ to bound the operator $A^{1,1}_\alpha$. We have

$$A^{1,1}_\alpha(f)(n) \leq \sup_{r\leq 10N} \frac{1}{r^{d-2}} \sum_{|m|=r} |f(n-m)| \leq 100N^2\mathcal{M}(f)(n).$$

The desired bound

$$||A^{1,1}_\alpha(f)||_{L^{1,\infty}} \leq C\alpha^{-2}||f||_{L^1} \tag{2.6}$$

follows from (2.5) and the fact that $N^2 = \alpha^{-2}$.

For the operator $A^{2,1}_\alpha$ we use Proposition 4.1 in [5], which can be written in the form

$$||\sup_{r\in[R,2R]} |A_r(f) - M_r(f)|\,||_{L^2} \leq CR^{-(d-4)/2}||f||_{L^2}$$

for $R \geq 1$ and $d \geq 5$. Since the supremum in the definition of the operator $A^{2,1}_\alpha$ is taken over $r \geq 10N$, it follows that

$$||A^{2,1}_\alpha(f)||_{L^2} \leq C\alpha^{(d-4)/2}||f||_{L^2} \tag{2.7}$$

as desired.

It remains to decompose the operator

$$f \to \sup_{r\geq 10N} |M_r(f)(n)|.$$

For this we write first

$$\sup_{r\geq 10N} |M_r(f)(n)| \leq C \sup_{r\geq 10N} \sum_{q=1}^{N/10} \sum_{1\leq a\leq q,\,(a,q)=1} \left|M^{a/q}_r(f)(n)\right| + CA^{2,2}_\alpha(f)(n)$$

where

$$A^{2,2}_\alpha(f)(n) = \sum_{q\geq N/10} \sum_{1\leq a\leq q,\,(a,q)=1} \sup_r |M^{a/q}_r(f)(n)|.$$

To bound the operator $A^{2,2}_\alpha$ we use Proposition 3.1(a) in [5] for $p = 2$:

$$||\sup_r M^{a/q}_r(f)||_{L^2} \leq Cq^{-d/2}||f||_{L^2}.$$

We can sum this bound over $q \geq N/10$ and $a \in [1, q] \cap \mathbb{Z}$ to obtain

$$||A^{2,2}_\alpha(f)||_{L^2} \leq C\alpha^{(d-4)/2}||f||_{L^2} \tag{2.8}$$

as desired.

It remains to decompose the operators $M^{a/q}_r$ for integers $q \in [1, N/10]$. For this let $M^{a,q,1}_{r,\alpha}$ denote the convolution operator given by the multiplier

$$\sum_{\ell\in\mathbb{Z}^d} G(a/q,\ell)\Psi_q(\xi-\ell/q)\widehat{d\sigma_r}(\xi-\ell/q)\Psi_{rq/N}(\xi-\ell/q),$$

and let $M^{a,q,2}_{r,\alpha}$ denote the convolution operator given by the multiplier

$$\sum_{\ell\in\mathbb{Z}^d} G(a/q,\ell)\Psi_q(\xi-\ell/q)\widehat{d\sigma_r}(\xi-\ell/q)(1-\Psi)_{rq/N}(\xi-\ell/q).$$

The notation is, as before, $F_\lambda(\eta) = F(\lambda\eta)$. Clearly, $M_r^{a/q} = M_{r,\alpha}^{a,q,1} + M_{r,\alpha}^{a,q,2}$. Let

$$M_\alpha^1(f)(n) = \sup_{r\geq 10N} \sum_{q=1}^{N/10} \sum_{1\leq a\leq q,\,(a,q)=1} \left|M_{r,\alpha}^{a,q,1}(f)(n)\right|$$

and

$$M_\alpha^2(f)(n) = \sum_{q=1}^{N/10} \sum_{1\leq a\leq q,\,(a,q)=1} \sup_{r\geq 10N} \left|M_{r,\alpha}^{a,q,2}(f)(n)\right|.$$

This is the decomposition of the remaining operator into an L^1 part and an L^2 part.

For the L^1 estimate we will prove that

$$||M_\alpha^1(f)||_{L^{1,\infty}} \leq C\alpha^{-2}||f||_{L^1}. \tag{2.9}$$

For this we need an estimate on the kernel of the operator $M_{r,\alpha}^{a,q,1}$. Let $K_{r,\alpha}^{a,q,1}$ denote this kernel. Notice that $\Psi_q(\xi-\ell/q)\Psi_{rq/N}(\xi-\ell/q) = \Psi_{rq/N}(\xi-\ell/q)$ since $r \geq 10N$. Let Q denote the standard cube $Q = \{\xi = (\xi_1, \dots, \xi_d) : -1/2 < \xi_j \leq 1/2\}$. Then by letting $\ell = \ell' + qs$, $\ell' \in \{0, 1, \dots, q-1\}^d$, $s \in \mathbb{Z}^d$ we have

$$\begin{aligned} K_{r,\alpha}^{a,q,1}(m) &= \int_Q e^{2\pi i m\cdot\xi} \sum_{\ell\in\mathbb{Z}^d} G(a/q,\ell)\widehat{d\sigma}_r(\xi-\ell/q)\Psi_{rq/N}(\xi-\ell/q)\,d\xi \\ &= \sum_{\ell'\in\mathbb{Z}^d/(q\mathbb{Z})^d} G(a/q,\ell') \sum_{s\in\mathbb{Z}^d} \int_Q e^{2\pi i m\cdot\xi}\widehat{d\sigma}_r(\xi-s-\ell'/q)\Psi_{rq/N}(\xi-s-\ell'/q)\,d\xi \\ &= \left(\sum_{\ell'\in\mathbb{Z}^d/(q\mathbb{Z})^d} G(a/q,\ell')e^{2\pi i m\cdot\ell'/q}\right) \int_{\mathbb{R}^d} e^{2\pi i m\cdot\eta}\widehat{d\sigma}_r(\eta)\Psi_{rq/N}(\eta)\,d\eta \\ &= e^{2\pi i|m|^2 a/q} r^{-d} \int_{\mathbb{R}^d} e^{2\pi i\eta\cdot m/r}\widehat{d\sigma}(\eta)\Psi_{q/N}(\eta)\,d\eta = e^{2\pi i|m|^2 a/q} r^{-d} d\sigma * \psi^{q/N}(m/r). \end{aligned}$$

Here ψ is the inverse (Euclidean) Fourier transform of Ψ, the convolution denotes the Euclidean convolution and $\psi^{q/N}(x) = (N/q)^d\psi(xN/q)$. Since ψ is a Schwartz function, it is easy to see that

$$|d\sigma * \psi^{q/N}(x)| \leq C(1+|x|)^{-(d+1)}N/q$$

for any $x \in \mathbb{R}^d$. Thus,

$$|K_{r,\alpha}^{a,q,1}(m)| \leq Cr^{-d}(1+|m|/r)^{-(d+1)}N/q.$$

By summing this bound over $a \in [1,q]\cap\mathbb{Z}$ and $q \in [1, N/10]\cap\mathbb{Z}$ we have

$$\sum_{q=1}^{N/10} \sum_{1\leq a\leq q,\,(a,q)=1} \left|M_{r,\alpha}^{a,q,1}(f)(n)\right| \leq C|f| * K_{r,\alpha}(n)$$

where

$$K_{r,\alpha}(m) = N^2 r^{-d}(1+|m|/r)^{-(d+1)}.$$

Thus,

$$M_\alpha^1(f)(n) \leq CN^2\mathcal{M}(f)(n)$$

and the estimate (2.9) follows from (2.5).

For the L^2 estimate we will prove that

$$||M_\alpha^2(f)||_{L^2} \leq C\alpha^{(d-4)/2}||f||_{L^2}. \tag{2.10}$$

By the formula of M^2_α, it suffices to prove that

$$||M^{a,q,2}_{r,\alpha}(f)||_{L^2_{L^\infty(\Lambda)}} \leq Cq^{-1}N^{-(d-2)/2}||f||_{L^2}, \tag{2.11}$$

where $L^2_{L^\infty(\Lambda)}$ denotes the space of L^2 functions on $\mathbb{Z}^d$ with values in the Banach space $L^\infty(\Lambda)$. We first argue as in [5]. Let $\widetilde{\Psi}$ be a smooth function supported in Q with the property that $\widetilde{\Psi}(\xi) \equiv 1$ in $Q/2$. The operator $M^{a,q,2}_{r,\alpha}$ can be written as the composition of two operators with multipliers

$$\sum_{\ell\in\mathbb{Z}^d} G(a/q,\ell)\widetilde{\Psi}_q(\xi-\ell/q)$$

and

$$\sum_{\ell\in\mathbb{Z}^d} \Psi_q(\xi-\ell/q)\widehat{d\sigma}_r(\xi-\ell/q)(1-\Psi)_{rq/N}(\xi-\ell/q),$$

respectively. Let $S^{a,q}$ and $T^q_{r,\alpha}$ denote the two operators. Since $|G(a/q,\ell)| \leq Cq^{-d/2}$, we have

$$||S^{a,q}||_{L^2\to L^2} \leq Cq^{-d/2}. \tag{2.12}$$

Let

$$m^q_{r,\alpha}(\eta) = \Psi_q(\eta)\widehat{d\sigma}_r(\eta)(1-\Psi)_{rq/N}(\eta).$$

The multiplier $m^q_{r,\alpha}$ is supported in Q/q. By the transference principle of Magyar, Stein and Wainger (Corollary 2.1 in [5]), we have

$$||T^q_{r,\alpha}||_{L^2(\mathbb{Z}^d)\to L^2_{L^\infty(\Lambda)}(\mathbb{Z}^d)} \leq C||\widetilde{T}^q_{r,\alpha}||_{L^2(\mathbb{R}^d)\to L^2_{L^\infty(\Lambda)}(\mathbb{R}^d)} \tag{2.13}$$

where $\widetilde{T}^q_{r,\alpha}$ denotes the operator with multiplier $m^q_{r,\alpha}$ acting on functions in $L^2(\mathbb{R}^d)$. It remains to prove that

$$||\widetilde{T}^q_{r,\alpha}||_{L^2(\mathbb{R}^d)\to L^2_{L^\infty(\Lambda)}(\mathbb{R}^d)} \leq C(q/N)^{(d-2)/2}. \tag{2.14}$$

The operator $\widetilde{T}^q_{r,\alpha}$ can be written as the composition of two operators with multipliers $\eta \to \Psi_q(\eta)$ and $\eta \to \widetilde{m}_r(\eta)$, respectively, where

$$\widetilde{m}(\eta) = \widehat{d\sigma}(\eta)(1-\Psi)(q\eta/N)$$

and $\widetilde{m}_r(\eta) = \widetilde{m}(r\eta)$. The operator defined by the multiplier $\eta \to \Psi_q(\eta)$ is bounded on $L^2(\mathbb{R}^d)$ uniformly in q. Let $\widetilde{U}_r$ denote the operator with multiplier $\widetilde{m}_r$ acting on functions in $L^2(\mathbb{R}^d)$. We will use the following lemma of Bourgain (Proposition 2 in [1]):

Lemma (Bourgain [1]). *Assume that $m : \mathbb{R}^d \to \mathbb{C}$ is a smooth function and U_r is the operator defined by the multiplier $\eta \to m_r(\eta) = m(r\eta)$. Then*

$$||\sup_{r>0}|U_r f|\,||_{L^2} \leq C\Gamma(m)||f||_{L^2}$$

for any Schwartz function f where

$$\Gamma(m) = \sum_{j\in\mathbb{Z}} \alpha_j^{1/2}(\alpha_j^{1/2}+\beta_j^{1/2})$$

with

$$\alpha_j = \sup_{|\eta|\in[2^j,2^{j+1}]} |m(\eta)| \quad \textit{and} \quad \beta_j = \sup_{|\eta|\in[2^j,2^{j+1}]} |\nabla m(\eta)\cdot\eta|\,.$$

In our case, the multiplier $\widetilde{m}$ is supported in the set $|\eta| \geq N/(8q)$ since $\Psi(\eta) = 1$ is $|\eta| \leq 1/8$. In addition, $\left|\widehat{d\sigma}(\eta)\right| \leq C(1+|\eta|)^{-(d-1)/2}$ and $\left|\nabla\widehat{d\sigma}(\eta)\right| \leq C(1+|\eta|)^{-(d-1)/2}$. Thus in our case, $\alpha_j \leq C2^{-j(d-1)/2}$ and $\beta_j \leq C2^{-j(d-3)/2}$ if $2^j \geq N/(16q)$ and $\alpha_j = \beta_j = 0$ if $2^j < N/(16q)$. Thus $\Gamma(\widetilde{m}) \leq C(q/N)^{(d-2)/2}$. By Bourgain's lemma,

$$||\widetilde{U}_r||_{L^2(\mathbb{R}^d)\to L^2_{L^\infty(\mathbb{R}_+)}(\mathbb{R}^d)} \leq C(q/N)^{(d-2)/2}.$$

This proves (2.14). The estimate (2.11) follows from (2.12), (2.13) and (2.14), and the estimate (2.10) follows by summing over q and a.

We can now finish the proof of Lemma 2. Let $A^1_\alpha(f) = A^{1,1}_\alpha(f) + M^1_\alpha(f)$ and $A^2_\alpha(f) = A^{2,1}_\alpha(f) + A^{2,2}_\alpha(f) + M^2_\alpha(f)$. The estimate (2.1) follows from (2.6) and (2.9), and the estimate (2.2) follows from (2.7), (2.8) and (2.10).

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