

SECOND COHOMOLOGY GROUP OF GROUP ALGEBRAS WITH COEFFICIENTS IN ITERATED DUALS

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ABSTRACT. In this paper we show that the first cohomology group $\mathcal{H}^1(\ell^1(G), (\ell^1(S))^{(n)})$ is zero for every odd $n \in \mathbb{N}$ and for every G -set S . In the case when G is a discrete group, this is a generalization of the following result of Dales et al.: for any locally compact group G , $L^1(G)$ is $(2n + 1)$ -weakly amenable.

Next we show that the second cohomology group $\mathcal{H}^2(\ell^1(G), (\ell^1(S))^{(n)})$ is a Banach space. Finally, for every locally compact group G we show that $\mathcal{H}^2(L^1(G), (L^1(G))^{(n)})$ is a Banach space for every odd $n \in \mathbb{N}$.

1. INTRODUCTION

In this paper we shall be concerned with the structure of the second cohomology group of a group algebra $L^1(G)$ with coefficients in the n th dual space $(L^1(G))^{(n)}$. We begin by recalling some terminology.

Suppose that \mathcal{A} is a Banach algebra and that \mathcal{X} is a Banach \mathcal{A} -bimodule, so that \mathcal{X} is an \mathcal{A} -bimodule, \mathcal{X} is a Banach space for a norm $\|\cdot\|$, and $\|a \cdot x\| \leq \|a\| \|x\|$ and $\|x \cdot a\| \leq \|a\| \|x\|$ for $a \in \mathcal{A}$ and $x \in \mathcal{X}$. For example, \mathcal{A} itself is a Banach \mathcal{A} -bimodule. Let \mathcal{A} be a Banach algebra and let \mathcal{X} be a Banach \mathcal{A} -bimodule. Then the \mathcal{X}' is also a Banach \mathcal{A} -bimodule for the products $a \cdot \lambda$ and $\lambda \cdot a$ specified by

$$a \cdot \lambda(x) = \lambda(x \cdot a), \quad \lambda \cdot a(x) = \lambda(a \cdot x) \quad (a \in \mathcal{A}, x \in \mathcal{X}, \lambda \in \mathcal{X}').$$

Similarly, the higher duals $\mathcal{X}^{(n)}$ are Banach \mathcal{A} -bimodules. The canonical embedding of \mathcal{X} in $\mathcal{X}^{(2)} = \mathcal{X}''$ is denoted by $\widehat{\cdot}$. Since $\widehat{a \cdot x} = a \cdot \widehat{x}$, then $\mathcal{X}^{(n)}$ is a submodule of $\mathcal{X}^{(n+2)}$ for every $n \in \mathbb{Z}^+$. We also set $\mathcal{X}^{(0)} = \mathcal{X}$. If S is a topological space and G is a (discrete) group, then we say that S is a G -set if the product gx is defined for all g in G and x in S in such a way that $g(hx) = (gh)x$ ($g, h \in G, x \in S$) and $x \mapsto gx$ is a homeomorphism of S onto S for every g in S . The cohomology complex is

$$0 \longrightarrow \mathcal{X} \xrightarrow{\delta^0} \mathcal{C}^1(\mathcal{A}, \mathcal{X}) \xrightarrow{\delta^1} \mathcal{C}^2(\mathcal{A}, \mathcal{X}) \xrightarrow{\delta^2} \dots,$$

where for $n \in \mathbb{Z}^+$, $\mathcal{C}^n(\mathcal{A}, \mathcal{X})$ is the set of all bounded n -linear maps from \mathcal{A} to \mathcal{X} . The map $\delta^0 : \mathcal{X} \longrightarrow \mathcal{C}^1(\mathcal{A}, \mathcal{X})$ is given by $\delta^0(x)(a) = a \cdot x - x \cdot a$ and for $n \in \mathbb{Z}^+$,

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the map $\delta^n : \mathcal{C}^n(\mathcal{A}, \mathcal{X}) \longrightarrow \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{X})$ is given by

$$\begin{aligned} \delta^n T(a_1, \dots, a_{n+1}) &= a_1 \cdot T(a_2, \dots, a_{n+1}) + \sum_{i=1}^n (-1)^i T(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ &\quad + (-1)^{n+1} T(a_1, \dots, a_n) \cdot a_{n+1}, \end{aligned}$$

where $T \in \mathcal{C}^n(\mathcal{A}, \mathcal{X})$ and $a_1, \dots, a_{n+1} \in \mathcal{A}$. The space $\ker \delta^n$ of bounded n -cocycle is denoted by $\mathcal{Z}^n(\mathcal{A}, \mathcal{X})$ and the space $\text{Im } \delta^{n-1}$ of bounded n -coboundary is denoted by $\mathcal{B}^n(\mathcal{A}, \mathcal{X})$. We recall that $\mathcal{B}^n(\mathcal{A}, \mathcal{X})$ is a subspace of $\mathcal{Z}^n(\mathcal{A}, \mathcal{X})$ and that the n th cohomology group $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ is defined by the quotient

$$\mathcal{H}^n(\mathcal{A}, \mathcal{X}) = \frac{\mathcal{Z}^n(\mathcal{A}, \mathcal{X})}{\mathcal{B}^n(\mathcal{A}, \mathcal{X})},$$

which is called the n th continuous or Banach cohomology group of \mathcal{A} with coefficients in \mathcal{X} . The space $\mathcal{Z}^n(\mathcal{A}, \mathcal{X})$ is a Banach space, but in general $\mathcal{B}^n(\mathcal{A}, \mathcal{X})$ is not closed; we regard $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ as a complete seminormed space with respect to the quotient seminorm. This seminorm is a norm if and only if $\mathcal{B}^n(\mathcal{A}, \mathcal{X})$ is a closed subspace of $\mathcal{C}^n(\mathcal{A}, \mathcal{X})$, which means that $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ is a Banach space. It is unknown whether or not $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ is a Banach space for every n .

The Banach algebra \mathcal{A} is amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{X}') = 0$ for every Banach \mathcal{A} -bimodule \mathcal{X}' . This definition of amenability was introduced by Johnson in (1972) [9]. The Banach algebra \mathcal{A} is weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}') = 0$. This definition generalizes the one that was introduced by Bade, Curtis and Dales in [1], where it was noted that a commutative Banach algebra \mathcal{A} is weakly amenable if and only if $\mathcal{H}^1(\mathcal{A}, \mathcal{X}) = 0$ for every symmetric Banach \mathcal{A} -bimodule \mathcal{X} . The Banach algebra \mathcal{A} is permanently weakly amenable if \mathcal{A} is n -weakly amenable for every $n \in \mathbb{N}$, that is, $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(n)}) = 0$. This notion was introduced by Dales, Ghahramani and Grønbaek in [2].

It was shown in [9] that the group algebra $L^1(G)$ is amenable if and only if G is an amenable group, and in [11] that $L^1(G)$ is weakly amenable for every locally compact group G ; see also [3] for a shorter proof. Also it was shown in [2] that $L^1(G)$ is $(2n+1)$ -weakly amenable for every locally compact group G . Johnson [9] proved that for the free group on two generators $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \mathbb{C}) \neq 0$ which by [15, Theorem 8.3.1] implies that $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \ell^1(\mathbb{F}_2)) \neq 0$ and $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \ell^\infty(\mathbb{F}_2)) \neq 0$.

In [8] Ivanov and in [12] Matsumoto and Morita showed that $\mathcal{H}^2(\ell^1(G), \mathbb{C})$ is a Banach space for every discrete group G with trivial action on \mathbb{C} . A. Pourabbas and M. C. White [13] showed that the second cohomology group of $L^1(G)$ with coefficients in $L^\infty(G)$ is a Banach space for every locally compact group G . In this paper first we show that $\mathcal{H}^1(\ell^1(G), (\ell^1(S))^{(2n+1)}) = 0$ for every discrete group G and every G -set S . Next we show that the second cohomology group of $L^1(G)$ with coefficients in $(L^1(G))^{(2n+1)}$ is a Banach space for every locally compact group G and every $n \in \mathbb{N}$.

The following proposition is standard. We include the proof as a base for later results.

Proposition 1.1. *Let X and Y be Banach spaces and let $\Phi : X \rightarrow Y$ be a bounded linear map. If there exists a constant M such that for every $y \in \text{Im } \Phi$ there exists an element $x \in X$ such that $\|x\| \leq M \|y\|$ and $y = \Phi(x)$, then $\text{Im } \Phi$ is closed.*

Proof. Fix $y \in \overline{\text{Im } \Phi}$ and $\epsilon > 0$. There exists an element $y_1 \in \text{Im } \Phi$ such that

$$\|y - y_1\| \leq \frac{1}{2}\epsilon.$$

By considering $y - y_1$, there exists an element $y_2 \in \text{Im } \Phi$ such that

$$\|y - y_1 - y_2\| \leq \frac{1}{2^2}\epsilon.$$

By induction, this process defines a sequence $\{y_n\} \subseteq \text{Im } \Phi$ such that

$$(1.1) \quad \|y - y_1 - \cdots - y_n\| \leq \frac{1}{2^n}\epsilon.$$

Note that

$$(1.2) \quad \begin{aligned} \|y_{n+1}\| &\leq \|y - y_1 - \cdots - y_n\| + \frac{1}{2^{n+1}}\epsilon \\ &\leq \frac{1}{2^{n+1}}\epsilon + \frac{1}{2^n}\epsilon \leq \frac{2}{2^n}\epsilon. \end{aligned}$$

By the hypothesis, for every $y_n \in \text{Im } \Phi$, there exists an $x_n \in X$ such that

$$(1.3) \quad \Phi(x_n) = y_n \quad \text{and} \quad \|x_n\| \leq M \|y_n\|.$$

For every $n \in \mathbb{N}$ by (1.3) and (1.2) we have

$$(1.4) \quad \|x_{n+1}\| \leq M \|y_{n+1}\| \leq \frac{M}{2^{n-1}}\epsilon.$$

If we set $s_n = x_1 + \cdots + x_n$, then (1.4) shows that $\{s_n\}$ is a Cauchy sequence in X . Since X is complete, there exists an $x \in X$ such that $s_n \rightarrow \sum_{n=1}^\infty x_n = x$. Since Φ is continuous $\Phi(s_n) \rightarrow \Phi(x)$. By (1.1),

$$\Phi(s_n) = y_1 + \cdots + y_n \rightarrow y.$$

Hence $\Phi(x) = y$ so that $y \in \text{Im } \Phi$. Thus the range of Φ is closed. □

2. SECOND COHOMOLOGY GROUP OF DISCRETE GROUP ALGEBRAS

In this section we will show that the first cohomology group of $\ell^1(G)$ with coefficients in $(\ell^1(S))^{(2n+1)}$ for an arbitrary G -set S is trivial. Next we will show that $\mathcal{H}^2(\ell^1(G), (\ell^1(S))^{(2n+1)})$ is a Banach space for every $n \in \mathbb{N}$.

Remark 2.1. Set $\mathcal{X} = (\ell^1(S))^{(2n)}$. We note that $(\ell^1(S))' = \ell^\infty(S)$ is a commutative unital C^* -algebra. Because the second dual of a commutative unital C^* -algebra is a commutative von Neumann algebra, then $\mathcal{X}' = (\ell^1(S))^{(2n+1)}$ is the underlying space of a commutative von Neumann algebra, and hence it is an L^∞ -space. The space $\mathcal{X}'_{\mathbb{R}}$ of real-valued functions in \mathcal{X}' forms a complete lattice in the sense that every nonempty subset of $\mathcal{X}'_{\mathbb{R}}$ that is bounded above has a supremum.

Theorem 2.2. *Let G be a discrete group. Then $\mathcal{H}^1(\ell^1(G), (\ell^1(S))^{(2n+1)}) = 0$ for every G -set S and for every $n \in \mathbb{N}$.*

Proof. Let $\mathcal{X} = (\ell^1(S))^{(2n)}$ and $D \in \mathcal{Z}^1(\ell^1(G), \mathcal{X}')$. We show that there exists a function ϕ in \mathcal{X}' that satisfies

$$D(\delta_g)(x) = \phi \cdot \delta_g(x) - \delta_g \cdot \phi(x),$$

where δ_g is the point mass at g . Let $\text{Re } D$ denote the real part of D and let $S = \{\text{Re } D(\delta_h) \cdot \delta_{h^{-1}} : h \in G\}$. Then S is a subset of $\mathcal{X}'_{\mathbb{R}}$ and is bounded above by $\|D\|$ in $\mathcal{X}'_{\mathbb{R}}$. Since $\mathcal{X}'_{\mathbb{R}}$ is a complete lattice, $\phi_r = \sup(S)$ exists in $\mathcal{X}'_{\mathbb{R}}$.

For every $g \in G$ and $x \in \mathcal{X}$ we have

$$\begin{aligned} \delta_g \cdot \phi_r(x) &= \sup_{h \in G} \{ \delta_g \cdot \operatorname{Re} D(\delta_h)(\delta_{h^{-1}} \cdot x) \} \\ &= \sup_{h \in G} \{ \operatorname{Re} D(\delta_g * \delta_h)(\delta_{h^{-1}} \cdot x) - \operatorname{Re} D(\delta_g) \cdot \delta_h(\delta_{h^{-1}} \cdot x) \} \\ &= \sup_{gh \in G} \{ \operatorname{Re} D(\delta_{gh})(\delta_{(gh)^{-1}} \cdot (\delta_g \cdot x)) - \operatorname{Re} D(\delta_g)(x) \} \\ &= \phi_r(\delta_g \cdot x) - \operatorname{Re} D(\delta_g)(x). \end{aligned}$$

Therefore, $\operatorname{Re} D(\delta_g)(x) = \phi_r \cdot \delta_g(x) - \delta_g \cdot \phi_r(x)$. A similar result holds for the imaginary part of $D(\delta_g)$, and so we see that there exists $\phi \in \mathcal{X}'$ such that

$$D(\delta_g)(x) = \phi \cdot \delta_g(x) - \delta_g \cdot \phi(x).$$

□

Theorem 2.3. *Let G be a discrete group. Then $\mathcal{H}^2(\ell^1(G), (\ell^1(S))^{(2n+1)})$ is a Banach space for every G -set S and for every $n \in \mathbb{N}$.*

Proof. Let $\mathcal{X} = (\ell^1(S))^{(2n)}$ and $\psi \in \mathcal{C}^1(\ell^1(G), \mathcal{X}')$. Then for every $g, h \in G$ and $x \in \mathcal{X}$ with $\|x\| \leq 1$ we have

$$(2.1) \quad |\delta\psi(\delta_g, \delta_h)(x)| = |\psi(\delta_g) \cdot \delta_h(x) - \psi(\delta_{gh})(x) + \delta_g \cdot \psi(\delta_h)(x)| \leq \|\delta\psi\|.$$

Since $\{\operatorname{Re} \psi(\delta_g) \cdot \delta_g^{-1} : g \in G\}$ is bounded above by $\|\psi\|$ in $\mathcal{X}'_{\mathbb{R}}$, then

$$f_r = \sup_{g \in G} \{ \operatorname{Re} \psi(\delta_g) \cdot \delta_g^{-1} \}$$

exists in $\mathcal{X}'_{\mathbb{R}}$. For every $h \in G$ by (2.1) we have

$$\begin{aligned} f_r \cdot \delta_h(x) &= \sup_{g \in G} \{ \operatorname{Re} \psi(\delta_g) \cdot \delta_{g^{-1}h}(x) \} \\ &= \sup_{k \in G} \{ \operatorname{Re} \psi(\delta_{hk}) \cdot \delta_{k^{-1}}(x) \} \\ (2.2) \quad &\leq \sup_{k \in G} \{ \operatorname{Re} \psi(\delta_h)(x) + \operatorname{Re} \delta_h \cdot \psi(\delta_k) \cdot \delta_{k^{-1}}(x) + \|\delta\psi\| \} \\ &= \operatorname{Re} \psi(\delta_h)(x) + \delta_h \cdot f_r(x) + \|\delta\psi\|, \end{aligned}$$

where $g^{-1}h = k^{-1}$. On the other hand,

$$\begin{aligned} f_r \cdot \delta_h(x) &= \sup_{g \in G} \{ \operatorname{Re} \psi(\delta_g) \cdot \delta_{g^{-1}h}(x) \} \\ (2.3) \quad &\geq \operatorname{Re} \psi(\delta_h)(x) + \delta_h \cdot f_r(x) - \|\delta\psi\|. \end{aligned}$$

From (2.2) and (2.3) we have

$$|\delta_h \cdot f_r(x) - f_r \cdot \delta_h(x) + \operatorname{Re} \psi(\delta_h)(x)| \leq \|\delta\psi\|.$$

Similarly, by considering imaginary parts we have

$$|\delta_h \cdot f_i(x) - f_i \cdot \delta_h(x) + \operatorname{Im} \psi(\delta_h)(x)| \leq \|\delta\psi\|.$$

By putting $f = f_r + if_i$ we obtain

$$|\delta_h \cdot f(x) - f \cdot \delta_h(x) + \psi(\delta_h)(x)| \leq 2\|\delta\psi\|.$$

Now let us define

$$\bar{\psi}(\delta_h)(x) = (\delta f)(\delta_h)(x) + \psi(\delta_h)(x).$$

It is clear that $\bar{\psi} \in \mathcal{C}^1(\ell^1(G), \mathcal{X}')$, $\delta\bar{\psi} = \delta\psi$ and $|\bar{\psi}(\delta_h)(x)| \leq 2\|\delta\psi\| \|x\|$ for every $h \in G$ and $x \in \mathcal{X}$. Thus $\|\bar{\psi}\| \leq 2\|\delta\psi\|$ and this completes the proof. □

3. SECOND COHOMOLOGY GROUP OF LOCALLY COMPACT GROUP ALGEBRAS

Now we state the final result of this paper. We show $\mathcal{H}^2(L^1(G), (L^1(G))^{(2n+1)})$ is a Banach space for every locally compact group G . Note that $L^\infty(G) = (L^1(G))'$ is also an $M(G)$ -module, where for every $f \in L^\infty(G), a \in L^1(G)$ and $\mu \in M(G)$ the module actions are defined by

$$(f\mu)(a) = f(\mu * a) \quad \text{and} \quad (\mu f)(a) = f(a * \mu).$$

Thus as noted in the Introduction the higher duals $(L^1(G))^{(n)}$ are Banach $M(G)$ -bimodules.

Proposition 3.1. *Let \mathcal{A} be a Banach algebra with a bounded approximate identity, and let \mathcal{X} be a Banach \mathcal{A} -bimodule. Let $\psi \in \mathcal{C}^1(\mathcal{A}, \mathcal{X}')$ such that $|\psi(a)(b \cdot x \cdot c)| \leq \|\delta\psi\|$ for every $x \in \mathcal{X}$ with $\|x\| \leq 1$ and $a, b, c \in \mathcal{A}$ with $\|a\| \leq 1, \|b\| \leq 1$ and $\|c\| \leq 1$. Then there exists $\widehat{\psi} \in \mathcal{X}'$ such that*

$$|\psi(a)(x) - \delta\widehat{\psi}(a)(x)| < 5\|\delta\psi\|.$$

Proof. For every $x \in \mathcal{X}$ with $\|x\| \leq 1$ and $a, b \in \mathcal{A}$ with $\|a\| \leq 1$ and $\|b\| \leq 1$, we have

$$(3.1) \quad |\delta\psi(a, b)(x)| = |a \cdot \psi(b)(x) - \psi(ab)(x) + \psi(a) \cdot b(x)| \leq \|\delta\psi\|.$$

Choose a bounded approximate identity $\{e_\alpha\}$ in \mathcal{A} such that the iterated weak*-limit $\widehat{\psi} = \lim_\alpha \lim_\beta (e_\alpha \cdot \psi(e_\beta) - \psi(e_\beta) \cdot e_\alpha)$ exists. Then for every $x \in \mathcal{X}$ and $a \in \mathcal{A}$, by (3.1) we have

$$\begin{aligned} \psi(a)(x) &= \lim_\alpha \lim_\beta [\psi(e_\alpha a e_\beta)(x)] \\ &\leq \lim_\alpha \lim_\beta [\psi(e_\alpha) \cdot a e_\beta(x) + e_\alpha \cdot \psi(a e_\beta)(x)] + \|\delta\psi\| \\ &\leq \lim_\alpha \lim_\beta [\psi(e_\alpha) \cdot a e_\beta(x) + e_\alpha a \cdot \psi(e_\beta)(x) + e_\alpha \cdot \psi(a) \cdot e_\beta(x)] + 2\|\delta\psi\| \\ &\leq \lim_\beta [\psi(e_\beta) \cdot a + a \cdot \psi(e_\beta)](x) + 3\|\delta\psi\| \\ &\leq \lim_\alpha \lim_\beta a \cdot (e_\alpha \cdot \psi(e_\beta) - \psi(e_\beta) \cdot e_\alpha)(x) \\ &\quad - \lim_\alpha \lim_\beta (e_\alpha \cdot \psi(e_\beta) - \psi(e_\beta) \cdot e_\alpha) \cdot a(x) + 5\|\delta\psi\| \\ &= a \cdot \widehat{\psi}(x) - \widehat{\psi} \cdot a(x) + 5\|\delta\psi\|, \end{aligned}$$

where we have used several times the fact that $|\psi(a)(b \cdot x \cdot c)| \leq \|\delta\psi\|$. Then $\psi(a)(x) - \delta\widehat{\psi}(a)(x) \leq 5\|\delta\psi\|$. On the other hand,

$$\begin{aligned} \psi(a)(x) &= \lim_\alpha \lim_\beta [\psi(e_\alpha a e_\beta)(x)] \\ &\geq \lim_\alpha \lim_\beta [\psi(e_\alpha) \cdot a e_\beta(x) + e_\alpha \cdot \psi(a e_\beta)(x)] - \|\delta\psi\| \\ &\geq \lim_\alpha \lim_\beta [\psi(e_\alpha) \cdot a e_\beta(x) + e_\alpha a \cdot \psi(e_\beta)(x)] + 3\|\delta\psi\| \\ &\geq a \cdot \widehat{\psi}(x) - \widehat{\psi} \cdot a(x) - 5\|\delta\psi\|. \end{aligned}$$

Then $\psi(a)(x) - \delta\widehat{\psi}(a)(x) \geq -5\|\delta\psi\|$. Thus

$$|\psi(a)(x) - \delta\widehat{\psi}(a)(x)| \leq 5\|\delta\psi\|.$$

□

For the rest of this section we set $\mathcal{X} = (L^1(G))^{(2n)}$. Observe that, as in Remark 2.1, since the dual space \mathcal{X}' is an L^∞ -space, then the space $\mathcal{X}'_{\mathbb{R}}$ of real-valued functions in \mathcal{X}' forms a complete lattice.

Proposition 3.2. *Let $\psi \in C^1(L^1(G), \mathcal{X}')$. Then $\tilde{\psi} \in C^1(M(G), \mathcal{X}')$ with*

- (i) $\tilde{\psi}|_{L^1(G)} = \psi$ and $\delta\tilde{\psi}|_{L^1(G) \times L^1(G)} = \delta\psi$.
- (ii) *Let μ and ν be in $M(G)$ with $\|\mu\|, \|\nu\| \leq 1$, and let x be in \mathcal{X} with $\|x\| \leq 1$. If $\{\mu_\alpha\}$ is a net in $M(G)$ with $\|\mu_\alpha\| \leq 1$ such that $\text{so-lim } \mu_\alpha = \mu$, then there exists $\hat{\psi} \in \mathcal{X}'$ such that*

$$\left| (\overline{\lim}_\alpha \operatorname{Re} \tilde{\psi}(\mu_\alpha)(x) + i \overline{\lim}_\alpha \operatorname{Im} \tilde{\psi}(\mu_\alpha)(x)) - \tilde{\psi}(\mu)(x) - \delta\hat{\psi}(\mu)(x) \right| \leq 15 \left\| \delta\tilde{\psi} \right\|.$$

Proof. (i) We follow the proof of [9, Lemma 1.10] for this particular case. Let $\mu \in M(G)$ and let $\{e_\alpha\}$ be a bounded approximate identity of norm one for $L^1(G)$. Defining

$$\psi_\alpha(\mu) = \psi(\mu * e_\alpha),$$

we see that $\{\psi_\alpha\}$ is a bounded net in $C^1(M(G), \mathcal{X}')$ and so has a subnet $\{\psi_\beta\}$ convergent to a limit $\tilde{\psi}$ in the weak*-topology induced by identifying $C^1(M(G), \mathcal{X}')$ with $\mathcal{C}_1(M(G), \mathcal{X}')$. Thus

$$\lim_\beta \psi(\mu * e_\beta)(x) = \tilde{\psi}(\mu)(x)$$

for all $\mu \in M(G)$, $x \in \mathcal{X}$. Since for all $a \in L^1(G)$, $\psi(a * e_\beta) \rightarrow \psi(a)$ in norm, then $\tilde{\psi}|_{L^1(G)} = \psi$. Also, $\delta\tilde{\psi}|_{L^1(G) \times L^1(G)} = \delta\psi$.

To prove (ii) let us consider $\mu, \nu \in M(G)$ with $\|\mu\|, \|\nu\| \leq 1$ and $x \in \mathcal{X}$ with $\|x\| \leq 1$. Then

$$(3.2) \quad \left| \delta\tilde{\psi}(\mu, \nu)(x) \right| = \left| \mu \cdot \tilde{\psi}(\nu)(x) - \tilde{\psi}(\mu * \nu)(x) + \tilde{\psi}(\mu) \cdot \nu(x) \right| \leq \left\| \delta\tilde{\psi} \right\|.$$

For $a, b \in L^1(G)$ with $\|a\|_1 \leq 1, \|b\|_1 \leq 1$ and $x \in \mathcal{X}$ with $\|x\| \leq 1$ by (3.2) we have

$$\begin{aligned} -\operatorname{Re} \tilde{\psi}(\mu_\alpha)(a \cdot x \cdot b) &= -\operatorname{Re} \tilde{\psi}(\mu_\alpha) \cdot a(x \cdot b) \\ &\leq \operatorname{Re} \mu_\alpha \cdot \psi(a)(x \cdot b) - \operatorname{Re} \psi(\mu_\alpha * a)(x \cdot b) + \left\| \delta\tilde{\psi} \right\| \end{aligned}$$

and so

$$\begin{aligned} -\overline{\lim} \operatorname{Re} \tilde{\psi}(\mu_\alpha)(a \cdot x \cdot b) &\leq \underline{\lim} \left\{ \operatorname{Re} \mu_\alpha \cdot \psi(a)(x \cdot b) - \operatorname{Re} \psi(\mu_\alpha * a)(x \cdot b) + \left\| \delta\tilde{\psi} \right\| \right\} \\ &= \operatorname{Re} \mu \cdot \psi(a)(x \cdot b) - \operatorname{Re} \psi(\mu * a)(x \cdot b) + \left\| \delta\tilde{\psi} \right\|. \end{aligned}$$

On the other hand,

$$-\overline{\lim} \operatorname{Re} \tilde{\psi}(\mu_\alpha)(a \cdot x \cdot b) \geq \operatorname{Re} \mu \cdot \psi(a)(x \cdot b) - \operatorname{Re} \psi(\mu * a)(x \cdot b) - \left\| \delta\tilde{\psi} \right\|.$$

Hence

$$\left| \mu \cdot \operatorname{Re} \psi(a)(x \cdot b) - \operatorname{Re} \psi(\mu * a)(x \cdot b) + \overline{\lim} \operatorname{Re} \tilde{\psi}(\mu_\alpha)(a \cdot x \cdot b) \right| \leq \left\| \delta\tilde{\psi} \right\|.$$

Similarly for imaginary parts we have

$$\left| \mu \cdot \operatorname{Im} \psi(a)(x \cdot b) - \operatorname{Im} \psi(\mu * a)(x \cdot b) + \overline{\lim} \operatorname{Im} \tilde{\psi}(\mu_\alpha)(a \cdot x \cdot b) \right| \leq \left\| \delta\tilde{\psi} \right\|.$$

Therefore,

$$(3.3) \quad \left| \mu \cdot \psi(a)(x \cdot b) - \psi(\mu * a)(x \cdot b) + \left(\overline{\lim} \operatorname{Re} \tilde{\psi}(\mu_\alpha) - i \overline{\lim} \operatorname{Im} \tilde{\psi}(\mu_\alpha) \right) (a \cdot x \cdot b) \right| \leq 2 \left\| \delta \tilde{\psi} \right\|.$$

But from (3.2) we also have

$$(3.4) \quad \left| \mu \cdot \psi(a)(x \cdot b) - \psi(\mu * a)(x \cdot b) + \tilde{\psi}(\mu)(a \cdot x \cdot b) \right| \leq \left\| \delta \tilde{\psi} \right\|.$$

Hence (3.3) and (3.4) imply that

$$\left| \left(\overline{\lim} \operatorname{Re} \tilde{\psi}(\mu_\alpha) - i \overline{\lim} \operatorname{Im} \tilde{\psi}(\mu_\alpha) \right) (a \cdot x \cdot b) - \tilde{\psi}(\mu)(a \cdot x \cdot b) \right| \leq 3 \left\| \delta \tilde{\psi} \right\|.$$

By Proposition 3.1 there exists $\hat{\psi} \in \mathcal{X}'$ such that

$$\left| \left(\overline{\lim} \operatorname{Re} \tilde{\psi}(\mu_\alpha) - i \overline{\lim} \operatorname{Im} \tilde{\psi}(\mu_\alpha) \right) (x) - \tilde{\psi}(\mu)(x) - \delta \hat{\psi}(\mu)(x) \right| \leq 15 \left\| \delta \tilde{\psi} \right\|.$$

□

Theorem 3.3. *Let G be a locally compact group. Then $\mathcal{H}^2(L^1(G), \mathcal{X}')$ is a Banach space, where $\mathcal{X} = (L^1(G))^{(2n)}$.*

Proof. Let $\phi \in \mathcal{C}^1(L^1(G), \mathcal{X}')$ and consider $\tilde{\phi} \in \mathcal{C}^1(M(G), \mathcal{X}')$ as in Proposition 3.2. Set

$$S = \left\{ \operatorname{Re} \delta_{g^{-1}} \tilde{\phi}(\delta_g) : g \in G \right\}.$$

Since S is bounded above by $\left\| \tilde{\phi} \right\|$ in $\mathcal{X}'_{\mathbb{R}}$, then $\psi_r = \sup_{g \in G} S$ exists in $\mathcal{X}'_{\mathbb{R}}$. For every $h \in G$ and $x \in \mathcal{X}$ with $\|x\| \leq 1$ by (3.2) we have

$$\begin{aligned} \delta_h \cdot \psi_r(x) &= \sup_{k \in G} \left\{ \operatorname{Re}(\delta_h * \delta_{k^{-1}}) \cdot \tilde{\phi}(\delta_k)(x) \right\} = \sup_{g \in G} \left\{ \operatorname{Re} \delta_{g^{-1}} \cdot \tilde{\phi}(\delta_g * \delta_h)(x) \right\} \\ &\leq \sup_{g \in G} \left\{ \operatorname{Re}(\delta_{g^{-1}} * \delta_g) \cdot \tilde{\phi}(\delta_h)(x) + \operatorname{Re} \delta_{g^{-1}} \cdot \tilde{\phi}(\delta_g) \cdot \delta_h(x) + \left\| \delta \tilde{\phi} \right\| \right\} \\ &= \operatorname{Re} \tilde{\phi}(\delta_h)(x) + \psi_r \cdot \delta_h(x) + \left\| \delta \tilde{\phi} \right\|, \end{aligned}$$

where $hk^{-1} = g^{-1}$. On the other hand,

$$\delta_h \cdot \psi_r(x) \geq \operatorname{Re} \tilde{\phi}(\delta_h)(x) + \psi_r \cdot \delta_h(x) - \left\| \delta \tilde{\phi} \right\|.$$

Therefore,

$$(3.5) \quad \left| \delta_h \cdot \psi_r(x) - \psi_r \cdot \delta_h(x) - \operatorname{Re} \tilde{\phi}(\delta_h)(x) \right| \leq \left\| \delta \tilde{\phi} \right\|.$$

Similarly, by considering imaginary parts we obtain ψ_i such that

$$(3.6) \quad \left| \delta_h \cdot \psi_i(x) - \psi_i \cdot \delta_h(x) - \operatorname{Im} \tilde{\phi}(\delta_h)(x) \right| \leq \left\| \delta \tilde{\phi} \right\|.$$

Since every measure μ in $M(G)$ is the so-lim of a net $\{\mu_\alpha\}$ with $\|\mu_\alpha\| \leq 1$, where every μ_α is a linear combination of point masses [4, 1.1.3], then by (3.5), (3.6) and Proposition 3.1 we have

$$\left| \mu \cdot \psi(x) - \psi \cdot \mu(x) - \left[\overline{\lim} \operatorname{Re} \tilde{\phi}(\mu_\alpha)(x) + i \overline{\lim} \operatorname{Im} \tilde{\phi}(\mu_\alpha)(x) \right] \right| \leq 2 \left\| \delta \tilde{\phi} \right\|$$

where $\psi = \psi_r + i\psi_i$. Now by Proposition 3.2 (ii) there exists $\widehat{\phi} \in \mathcal{X}'$ such that

$$\begin{aligned} & \left| \mu \cdot \psi(x) - \psi \cdot \mu(x) - \delta\widehat{\phi}(\mu)(x) - \widetilde{\phi}(\mu)(x) \right| \\ & \leq \left| \mu \cdot \psi(x) - \psi \cdot \mu(x) - \left[\overline{\lim} \operatorname{Re} \widetilde{\phi}(\mu_\alpha)(x) + i \overline{\lim} \operatorname{Im} \widetilde{\phi}(\mu_\alpha)(x) \right] \right| \\ & \quad + \left| \overline{\lim} \operatorname{Re} \widetilde{\phi}(\mu_\alpha)(x) + i \overline{\lim} \operatorname{Im} \widetilde{\phi}(\mu_\alpha)(x) - \delta\widehat{\phi}(\mu)(x) - \widetilde{\phi}(\mu)(x) \right| \\ & \leq 17 \left\| \delta\widetilde{\phi} \right\|. \end{aligned}$$

Define

$$\bar{\psi}(\mu)(x) = -\delta\psi(\mu)(x) - \delta\widehat{\phi}(\mu)(x) + \widetilde{\phi}(\mu)(x).$$

Then $\delta\bar{\psi} = \delta\widetilde{\phi}$ and $|\bar{\psi}(\mu)(x)| \leq 17 \left\| \delta\widetilde{\phi} \right\|$ for every $\mu \in M(G)$ with $\|\mu\| \leq 1$ and $x \in \mathcal{X}$ with $\|x\| \leq 1$. So $\|\bar{\psi}\| \leq 17 \left\| \delta\widetilde{\phi} \right\|$ and by Proposition 3.2 (i), this completes the proof. \square

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