

ON A SPECTRAL PROPERTY OF JACOBI MATRICES

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ABSTRACT. Let J be a Jacobi matrix with elements b_k on the main diagonal and elements a_k on the auxiliary ones. We suppose that J is a compact perturbation of the free Jacobi matrix. In this case the essential spectrum of J coincides with $[-2, 2]$, and its discrete spectrum is a union of two sequences $\{x_j^\pm\}$, $x_j^+ > 2, x_j^- < -2$, tending to ± 2 . We denote sequences $\{a_{k+1} - a_k\}$ and $\{a_{k+1} + a_{k-1} - 2a_k\}$ by ∂a and $\partial^2 a$, respectively.

The main result of the note is the following theorem.

Theorem. Let J be a Jacobi matrix described above and σ be its spectral measure. Then $a - 1, b \in l^4$, $\partial^2 a, \partial^2 b \in l^2$ if and only if

$$\text{i) } \int_{-2}^2 \log \sigma'(x)(4 - x^2)^{5/2} dx > -\infty, \quad \text{ii) } \sum_j (x_j^\pm \mp 2)^{7/2} < \infty.$$

INTRODUCTION

It has been observed recently [4] that the sum rules [1], [2] for a Jacobi matrix J lead to direct relations between elements of J and its spectral properties. However, the sum rules become considerably more complicated when their order increases. In this note, we carry out a complete analysis of a sum rule of order six.

Let us consider a Jacobi matrix

$$J = J(a, b) = \begin{bmatrix} b_0 & a_0 & 0 \\ a_0 & b_1 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix},$$

defined by two sequences $a = \{a_k\}$, $a_k > 0$, and $b = \{b_k\}$, $b_k \in \mathbb{R}$. We suppose that J is a compact perturbation of the so-called free Jacobi matrix J_0 ,

$$J_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix}.$$

The (scalar) spectral measure σ of J is defined by the relation

$$((J - z)^{-1} e_0, e_0) = \int_{\mathbb{R}} \frac{d\sigma(x)}{x - z},$$

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with $z \in \mathbb{C} \setminus \mathbb{R}$. It is well known that in this case the continuous part σ_c of the measure is supported by $[-2, 2]$, and its discrete part σ_d lives on the union of two sequences $\{x_j^\pm\}$ such that $x_j^+ > 2$, $x_j^- < -2$, and $x_j^\pm \rightarrow \pm 2$, as $j \rightarrow \infty$.

Let us put $\partial a = \{a_{k+1} - a_k\}$ and $\partial^2 a = \{a_{k+1} + a_{k-1} - 2a_k\}$, and denote by 1 the sequence consisting of ones. The following theorem holds.

Theorem 0.1. *Let $J = J(a, b)$ and σ be its spectral measure. Then*

$$(0.1) \quad a - 1, b \in l^4, \partial^2 a, \partial^2 b \in l^2$$

if and only if

$$(0.2) \quad \text{i) } \int_{-2}^2 \log \sigma'(x)(4 - x^2)^{5/2} dx > -\infty, \quad \text{ii) } \sum_j (x_j^\pm \mp 2)^{7/2} < \infty.$$

The theorem has an immediate corollary.

Corollary 0.2. *Let $J = J(a, b)$, and let σ be its spectral measure. If $a - 1, b \in l^4$, $\partial a, \partial b \in l^2$, then relations (0.2) hold.*

In this direction we also have (see [7]) that

$$\text{i) } \int_{-2}^2 \log \sigma'(x)(4 - x^2)^{3/2} dx > -\infty, \quad \text{ii) } \sum_j (x_j^\pm \mp 2)^{5/2} < \infty,$$

when $a - 1, b \in l^3$, $\partial a, \partial b \in l^2$. Together with Corollary 0.2, this observation leads to a natural conjecture. The conjecture would yield a right discrete counterpart of results, obtained in [6] for Schrödinger operators on the half-line.

Conjecture 0.3. For an integer $k \geq 1$, conditions $a - 1, b \in l^{k+1}$, $\partial a, \partial b \in l^2$ imply

$$\text{i) } \int_{-2}^2 \log \sigma'(x)(4 - x^2)^{k-1/2} dx > -\infty, \quad \text{ii) } \sum_j (x_j^\pm \mp 2)^{k+1/2} < \infty.$$

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1. PROOF OF THEOREM 0.1

It is convenient to map the domain $\mathbb{C} \setminus [-2, 2]$ onto the unit disk with the help of the transformation $\zeta(z) = \frac{1}{2}(z - \sqrt{z^2 - 4})$. We construct a measure μ on the unit circle by letting $d\mu(\theta) = d\sigma(x)/(2 \sin \theta)$, where $x = z(e^{i\theta})$ and $z(\zeta) = \zeta + 1/\zeta$. The sequences $\{x_j^\pm\}$ are mapped to points $\{\zeta_j\}$, lying on the real diameter of the unit disk. In these new terms Theorem 0.1 says that $a - 1, b \in l^4$, $\partial^2 a, \partial^2 b \in l^2$ if and only if

$$(1.1) \quad \text{i) } \int_0^{2\pi} \log \mu'(\theta) \sin^6 \theta d\theta > -\infty, \quad \text{ii) } \sum_j (1 - |\zeta_j|)^7 < \infty.$$

Let us assume first that $\text{rank}(J - J_0) < \infty$. For $j \geq 1$, we define Chebyshev polynomials by recurrence relations $T_{j+1}(z) = zT_j(z) - T_{j-1}(z)$, and we take

$T_0(z) = 2, T_1(z) = z$. We have the so-called sum rules [1], [2], [4], [8], [9]

$$\begin{aligned}
 0) \quad & \frac{1}{4\pi} \int_0^{2\pi} \log \frac{2 \sin \theta}{\mu'(\theta)} d\theta = \sum_j \log \beta_j - \sum_j \log a_j, \\
 n) \quad & \frac{1}{2\pi} \int_0^{2\pi} \log \frac{2 \sin \theta}{\mu'(\theta)} \cos n\theta d\theta = \frac{1}{n} \sum_j (\beta_j^n - \beta_j^{-n}) - \frac{1}{n} \operatorname{tr} \{T_n(J) - T_n(J_0)\},
 \end{aligned}$$

where we let $\beta_j = 1/|\zeta_j|$. Taking into account that $\sin^6 \theta = (1/2^5)(10 - 15 \cos 2\theta + 6 \cos 4\theta - \cos 6\theta)$, we get

$$\frac{1}{2\pi} \int_0^{2\pi} \log \frac{2 \sin \theta}{\mu'(\theta)} \sin^6 \theta d\theta + \frac{1}{6} \sum_j F(\beta_j) = \frac{1}{6} \operatorname{tr} \{G(J) - G(J_0) - 120 \log A\},$$

where $A = \operatorname{diag} \{a_k\}$, and

$$\begin{aligned}
 F(\beta) &= \frac{1}{2^5} ((\beta^6 - \beta^{-6}) - 9(\beta^4 - \beta^{-4}) + 45(\beta^2 - \beta^{-2}) - 120 \log \beta), \\
 (1.2) \quad G(J) &= \frac{1}{2^5} (T_6(J) - 9T_4(J) + 45T_2(J)) = \frac{1}{2^5} (J^6 - 15J^4 + 90J^2 - 110).
 \end{aligned}$$

Notice that $F(\beta) = C_0(\beta - 1)^7 + O((\beta - 1)^8)$, where $\beta \geq 1$ and C_0 is a positive constant.

Let

$$(1.3) \quad \Psi(a, b) = \Psi(J(a, b)) = \operatorname{tr} \{G(J) - G(J_0) - 120 \log A\}.$$

The following lemma is a cornerstone of the proof of the theorem.

Main Lemma. *Let $a = \{a_k\}$, $b = \{b_k\}$ and $a_k \neq 1, b_k \neq 0$, for finitely many indices only. Also let norms $\|a - 1\|_\infty, \|b\|_\infty$ be small enough. Then*

$$(1.4) \quad \Psi(a, b) \asymp (\|a - 1\|_4^4 + \|b\|_4^4 + \|\partial^2 a\|_2^2 + \|\partial^2 b\|_2^2).$$

In particular, $\Psi(a, b) \geq 0$ for these a and b .

The norms $\|\cdot\|_p$ refer to the standard l^p -space norms. The sign “ \asymp ” means a two-sided estimate with positive constants depending on $\|a - 1\|_\infty, \|b\|_\infty$. The lemma will be proved in the next section.

With the exception of this lemma, the proof of Theorem 0.1 follows a well-known scheme [4], [5]. We give only its sketch.

Proof of Theorem 0.1. Since the absolutely continuous spectrum of $J = J(a, b)$ is the interval $[-2, 2]$, we have (see [3]) that $a_k \rightarrow 1$ and $b_k \rightarrow 0$. Consequently, discarding the first N_0 rows and columns of J (see [8], Section 3), we may make norms $\|a - 1\|_\infty$ and $\|b\|_\infty$ small enough to apply the Main Lemma.

First, we show that condition (0.1) implies (1.1), which is equivalent to showing (0.2). Denote

$$\Phi(J) = \Phi(\mu) = \Phi_1(\mu) + \Phi_2(\mu) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{2 \sin \theta}{\mu'(\theta)} \sin^6 \theta d\theta + \frac{1}{6} \sum_j F\left(\frac{1}{|\zeta_j|}\right).$$

Let $a_N = \{(a_N)_k\}$ and $a'_N = \{(a'_N)_k\}$, where

$$(a_N)_k = \begin{cases} a_k, & k \leq N, \\ 1, & k > N, \end{cases} \quad (a'_N)_k = \begin{cases} 1, & k \leq N, \\ a_k, & k > N. \end{cases}$$

We also consider b_N, b'_N , constructed in the same way (of course, with 1's replaced by 0's). Let $J_N = J(a_N, b_N)$ and μ_N be the corresponding measure on the unit circle. Obviously, we have that $a'_N - 1, b'_N \rightarrow 0$ and $\partial^2 a'_N, \partial^2 b'_N \rightarrow 0$ in corresponding norms. Since

$$\Psi(a, b) = \sum_k \psi(a_k, a_{k+1}, a_{k+2}; b_k, b_{k+1}, b_{k+2})$$

for a function ψ (see Section 2), we have that

$|\Psi(J) - \Psi(J_N)| \leq \Psi(a'_N, b'_N) \leq C_1(\|a'_N - 1\|_4^4 + \|b'_N\|_4^4 + \|\partial^2 a'_N\|_2^2 + \|\partial^2 b'_N\|_2^2) \rightarrow 0$, or $\Psi(J_N) \rightarrow \Psi(J)$. On the other hand, $(J_N - z(\zeta))^{-1} \rightarrow (J - z(\zeta))^{-1}$, for ζ inside the unit disk and, consequently, $\mu_N \rightarrow \mu$ weakly. Using [4], Corollary 5.3 and Theorem 6.2, we get

$$\Phi_1(\mu) \leq \liminf_N \Phi_1(\mu_N)$$

and

$$\lim_{N \rightarrow \infty} \Phi_2(\mu_N) = \Phi_2(\mu).$$

Summing up, we obtain

$$\Phi(\mu) \leq \limsup_N \Phi(\mu_N) = \limsup_N \Psi(J_N) = \lim_{N \rightarrow \infty} \Psi(J_N) = \Psi(J).$$

We now show the converse, i.e., condition (1.1) yields relations (0.1). Recall ([4], Proposition 8.4) that if $\mu = \mu(J)$ satisfies inequality $\mu \geq \delta m$, then

$$\Psi(J) \leq \Phi(J),$$

where $\delta > 0$ and m is the Lebesgue measure on the unit circle. We take a J with the property $\Psi(J) < \infty$, and we put $\mu_\delta = (1 - \delta)\mu + \delta m$ for $\mu = \mu(J)$ and $\delta \in (0, 1)$. The measure μ_δ defines a Jacobi matrix $J_\delta = J(a_\delta, b_\delta)$. Theorem 8.1 of [4] shows that $\Phi(J) = \lim_{\delta \rightarrow 0} \Phi(J_\delta)$. On the other hand (see [4], Proposition 8.4 and Theorem 8.1),

$$\Psi(J) \leq \liminf_\delta \Psi(J_\delta).$$

Hence,

$$0 \leq \Psi(J) \leq \liminf_\delta \Psi(J_\delta) \leq \liminf_\delta \Phi(J_\delta) = \lim_{\delta \rightarrow 0} \Phi(J_\delta) = \Phi(J).$$

The theorem is proved. □

2. PROOF OF THE MAIN LEMMA

We make a few simplifications before going into the proof. First, we drop the term $\text{tr } G(J_0)$ in (1.3), since it is finite. Second, we discard factors $1/2^5$ (see (1.2)). Having two sequences $a = \{a_k\}$ and $b = \{b_k\}$, we denote the sequence $\{a_k b_k\}$ by ab . We also write a^\pm for the sequence $\{a_{k \pm 1}\}$.

The starting point of the proof is the computation of $\text{tr } J^n$. According to formulas from [9], Section 6.1, we have $\text{tr } J^j = \sum_k (g_j)_k$, where sequences g_j are constructed as

$$(2.1) \quad \begin{aligned} g_{j+1} &= h_j + h_j^- + b g_j, \\ h_{j+1} &= a^2 \sum_{l=0}^j g_{j-l} g_l^- - \sum_{l=0}^j h_{j-l} h_l, \end{aligned}$$

with $g_0 = 1, h_0 = 0$.

We give the proof of the lemma in cases when $J = J(a, b)$ with $b = 0$, and $J = J(a, b)$ with $a = 1$. Since the proof in the general case follows along the same lines and is long and tedious, we omit it.

Let us consider the case when $b = 0$.

Lemma 2.1. *Let $a = \{a_k\}$ and $a_k \neq 1$ for finitely many indices. Then*

$$\begin{aligned} \Psi(a) = \Psi(J(a, 0)) &= \sum_k \left\{ -9\lambda_k^3 + 3\lambda_k^2\lambda_{k+1} + 3\lambda\lambda_{k+1}^2 + 3\lambda_{k-1}\lambda_k\lambda_{k+1} \right. \\ &\quad \left. + 9\lambda_k^2 - 12\lambda\lambda_{k+1} + 3\lambda_{k-1}\lambda_{k+1} + \frac{15}{2}\lambda_k^4 + O(|\lambda_k|^5) \right\}, \end{aligned}$$

where $\lambda = a^2 - 1$.

Proof. A straightforward computation using (2.1) yields $g_1 = g_3 = g_5 = 0$ and

$$\begin{aligned} g_2 &= a^2 + a^{-2}, \\ g_4 &= a^{-2}(a^{-2} + a^{-2} + a^2) + a^2(a^{-2} + a^2 + a^{+2}), \\ g_6 &= a^2(a^{-2}(a^{-2} + \dots + a^{+2}) + a^2(a^{-2} + \dots + a^{+2})) \\ &\quad + a^{+2}(a^2 + \dots + a^{++2}) \\ &\quad + a^{-2}(a^{-2}(a^{---2} \\ &\quad + \dots + a^2) + a^{-2}(a^{-2} + \dots + a^2) + a^2(a^{-2} + \dots + a^{+2})). \end{aligned}$$

We also have $\text{tr} \log A = \frac{1}{2} \sum_k \log(1 + \lambda_k)$. Rewriting in terms of $\lambda = a^2 - 1$ and recalling that $\text{tr} J^j = \sum_k (g_j)_k$, we get the conclusion of the lemma. \square

In particular, we see that

$$\Psi(a) = \sum_k \left\{ \psi_1(\lambda_k, \lambda_{k+1}, \lambda_{k+2}) + O(|\lambda_k|^5) \right\},$$

where

$$\begin{aligned} \psi_1(x, y, z) &= \frac{1}{2}(x^3 - 20y^3 + z^3 + 6xy^2 + 6y^2z + 6xyz) \\ &\quad + \frac{3}{2}(x^2 + 4y^2 + z^2 - 4xy - 4yz + 2xz) + \frac{5}{4}(x^4 + 4y^4 + z^4). \end{aligned}$$

Note also that $\lambda_k = 2(a_k - 1) + O((a_k - 1)^2)$.

Lemma 2.2. *We have*

$$\psi_1(x, y, z) \asymp (x^4 + y^4 + z^4 + (x + z - 2y)^2).$$

Proof. We notice that

$$\begin{aligned} (2.2) \quad \psi_1(x, y, z) &= \frac{1}{2}((x + z - 2y)(x^2 + 10y^2 + z^2 + 2xy + 2yz - zx) \\ &\quad + 3(x + z - 2y)^2 + \frac{5}{2}(x^4 + 4y^4 + z^4)), \end{aligned}$$

and the bound from above follows from the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ for $a, b \geq 0$. To get the bound from below, we estimate the first term in (2.2) as

$$|(x + z - 2y)(x^2 + 10y^2 + z^2 + 2xy + 2yz - zx)| \leq \frac{1}{2}(x + z - 2y)^2 + \frac{3}{2}(x^4 + 4y^4 + z^4),$$

and, consequently,

$$\psi_1(x, y, z) \geq \frac{1}{2} \left(\frac{5}{2}(x + z - 2y)^2 + (x^4 + 4y^4 + z^4) \right).$$

The lemma is proved. \square

Now, we turn to the second case $J = J(a, b)$ with $a = 1$.

Lemma 2.3. *Let $b = \{b_k\}$ and $b_k \neq 0$ for finitely many indices. Then*

$$\begin{aligned} \Psi(b) = \Psi(1, b) &= \sum_k \{b_k^6 + (-3b_k^4 + 6b_k b_{k+1}^3 + 6b_k^3 b_{k+1} + 6b_k^2 b_{k+1}^2) \\ &\quad + (18b_k^2 - 24b_k b_{k+1} + 6b_{k-1} b_{k+1})\}. \end{aligned}$$

Proof. The argument is exactly as in Lemma 2.1. We only quote expressions for g_5 and g_6 :

$$\begin{aligned} g_5 &= b^5 + (b^{-3} + 8b^3 + b^{+3} + 2b^{-2}b + 3b^{-b^2} + 3b^2b^+ + 2bb^{+2}) \\ &\quad + (b^{--} + 6b^- + 16b + 6b^+ + b^{++}), \\ g_6 &= b^6 + (b^{-4} + 10b^4 + b^{+4} \\ &\quad + 2b^{-3}b + 3b^{-2}b^2 + 4b^{-b^3} + 4b^3b^+ + 3b^2b^{+2} + 2bb^{+3}) \\ &\quad + (b^{--2} + 8b^{-2} + 30b^2 + 8b^{+2} + b^{++2} + 2b^{--b^-} + 2b^{--b} \\ &\quad + 16b^-b + 16bb^+ + 2bb^{++} + 2b^+b^{++} + 2b^+b^-) + 20. \end{aligned}$$

□

Hence, we obtain

$$\Psi(b) = \sum_k \{\psi_2(b_k, b_{k+1}, b_{k+2}) + O(b_k^6)\},$$

where

$$\psi_2(x, y, z) = -3y^4 + (6xy^3 + 6y^3z + 3y^2x^2 + 3y^2z^2) + 3(x + z - 2y)^2.$$

Lemma 2.4. *We have*

$$\psi_2(x, y, z) \asymp (y^4 + (x + z - 2y)^2).$$

Proof. One more time, the bound from above follows at once from inequality $ab \leq 1/(p a^p) + 1/(q b^q)$, where $a, b \geq 0$, and $1/p + 1/q = 1$.

Furthermore, we see that

$$\psi_2(x, y, z) = -3y^4 + y^2 \left(\frac{3}{2}(x - z)^2 + 3(x + y + z)^2 - \frac{1}{2}(x + z - 2y)^2 \right) + 3(x + z - 2y)^2.$$

Estimating moduli of negative terms as in Lemma 2.2 and taking into account

$$-3y^4 + 3y^2(x + y + z)^2 - \frac{1}{2}y^2(x + z - 2y)^2 + 3(x + z - 2y)^2 \geq \frac{1}{2}(y^4 + (x + z - 2y)^2),$$

finishes the proof. □

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