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POLAROID OPERATORS AND WEYL'S THEOREM

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ABSTRACT. "Polaroid elements" represent an attempt to abstract part of the condition, "Weyl's theorem holds" for operators.

In the early days of operator theory Hermann Weyl [12] made an interesting observation about selfadjoint operators: "Weyl's theorem" says that, whenever $T = T^* \in B(X)$ for a Hilbert space X, we have the equality

$$\sigma(T) \setminus \omega_{ess}(T) = \pi_0^{left}(T) \ . \tag{0.1}$$

Here $\sigma(T)$ is the usual spectrum of the operator T, collecting complex numbers λ for which $T - \lambda I$ does not have an inverse; the "Weyl spectrum" $\omega_{ess}(T)$ consists of those $\lambda \in \mathbf{C}$ for which $T - \lambda I$ fails to be Fredholm of index zero, and $\pi_0^{left}(T)$ denotes the isolated points of the spectrum that are eigenvalues of finite multiplicity. In a curious choice of words, (0.1) is usually described as saying that "Weyl's theorem holds" for T: this has been extended to normal operators, both on Hilbert space and on Banach space, and to hyponormal, semi-hyponormal and p-hyponormal operators on Hilbert space.

In a felicitous misreading, the second author replaced (0.1) by

$$\sigma(T) \subseteq \omega_{ess}(T) \cup \pi_0^{left}(T) \ ; \tag{0.2}$$

in homage to the terminology above, this could be described [8] by saying "Browder's theorem holds" for T. The permanence properties of (0.2) are ([8], [9]) somewhat better than those of (0.1), but it of course misses the disjointness

$$\omega_{ess}(T) \cap \pi_0^{left}(T) = \emptyset \ . \tag{0.3}$$

In the present note we wish to explore a variant of this disjointness condition; we begin with an excursion into Banach algebras.

Suppose A is a semigroup, with identity 1 and invertible group A^{-1}, or more generally an abstract category: then we call $a \in A$ *simply polar* iff ([5]; [6], Definition 7.5.2) it has a commuting generalized inverse $a' \in A$ for which

$$a = aa'a \text{ with } aa' = a'a \ . \tag{0.4}$$

This is a strong condition: for example, when A is a ring and $a' \in A$ satisfies (0.4), then also

$$a'' = a'aa' + (1 - a'a) \tag{0.5}$$

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is an invertible generalized inverse for $a \in A$. More generally, $a \in A$ is said to be *polar* iff there is $n \in \mathbf{N}$ for which

$$a^n \text{ is simply polar ;} \tag{0.6}$$

more generally still, if A is a complex Banach algebra we call $a \in A$ *quasipolar* iff there is $p \in A$ for which

$$p = p^2 \ , \ ap = pa \ , \ p \in aA \cap Aa \text{ and } \|a^n(1-p)\|^{\frac{1}{n}} \to 0 \text{ as } n \to \infty \ . \tag{0.7}$$

Thus $ap \in pAp$ is invertible (not in A) while $a(1-p)$ is quasinilpotent. Equivalently, with $q = 1 - p$, we have [11]

$$q = q^2 \ , \ aq = qa \ , \ a + q \in A^{-1} \text{ and } \|a^n q\|^{\frac{1}{n}} \to 0 \text{ as } n \to \infty \ . \tag{0.8}$$

To attempt to do this in more general rings we need ([7], [11]) an algebraic version of "quasinilpotent". It is familiar ([5]; [6], Theorem 7.5.3) that if $a \in A$ is quasipolar, then the projection $p = a^\bullet$ of (0.7) is unique and double commutes with a, as is the relative inverse $a^\times \in A$ for which

$$a^\bullet = a^\times a = a a^\times \text{ and } a^\times = a^\times a^\bullet = a^\bullet a^\times \ . \tag{0.9}$$

We shall call the projection $a^\bullet$ the *support* of $a \in A$ and—a slight abuse of language [11]—the relative inverse $a^\times$ the *Drazin inverse*. We can very slightly improve the double commutivity:

1. Lemma. *If $a \in A$ and $b \in A$ are quasipolar, and if $v \in A$ satisfies*

$$bv = va \ , \tag{1.1}$$

then also

$$b^\bullet v = va^\bullet \textit{ and } b^\times v = va^\times \ . \tag{1.2}$$

Proof. If $p = a^\bullet$ and $q = b^\bullet$, then we claim

$$qv = qvp = vp \ : \tag{1.3}$$

for if $n \in \mathbf{N}$ is arbitrary,

$$qv - qvp = qv(1-p) = b^{\times n} b^n v(1-p) = b^{\times n} v a^n (1-p) \to 0 \ ,$$

and similarly for the second equality. Also,

$$b^\times v = b^\times qv = b^\times vp = b^\times vaa^\times = b^\times bva^\times = qva^\times = vpa^\times = va^\times .$$

□

A necessary and sufficient condition for $a \in A$ to be quasipolar in a Banach algebra is that $0 \in \mathbf{C}$ is not an accumulation point of the spectrum $\sigma(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}\}$:

$$0 \notin \operatorname{acc} \sigma(a) \ . \tag{1.4}$$

When (1.4) holds, then the support $a^\bullet$ and the Drazin inverse $a^\times$ are given by familiar Cauchy integrals.

Spectral inclusion is incorporated in the following "quasi-affine comparison" of elements: we shall write

$$a \prec_{left} b \tag{1.5}$$

to mean that there is $v \in A$ for which

$$\big(vx = 0 \Longrightarrow x = 0\big) \ , \ bv = va \text{ and } \sigma(b) \subseteq \sigma(a) \ . \tag{1.6}$$

This in turn interacts with our idea of a "polaroid" element:

2. Definition. We shall call the element $a \in A$ polaroid iff there is an implication, for arbitrary $\lambda \in \mathbf{C}$,

$$a - \lambda \text{ quasipolar} \implies a - \lambda \text{ polar} , \tag{2.1}$$

and simply polaroid if the implication is

$$a - \lambda \text{ quasipolar} \implies a - \lambda \text{ simply polar} . \tag{2.2}$$

For example, the argument of Stampfli ([8], Theorem 14) says that for $a = T \in B(X)$,

$$\text{normaloid} \implies \text{simply polaroid} \implies \text{reguloid} . \tag{2.3}$$

The first implication holds ([10], Lemma 2.5) if more generally the operator "satisfies the growth condition G_m", while the second is trivial. The "Property B" of Djordjevic, Jeon and Ko ([2], (4), page 324) implies the polaroid condition (2.2).

The quasi-affine comparison of (1.5) transmits polaroid and simple polaroid properties:

3. Theorem. *If $a \in A$ and $b \in A$, then*

$$a \prec_{left} b \textit{ polaroid} \implies a \textit{ polaroid} , \tag{3.1}$$

and

$$a \prec_{left} b \textit{ simply polaroid} \implies a \textit{ simply polaroid} . \tag{3.2}$$

Proof. Begin by checking that if $a \prec_{left} b$, then

$$\text{iso } \sigma(a) \subseteq \text{iso } \sigma(b) , \tag{3.3}$$

so that

$$a \prec_{left} b , \ a \text{ quasipolar} \implies b \text{ quasipolar} . \tag{3.4}$$

□

The joint spectrum argument ([10], Lemma 2.5) is taken from Fialkow ([3], Theorem 2.5): if $\lambda \in \text{iso } \sigma(a)$ then, with $p = (a - \lambda)^\bullet$ and $c = (a - \lambda)(1 - p)$,

$$v(1 - p)c = (b - \lambda)v(1 - p),$$

and hence

$$0 \neq v(1 - p) \in (R_c - L_{b-\lambda})^{-1}(0) .$$

Thus ([6], Theorem 11.6.2)

$$0 \in \sigma(R_c - L_{b-\lambda}) \subseteq \sigma(c) - \sigma(b - \lambda),$$

giving

$$\{0\} = \sigma(c) \subseteq \sigma(b) - \lambda .$$

Thus iso $\sigma(a) \subseteq \sigma(b) \subseteq \sigma(a)$, giving (3.3).

Towards (3.1) we can now argue, with $c = (a - \lambda)^n$ and $d = (b - \lambda)^n$,

$$v(c - cc^\times c) = (d - dd^\times d)v = 0 \implies c = cc^\times c ; \tag{3.5}$$

then, in particular, (3.2) is the case $n = 1$. □

For operators it is easy to see ([8], Theorem 9; [9], Theorem 2) that "Browder's theorem holds" for T in the sense of (0.2) if and only if

$$\text{acc } \sigma(T) \subseteq \omega_{ess}(T) ; \tag{3.6}$$

the polaroid condition is close to the reverse inclusion:

4. Theorem. *If $T \in B(X)$ for a Banach space X, then the inclusion*

$$\omega_{ess}(T) \subseteq \text{acc } \sigma(T) \tag{4.1}$$

is sufficient for the polaroid condition (2.1)*, which in turn is sufficient for the disjointness* (0.3)*. If T is polaroid and if $T - \lambda I$ has for arbitrary $\lambda \in \mathbf{C}$ either finite ascent or finite descent, then Weyl's theorem holds for T.*

Proof. If (4.1) holds, then if $T - \lambda I$ is quasipolar we have

$$\lambda \in \text{iso } \sigma(T) \Longrightarrow \lambda \in \text{iso } \sigma(T) \setminus \omega_{ess}(T) ,$$

which ([6], Theorem 9.8.4) by the punctured neighbourhood theorem makes it a "Riesz point" for T, so that $T - \lambda I$ is polar. Conversely, if T is polaroid and $\lambda \in \pi_0^{left}(T)$, then $T - \lambda I$ is polar with $0 < \dim\, (T - \lambda I)^{-1}(0) < \infty$. By the punctured neighbourhood theorem again this gives also $\dim X/(T-\lambda I)X = \dim\, (T-\lambda I)^{-1}(0)$, excluding λ from $\omega_{ess}(T)$. For the last part, if an operator T is Fredholm of index zero, then finite ascent and finite descent are equivalent. □

Neither of the implications in the first part of Theorem 4 is reversible: for example, Weyl's theorem holds for the Volterra operator $x(t) \mapsto \int_{s=0}^{t} x(s)ds$ on $C[0,1]$ while (4.1) fails. If (0.3) holds for T and fails for the dual operator $T^\dagger$, then $T \in B(X)$ is not polaroid: for a specific example, take $T = UW$ to be the product of the standard weight $W : (x_n) \mapsto (\frac{1}{n}x_n)$ and the forward shift U. For Weyl operators the ascent/descent condition is equivalent to the "single valued extension property" of Finch [4].

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