POLAROID OPERATORS AND WEYL'S THEOREM

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Abstract. “Polaroid elements” represent an attempt to abstract part of the condition, “Weyl’s theorem holds” for operators.

In the early days of operator theory Hermann Weyl [12] made an interesting observation about selfadjoint operators: “Weyl’s theorem” says that, whenever \( T = T^* \in B(X) \) for a Hilbert space \( X \), we have the equality

\[
\sigma(T) \setminus \omega_{ess}(T) = \pi_{0}^{left}(T).
\]

Here \( \sigma(T) \) is the usual spectrum of the operator \( T \), collecting complex numbers \( \lambda \) for which \( T - \lambda I \) does not have an inverse; the “Weyl spectrum” \( \omega_{ess}(T) \) consists of those \( \lambda \in \mathbb{C} \) for which \( T - \lambda I \) fails to be Fredholm of index zero, and \( \pi_{0}^{left}(T) \) denotes the isolated points of the spectrum that are eigenvalues of finite multiplicity. In a curious choice of words, (0.1) is usually described as saying that “Weyl’s theorem holds” for \( T \): this has been extended to normal operators, both on Hilbert space and on Banach space, and to hyponormal, semi-hyponormal and \( p \)-hyponormal operators on Hilbert space.

In a felicitous misreading, the second author replaced (0.1) by

\[
\sigma(T) \subseteq \omega_{ess}(T) \cup \pi_{0}^{left}(T);
\]

in homage to the terminology above, this could be described [8] by saying “Browder’s theorem holds” for \( T \). The permanence properties of (0.2) are ([8], [9]) somewhat better than those of (0.1), but it of course misses the disjointness

\[
\omega_{ess}(T) \cap \pi_{0}^{left}(T) = \emptyset.
\]

In the present note we wish to explore a variant of this disjointness condition; we begin with an excursion into Banach algebras.

Suppose \( A \) is a semigroup, with identity 1 and invertible group \( A^{-1} \), or more generally an abstract category: then we call \( a \in A \) simply polar [3]: [6], Definition 7.5.2) it has a commuting generalized inverse \( a' \in A \) for which

\[
a = aa'a \text{ with } aa' = a'.
\]

This is a strong condition: for example, when \( A \) is a ring and \( a' \in A \) satisfies (0.4), then also

\[
a'' = a'aa' + (1 - a'a)
\]
is an invertible generalized inverse for \( a \in A \). More generally, \( a \in A \) is said to be polar iff there is \( n \in \mathbb{N} \) for which
\[
\tag{0.6} a^n \text{ is simply polar ;}
\]
more generally still, if \( A \) is a complex Banach algebra we call \( a \in A \) quasipolar iff there is \( p \in A \) for which
\[
\tag{0.7} p = p^2 \, , \quad ap = pa \, , \quad p \in aA \cap Aa \text{ and } \|a^n(1 - p)\|^{1/n} \to 0 \quad \text{as } n \to \infty .
\]
Thus \( ap \in pAp \) is invertible (not in \( A \)) while \( a(1 - p) \) is quasinilpotent. Equivalently, with \( q = 1 - p \), we have
\[
\tag{0.8} q = q^2 \, , \quad aq = qa \, , \quad a + q \in A^{-1} \text{ and } \|a^nq\|^{1/n} \to 0 \quad \text{as } n \to \infty .
\]
To attempt to do this in more general rings we need (\[7\], \[11\]) an algebraic version of “quasinilpotent”. It is familiar (\[5\]; \[6\], Theorem 7.5.3) that if \( a \in A \) is quasipolar, then the projection \( p = a^\bullet \) of (0.7) is unique and double commutes with \( a \), as is the relative inverse \( a^\times \in A \) for which
\[
\tag{0.9} a^\bullet = a^\times a = aa^\times \text{ and } a^\times = a^\times a^\bullet = a^\bullet a^\times .
\]
We shall call the projection \( a^\bullet \) the support of \( a \) and—a slight abuse of language \[11\]—the relative inverse \( a^\times \) the Drazin inverse. We can very slightly improve the double commutivity:

1. **Lemma.** If \( a \in A \) and \( b \in A \) are quasipolar, and if \( v \in A \) satisfies
\[
\tag{1.1} bv = va ,
\]
then also
\[
\tag{1.2} b^\bullet v = va^\bullet \text{ and } b^\times v = va^\times .
\]

**Proof.** If \( p = a^\bullet \) and \( q = b^\bullet \), then we claim
\[
\tag{1.3} qv = qvp = vp ;
\]
for if \( n \in \mathbb{N} \) is arbitrary,
\[
qv - qvp = qv(1 - p) = b^\times a^\times v(1 - p) = b^\times a^\times va^n(1 - p) \to 0 ,
\]
and similarly for the second equality. Also,
\[
b^\times v = b^\times qv = b^\times vp = b^\times va^\times = b^\times bva^\times = a^\times vpa^\times = vpa^\times = va^\times .
\]

A necessary and sufficient condition for \( a \in A \) to be quasipolar in a Banach algebra is that \( 0 \notin \mathbb{C} \) is not an accumulation point of the spectrum \( \sigma (a) = \{ \lambda \in \mathbb{C} : a - \lambda \notin A^{-1} \} \):
\[
\tag{1.4} 0 \notin \text{acc } \sigma (a) .
\]

When (1.4) holds, then the support \( a^\bullet \) and the Drazin inverse \( a^\times \) are given by familiar Cauchy integrals.

Spectral inclusion is incorporated in the following “quasi-affine comparison” of elements: we shall write
\[
\tag{1.5} a \prec_{\text{aff}} b
\]
to mean that there is \( v \in A \) for which
\[
\tag{1.6} (vx = 0 \implies x = 0) \, , \quad bv = va \text{ and } \sigma (b) \subseteq \sigma (a) .
\]
This in turn interacts with our idea of a “polaroid” element:

2. Definition. We shall call the element \( a \in A \) polaroid iff there is an implication, for arbitrary \( \lambda \in \mathbb{C} \),

\[
\text{(2.1)} \quad a - \lambda \text{ quasipolar} \implies a - \lambda \text{ polar ,}
\]

and simply polaroid if the implication is

\[
\text{(2.2)} \quad a - \lambda \text{ quasipolar} \implies a - \lambda \text{ simply polar .}
\]

For example, the argument of Stampfli ([8], Theorem 14) says that for \( a = T \in B(X) \),

\[
\text{(2.3)} \quad \text{normaloid} \implies \text{simply polaroid} \implies \text{reguloid} .
\]

The first implication holds ([10], Lemma 2.5) if more generally the operator “satisfies the growth condition \( G_m \),” while the second is trivial. The “Property B” of Djordjevic, Jeon and Ko ([2], (4), page 324) implies the polaroid condition (2.2).

The quasi-affine comparison of (1.5) transmits polaroid and simple polaroid properties:

3. Theorem. If \( a \in A \) and \( b \in A \), then

\[
\text{(3.1)} \quad a \prec_{left} b \text{ polaroid} \implies a \text{ polaroid} ,
\]

and

\[
\text{(3.2)} \quad a \prec_{left} b \text{ simply polaroid} \implies a \text{ simply polaroid} .
\]

Proof. Begin by checking that if \( a \prec_{left} b \), then

\[
\text{(3.3)} \quad \text{iso } \sigma(a) \subseteq \text{iso } \sigma(b) ,
\]

so that

\[
\text{(3.4)} \quad a \prec_{left} b , a \text{ quasipolar} \implies b \text{ quasipolar} .
\]

The joint spectrum argument ([10], Lemma 2.5) is taken from Fialkow ([3], Theorem 2.5): if \( \lambda \in \text{iso } \sigma(a) \) then, with \( p = (a - \lambda)^n \) and \( c = (a - \lambda)(1 - p) \),

\[
v(1 - p)c = (b - \lambda)v(1 - p),
\]

and hence

\[
0 \neq v(1 - p) \in (R_c - L_{b - \lambda})^{-1}(0) .
\]

Thus ([6], Theorem 11.6.2)

\[
0 \in \sigma(R_c - L_{b - \lambda}) \subseteq \sigma(c) - \sigma(b - \lambda) ,
\]

giving

\[
\{0\} = \sigma(c) \subseteq \sigma(b - \lambda) .
\]

Thus \( \text{iso } \sigma(a) \subseteq \sigma(b) \subseteq \sigma(a) \), giving (3.3).

Towards (3.1) we can now argue, with \( c = (a - \lambda)^n \) and \( d = (b - \lambda)^n \),

\[
v(c - cc^x c) = (d - dd^x d)v = 0 \implies c = cc^x c ;
\]

then, in particular, (3.2) is the case \( n = 1 \).

For operators it is easy to see ([8], Theorem 9; [9], Theorem 2) that “Browder’s theorem holds” for \( T \) in the sense of (0.2) if and only if

\[
\text{(3.6)} \quad \text{acc } \sigma(T) \subseteq \omega_{ess}(T) ;
\]
the polaroid condition is close to the reverse inclusion:

**4. Theorem.** If \( T \in B(X) \) for a Banach space \( X \), then the inclusion

\[
\omega_{\text{ess}}(T) \subseteq \text{acc}(T)
\]

is sufficient for the polaroid condition (2.1), which in turn is sufficient for the disjointness (0.3). If \( T \) is polaroid and if \( T - \lambda I \) has for arbitrary \( \lambda \in \mathbb{C} \) either finite ascent or finite descent, then Weyl’s theorem holds for \( T \).

**Proof.** If (4.1) holds, then if \( T - \lambda I \) is quasipolar we have

\[
\lambda \in \text{iso}(T) \implies \lambda \in \text{iso}(T) \setminus \omega_{\text{ess}}(T),
\]

which (6, Theorem 9.8.4) by the punctured neighbourhood theorem makes it a “Riesz point” for \( T \), so that \( T - \lambda I \) is polar. Conversely, if \( T \) is polaroid and \( \lambda \in \pi_{0}^{\text{left}}(T) \), then \( T - \lambda I \) is polar with \( 0 < \dim (T - \lambda I)^{-1}(0) < \infty \). By the punctured neighbourhood theorem again this gives also \( \dim X/(T - \lambda I)X = \dim (T - \lambda I)^{-1}(0) \), excluding \( \lambda \) from \( \omega_{\text{ess}}(T) \). For the last part, if an operator \( T \) is Fredholm of index zero, then finite ascent and finite descent are equivalent. \( \square \)

Neither of the implications in the first part of Theorem 4 is reversible: for example, Weyl’s theorem holds for the Volterra operator \( x(t) \mapsto \int_{s=0}^{t} x(s)ds \) on \( C[0, 1] \) while (4.1) fails. If (0.3) holds for \( T \) and fails for the dual operator \( T^* \), then \( T \in B(X) \) is not polaroid: for a specific example, take \( T = UW \) to be the product of the standard weight \( W : (x_n) \mapsto (\frac{1}{n} x_n) \) and the forward shift \( U \). For Weyl operators the ascent/descent condition is equivalent to the “single valued extension property” of Finch [4].

**References**