

RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

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ABSTRACT. In this paper we prove that for any associative ring R , and for any left R -module M with finite projective dimension, the Gorenstein injective dimension $\text{Gid}_R M$ equals the usual injective dimension $\text{id}_R M$. In particular, if $\text{Gid}_R R$ is finite, then also $\text{id}_R R$ is finite, and thus R is Gorenstein (provided that R is commutative and Noetherian).

1. INTRODUCTION

It is well known that among the commutative local Noetherian rings (R, \mathfrak{m}, k) , the *Gorenstein rings* are characterized by the condition $\text{id}_R R < \infty$. From the dual of [10, Proposition (2.27)] ([6, Proposition 10.2.3] is a special case) it follows that the *Gorenstein injective dimension* $\text{Gid}_R(-)$ is a *refinement* of the usual injective dimension $\text{id}_R(-)$ in the following sense:

For any R -module M there is an inequality $\text{Gid}_R M \leq \text{id}_R M$, and if $\text{id}_R M < \infty$, then there is an equality $\text{Gid}_R M = \text{id}_R M$.

Now, since the injective dimension $\text{id}_R R$ of R measures Gorensteinness, it is only natural to ask what does the Gorenstein injective dimension $\text{Gid}_R R$ of R measure? As a consequence of Theorem (2.1) below, it turns out that:

An associative ring R with $\text{Gid}_R R < \infty$ also has $\text{id}_R R < \infty$ (and hence R is Gorenstein, provided that R is commutative and Noetherian).

This result is proved by Christensen [2, Theorem (6.3.2)] in the case where (R, \mathfrak{m}, k) is a commutative local Noetherian Cohen-Macaulay ring with a dualizing module. The aim of this paper is to prove Theorem (2.1), together with a series of related results. Among these results is Theorem (3.2), which has the nice, and easily stated, Corollary (3.3):

Assume that (R, \mathfrak{m}, k) is a commutative local Noetherian ring, and let M be an R -module of finite depth, that is, $\text{Ext}_R^m(k, M) \neq 0$ for some $m \in \mathbb{N}_0$ (this happens for example if $M \neq 0$ is finitely generated). If either

(i) $\text{Gfd}_R M < \infty$ and $\text{id}_R M < \infty$ or (ii) $\text{fd}_R M < \infty$ and $\text{Gid}_R M < \infty$,

then R is Gorenstein.

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This corollary is also proved by Christensen [2, Theorem (6.3.2)] in the case where (R, \mathfrak{m}, k) is Cohen-Macaulay with a dualizing module. However, Theorem (3.2) itself (dealing not only with local rings) is a generalization of [8, Proposition 2.10] (in the module case) by Foxby from 1979.

We should briefly mention the history of Gorenstein injective, projective and flat modules: *Gorenstein injective* modules over an arbitrary associative ring, and the related Gorenstein injective dimension, was introduced and studied by Enochs and Jenda in [3]. The dual concept, *Gorenstein projective* modules, was already introduced by Auslander and Bridger [1] in 1969, but only for finitely generated modules over a two-sided Noetherian ring. *Gorenstein flat* modules were also introduced by Enochs and Jenda; please see [5].

1.1. Setup and notation. Let R be any associative ring with a nonzero multiplicative identity. All modules are—if not specified otherwise—*left* R -modules. If M is any R -module, we use $\text{pd}_R M$, $\text{fd}_R M$, and $\text{id}_R M$ to denote the usual projective, flat, and injective dimension of M , respectively. Furthermore, we write $\text{Gpd}_R M$, $\text{Gfd}_R M$, and $\text{Gid}_R M$ for the Gorenstein projective, Gorenstein flat, and Gorenstein injective dimension of M , respectively.

2. RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

Theorem 2.1. *If M is an R -module with $\text{pd}_R M < \infty$, then $\text{Gid}_R M = \text{id}_R M$. In particular, if $\text{Gid}_R R < \infty$, then also $\text{id}_R R < \infty$ (and hence R is Gorenstein, provided that R is commutative and Noetherian).*

Proof. Since $\text{Gid}_R M \leq \text{id}_R M$ always, it suffices to prove that $\text{id}_R M \leq \text{Gid}_R M$. Naturally, we may assume that $\text{Gid}_R M < \infty$.

First consider the case where M is Gorenstein injective, that is, $\text{Gid}_R M = 0$. By definition, M is a kernel in a complete injective resolution. This means that there exists an exact sequence $\mathbf{E} = \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E_{-1} \rightarrow \cdots$ of injective R -modules, such that $\text{Hom}_R(I, \mathbf{E})$ is exact for every injective R -module I , and such that $M \cong \text{Ker}(E_1 \rightarrow E_0)$. In particular, there exists a short exact sequence $0 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 0$, where E is injective, and M' is Gorenstein injective. Since M' is Gorenstein injective and $\text{pd}_R M < \infty$, it follows by [4, Lemma 1.3] that $\text{Ext}_R^1(M, M') = 0$. Thus $0 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 0$ is split-exact; so M is a direct summand of the injective module E . Therefore, M itself is injective.

Next consider the case where $\text{Gid}_R M > 0$. By [10, Theorem (2.15)] there exists an exact sequence $0 \rightarrow M \rightarrow H \rightarrow C \rightarrow 0$ where H is Gorenstein injective and $\text{id}_R C = \text{Gid}_R M - 1$. As in the previous case, since H is Gorenstein injective, there exists a short exact sequence $0 \rightarrow H' \rightarrow I \rightarrow H \rightarrow 0$ where I is injective and H' is Gorenstein injective. Now consider the pull-back diagram with exact rows and

columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & M & \longrightarrow & H & \longrightarrow & C \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \parallel \\
 0 & \longrightarrow & P & \longrightarrow & I & \longrightarrow & C \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & H' & \xlongequal{\quad} & H' & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Since I is injective and $\text{id}_R C = \text{Gid}_R M - 1$ we get $\text{id}_R P \leq \text{Gid}_R M$ by the second row. Since H' is Gorenstein injective and $\text{pd}_R M < \infty$, it follows (as before) by [4, Lemma 1.3] that $\text{Ext}_R^1(M, H') = 0$. Consequently, the first column $0 \rightarrow H' \rightarrow P \rightarrow M \rightarrow 0$ splits. Therefore $P \cong M \oplus H'$, and hence $\text{id}_R M \leq \text{id}_R P \leq \text{Gid}_R M$. \square

The theorem above has, of course, a dual counterpart:

Theorem 2.2. *If M is an R -module with $\text{id}_R M < \infty$, then $\text{Gpd}_R M = \text{pd}_R M$.* \square

Theorem (2.6) below is a “flat version” of the two previous theorems. First recall the following.

Definition 2.3. The left finitistic projective dimension $\text{LeftFPD}(R)$ of R is defined as

$$\text{LeftFPD}(R) = \sup\{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}.$$

The right finitistic projective dimension $\text{RightFPD}(R)$ of R is defined similarly.

Remark 2.4. When R is commutative and Noetherian, we have that $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ equals the Krull dimension of R , by [9, Théorème (3.2.6) (Seconde partie)].

Furthermore, we will need the following result from [10, Proposition (3.11)]:

Proposition 2.5. *For any (left) R -module M the inequality*

$$\text{Gid}_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leq \text{Gfd}_R M$$

holds. If R is right coherent, then we have $\text{Gid}_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \text{Gfd}_R M$. \square

We are now ready to state:

Theorem 2.6. *For any R -module M , the following conclusions hold:*

- (i) *Assume that $\text{LeftFPD}(R)$ is finite. If $\text{fd}_R M < \infty$, then $\text{Gid}_R M = \text{id}_R M$.*
- (ii) *Assume that R is left and right coherent with finite $\text{RightFPD}(R)$. If $\text{id}_R M < \infty$, then $\text{Gfd}_R M = \text{fd}_R M$.*

Proof. (i) If $\text{fd}_R M < \infty$, then also $\text{pd}_R M < \infty$, by [11, Proposition 6] (since $\text{LeftFPD}(R) < \infty$). Hence the desired conclusion follows from Theorem (2.1) above.

(ii) Since R is left coherent, we have that $\text{fd}_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leq \text{id}_R M < \infty$, by [12, Lemma 3.1.4]. By assumption, $\text{RightFPD}(R) < \infty$, and therefore also

$\text{pd}_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) < \infty$, by [11, Proposition 6]. Now Theorem (2.1) gives that $\text{Gid}_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \text{id}_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. It is well known that

$$\text{fd}_R M = \text{id}_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$$

(without assumptions on R), and by Proposition (2.5) above, we also get $\text{Gfd}_R M = \text{Gid}_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, since R is right coherent. The proof is done. \square

3. A THEOREM ON GORENSTEIN RINGS BY FOXBY

We end this paper by generalizing a theorem [8, Proposition 2.10] on Gorenstein rings by Foxby from 1979. For completeness, we briefly recall:

3.1. The small support. Assume that R is commutative and Noetherian. For an R -module M , an integer n , and a prime ideal \mathfrak{p} in R , we write $\beta_n^R(\mathfrak{p}, M)$, respectively, $\mu_n^R(\mathfrak{p}, M)$, for the n th Betti number, respectively, n th Bass number, of M at \mathfrak{p} .

Foxby [8, Definition p. 157] or [7, (14.8)] defines the *small (or homological) support* of an R -module M to be the set

$$\text{supp}_R M = \{ \mathfrak{p} \in \text{Spec } R \mid \exists n \in \mathbb{N}_0 : \beta_n^R(\mathfrak{p}, M) \neq 0 \}.$$

Let us mention the most basic results about the small support, all of which can be found in [8, pp. 157 – 159] and [7, Chapter 14]:

- (a) The small support, $\text{supp}_R M$, is contained in the usual (large) support, $\text{Supp}_R M$, and $\text{supp}_R M = \text{Supp}_R M$ if M is finitely generated. Also, if $M \neq 0$, then $\text{supp}_R M \neq \emptyset$.
- (b) $\text{supp}_R M = \{ \mathfrak{p} \in \text{Spec } R \mid \exists n \in \mathbb{N}_0 : \mu_n^R(\mathfrak{p}, M) \neq 0 \}$.
- (c) Assume that (R, \mathfrak{m}, k) is local. If M is an R -module with finite depth, that is,

$$\text{depth}_R M := \inf \{ m \in \mathbb{N}_0 \mid \text{Ext}_R^m(k, M) \neq 0 \} < \infty$$

(this happens for example if $M \neq 0$ is finitely generated), then $\mathfrak{m} \in \text{supp}_R M$, by (b) above.

Now, given these facts about the small support, and the results in the previous section, the following generalization of [8, Proposition 2.10] is immediate:

Theorem 3.2. *Assume that R is commutative and Noetherian. Let M be any R -module, and assume that any of the following four conditions is satisfied:*

- (i) $\text{Gpd}_R M < \infty$ and $\text{id}_R M < \infty$,
- (ii) $\text{pd}_R M < \infty$ and $\text{Gid}_R M < \infty$,
- (iii) R has finite Krull dimension, and $\text{Gfd}_R M < \infty$ and $\text{id}_R M < \infty$,
- (iv) R has finite Krull dimension, and $\text{fd}_R M < \infty$ and $\text{Gid}_R M < \infty$.

Then $R_{\mathfrak{p}}$ is a Gorenstein local ring for all $\mathfrak{p} \in \text{supp}_R M$. \square

Corollary 3.3. *Assume that (R, \mathfrak{m}, k) is a commutative local Noetherian ring. If there exists an R -module M of finite depth, that is,*

$$\text{depth}_R M := \inf \{ m \in \mathbb{N}_0 \mid \text{Ext}_R^m(k, M) \neq 0 \} < \infty,$$

and which satisfies either

- (i) $\text{Gfd}_R M < \infty$ and $\text{id}_R M < \infty$, or
- (ii) $\text{fd}_R M < \infty$ and $\text{Gid}_R M < \infty$,

then R is Gorenstein. \square

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