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MINIMAL POLYNOMIALS OF ELEMENTS OF ORDER p IN p-MODULAR PROJECTIVE REPRESENTATIONS OF ALTERNATING GROUPS

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Abstract. Let F be an algebraically closed field of characteristic $p > 0$ and let G be a quasi-simple group with $G/Z(G) \cong A_n$. We describe the minimal polynomials of elements of order p in irreducible representations of G over F. If $p = 2$, we determine the minimal polynomials of elements of order 4 in 2-modular irreducible representations of A_n, S_n, $3 \cdot A_6$, $3 \cdot S_6$, $3 \cdot A_7$, and $3 \cdot S_7$.

1. Introduction

Throughout the paper, F is an algebraically closed field of characteristic $p > 0$ and all representations are F-representations unless otherwise stated. Let A_n and S_n denote the alternating and symmetric groups on n letters. We always assume that $n \geq 5$. Let G be a quasi-simple group with $G/Z(G) \cong A_n$, and let

$$\pi : G \to A_n$$

be the natural projection. Thus G is one of the following groups: A_n, $\tilde{A}_n := 2 \cdot A_n$, $k \cdot A_6$, or $k \cdot A_7$ for $k = 3, 6$.

Our goal is to determine the minimal polynomials of the elements $g \in G$ of order p in the irreducible representations of G. Minimal polynomials of such elements are always of the form $(x-1)^d$ for some $d \leq p$, and we determine all configurations where $d < p$.

Theorem 1.1. *Let G be a quasi-simple group with $G/Z(G) \cong A_n$, let $g \in G \setminus Z(G)$ be an element of order p, and let ϕ be a faithful irreducible representation of G over F. Then the degree d of the minimal polynomial of $\phi(g)$ is less than p if and only if one of the following happens:*

(i) *$\pi(g)$ is a product of two 3-cycles, $G = \tilde{A}_6$, $p = 3$, and ϕ is a basic spin representation of dimension 2.*

(ii) *$\pi(g)$ is a p-cycle and one of the following holds:*
   (a) *$G = A_p$, and ϕ is the "natural" representation of dimension $p - 2$;*
   (b) *$G = \tilde{A}_n$, $p = 3$ or 5, and ϕ is a basic spin representation;*
   (c) *$G = 3 \cdot A_7$ or $6 \cdot A_7$, $p = 7$, and $\dim \phi = 6$;*
   (d) *$G = \tilde{A}_7$, $p = 7$, and $\dim \phi = 4$;*
   (e) *$G = \tilde{A}_5$, $p = 5$, and $\dim \phi = 4$;*

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(f) $G = 3 \cdot A_6$ *or* $3 \cdot A_7$, $p = 5$, *and* $\dim \phi = 3$.

Moreover, $d = p - 1$ *in the case* (ii)(b) *above, and* $d = \dim \phi$ *in the remaining exceptional cases.*

In particular, we see that there are two "reasons" for the minimal polynomial of an element of order p to have degree less than p in an irreducible representation of G. One is trivial—the dimension of our representation might be less than p. The other is less obvious—$p = 3$ or 5 and the representation is a basic spin representation (these representations are known to be a source of many counterexamples and are pretty well-understood). We note that the degrees of the basic representations of $\tilde{A}_n$ in prime characteristic may differ from those in zero characteristic; see Lemma 2.6 below.

In the proofs we only have to deal with the case $p > 3$, since the case $p = 3$ of Theorem 1.1 has recently been settled by Chermak [3].

Obviously, the case $p = 2$ is trivial for elements of order 2. However, a version of the question for $g \in G$ of order 4 is of essential interest. Of course, when $p = 2$ we do not need to deal with two-fold coverings. However, the case $G = S_n$ does not automatically reduce to A_n since g may not belong to A_n. So we consider S_n as well.

Theorem 1.2. *Let* $p = 2$, $n \geq 5$, $G \in \{A_n, S_n, 3 \cdot A_6, 3 \cdot S_6, 3 \cdot A_7, 3 \cdot S_7\}$, $g \in G$ *be an element of order* 4, *and let* ϕ *be a faithful irreducible representation of* G *over* F. *Then the degree* d *of the minimal polynomial of* $\phi(g)$ *is less than* 4 *if and only if* $d = 3$ *and one of the following happens:*

(a) g *is of cycle type* $(4, 2)$, *and either* $G \cong 3 \cdot A_6$, $\dim \phi = 3$ *or* $G \cong 3 \cdot S_6$, $\dim \phi = 6$;

(b) $G \cong A_8 \cong SL(4, 2)$, g *is of cycle type* $(4, 4)$, *and* ϕ *is either the natural representation of* $SL(4, 2)$, *or its dual, or its exterior square;*

(c) $G \cong S_8$, g *is of cycle type* $(4, 4)$, *and* $\dim \phi = 8$ *or* 6.

2. Preliminaries

If M is a matrix, we denote by $\deg M$ the degree of the minimal polynomial of M and by $\operatorname{Jord} M$ the Jordan normal form of M (defined up to the ordering of Jordan blocks). The Jordan block of size k with eigenvalue 1 is denoted by J_k. The symbol $\operatorname{diag}(a_1, \dots, a_k)$ denotes the block-diagonal matrix with square matrices $a_1, \dots, a_k$ along the diagonal.

If G is any group, we denote by 1_G the trivial FG-module (or the corresponding representation). If M is an FG-module (resp. $\phi : G \to GL(M)$ is a representation of G), and $H < G$ is a subgroup, then $M|H$ (resp. $\phi|H$) stands for the restriction of M (resp. ϕ) to H.

We record the following obvious fact.

Lemma 2.1. *Let* G *be a finite group and* $g \in G$. *If* ρ *and* ϕ *are representations of* G *such that* ρ *is a subfactor of* ϕ, *then* $\deg \rho(g) \leq \deg \phi(g)$.

Let $m < n$. Throughout the paper we will often consider S_m as a subgroup of S_n, $\tilde{A}_m$ as a subgroup of $\tilde{A}_n$, etc. Unless otherwise stated, the embeddings are assumed to be *natural*, i.e., the subgroup acts on the *first* m letters.

Now, let $G = A_n$ or S_n. We will refer to the nontrivial composition factor of the natural n-dimensional permutation FG-module as the *natural irreducible module* and denote it by E_n. Denote by ε_n the corresponding representation. We have $\dim \varepsilon_n = n-2$ if $p|n$ and $\dim \varepsilon_n = n-1$ otherwise.

Lemma 2.2. *Let $G = S_n$ or A_n and let $g \in G$ be an element of order p. Then the degree d of the minimal polynomial of $\varepsilon_n(g)$ is p, unless $n = p$, in which case $d = p-2$.*

Proof. An easy explicit calculation (see, e.g., [8, Lemmas 2.1 and 2.2]). ☐

Let $1^-_{S_n}$ be the sign module over FS_n, and set $E^-_n = E_n \otimes 1^-_{S_n}$. Define

$$\mathcal{E}_n := \{1_{S_n}, 1^-_{S_n}, E_n, E^-_n\}.$$

If λ is a p-regular partition, D^λ denotes the irreducible FS_n-module corresponding to λ; see [6]. The following is a useful inductive characterization of the FS_n-modules from $\mathcal{E}_n$.

Proposition 2.3. *Let $n \geq 6$ and let D be an irreducible FS_n-module. Suppose that all composition factors of the restriction $D|S_{n-1}$ belong to $\mathcal{E}_{n-1}$. Then $D \in \mathcal{E}_n$, unless $n = 6, p = 3$ and $D \in \{D^{(4,2)}, D^{(2^2,1^2)}\}$, or $n = 6, p = 5$ and $D \in \{D^{(4,1^2)}, D^{(3,1^3)}\}$.*

Proof. By tensoring with $1^-_{S_n}$ if necessary, we may assume that $1_{S_{n-1}}$ or E_{n-1} occurs in the socle of $D|S_{n-1}$. Then it follows from [7, Theorem 0.5] that either $D \in \mathcal{E}_n$ or $D \in \{D^{(n-2,2)}, D^{(n-2,1^2)}\}$. However, by [7, Theorem 0.4(ii)], $D^{(n-2,2)}|S_{n-1}$ contains $D^{(n-3,2)}$ as a composition factor, and $D^{(n-3,2)} \notin \mathcal{E}_{n-1}$ unless $n = 6$ and $p = 3$. Similarly, $D^{(n-2,1^2)}|S_{n-1}$ contains $D^{(n-3,1^2)}$ as a composition factor and $D^{(n-3,1^2)} \notin \mathcal{E}_{n-1}$ unless $n = 6$ and $p = 5$. ☐

Corollary 2.4. *Let $n \geq 7$, and let V be an irreducible FA_n-module such that all composition factors of the restriction $V|A_{n-1}$ belong to $\{1_{A_{n-1}}, E_{n-1}\}$. Then $V \in \{1_{A_n}, E_n\}$.*

Proof. Follows from Clifford theory and Proposition 2.3. ☐

We need the following result of Benson in characteristic 2:

Lemma 2.5 ([1]). *Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a 2-regular partition. Then $D^\lambda|A_n$ splits as a direct sum of two non-equivalent irreducible FA_n-modules if and only if for all j with $\lambda_{2j-1} > 0$ we have $\lambda_{2j-1} - \lambda_{2j} = 1$ or 2, and $\lambda_{2j-1} + \lambda_{2j} \equiv 2 \pmod 4$. Otherwise, $D^\lambda|A_n$ is irreducible.*

Let $\tilde{S}_n$ denote a (nontrivial) two-fold central cover of S_n. Of course, $\tilde{A}_n$ is a subgroup in $\tilde{S}_n$ of index 2. The group $\tilde{S}_n$ has (one or two) remarkable complex representations called *basic (spin) representations*. These can be characterized as its faithful complex representations of minimal degree and can be constructed using Clifford algebras. A basic spin representation can also be defined as an irreducible representation of $\tilde{S}_n$ whose character is labelled by the partition (n) in Schur's parametrization of irreducible characters. The degree of a basic representation of $\tilde{S}_n$ is $2^{(n-1)/2}$ if n is odd, and $2^{(n-2)/2}$ if n is even. On restriction to $\tilde{A}_n$, basic representations remain irreducible if n is even and split as a direct sum of two non-equivalent irreducibles if n is odd. In both cases the corresponding complex representations of $\tilde{A}_n$ are also called *basic*.

Finally, for both $\tilde{S}_n$ and $\tilde{A}_n$, every irreducible constituent of Brauer reduction of a basic representation modulo p is called a (modular) *basic* representation. Dimensions of modular basic representations of $\tilde{S}_n$ have been determined by Wales [11]. For $p > 2$, these are the same as for complex representations, unless p divides n, in which case they are twice as small. Moreover, in [11, Table III], Wales provides complete information concerning tensoring basic modular representations with sign, from which the dimensions of basic modular representations of $\tilde{A}_n$ also follow, at least for $p > 2$. If $p = 2$, one can use Benson [1]. To summarize, we have:

Lemma 2.6. *Let $d_n(p)$ be the dimension of a modular basic representation of $\tilde{A}_n$.*

(i) *Let $p > 2$ and $p \nmid n$. Then $d_n(p) = 2^{(n-3)/2}$ if n is odd, and $2^{(n-2)/2}$ if n is even.*
(ii) *Let $p > 2$ and $p | n$. Then $d_n(p) = 2^{(n-3)/2}$ if n is odd, and $2^{(n-4)/2}$ if n is even.*
(iii) *Let $p = 2$. Then $d_n(2) = 2^{(n-3)/2}$ if n is odd, $2^{(n-2)/2}$ if $n \equiv 2 \pmod 4$, and $2^{(n-4)/2}$ if $n \equiv 0 \pmod 4$.*

We cite another result of Wales for future reference:

Proposition 2.7. *Let $n > 5$ and let ϕ be a faithful irreducible r of $\tilde{A}_n$. Then ϕ is basic if and only if all composition factors of $\phi|\tilde{A}_{n-1}$ are basic.*

Proof. For $\tilde{S}_n$ a similar result is contained in the proof of [11, Theorem 8.1]. Then Clifford theory implies the result for $\tilde{A}_n$. □

Finally, we record a lemma of G. Higman which is often used below.

Lemma 2.8 ([2, Ch. IX, Theorem 1.10])**.** *Let $G \subset GL(n, F)$ be a finite subgroup with abelian normal subgroup A of order coprime to p. Let $g \in G$ be an element of order p^k such that $g^{p^{k-1}} \notin C_G(A)$. Then $\deg g = p^k$.*

3. Main results

The following result of the second author provides us with an induction base for future arguments:

Lemma 3.1 ([12, Lemma 2.12])**.** *Let $n < 2p$, let G be a quasi-simple group with $G/Z(G) \cong A_n$, and let $g \in G$ be an element with $g^p \in Z(G)$. Suppose that ϕ is a faithful irreducible representation of G such that $\deg \phi(g) < p$. Then one of the following holds:*

(i) *$Z(G) = 1$, $n = p$, and $\phi = \varepsilon_n$ with $\dim \phi = p - 2$;*
(ii) *$p = 3$, $G = \tilde{A}_5$, and $\dim \phi = 2$;*
(iii) *either $p = 5$, $G \cong \tilde{A}_6$, or $p = 5, 7$, $G \cong \tilde{A}_7$, and in both cases $\dim \phi = 4$;*
(iv) *$p = 5$, $G = \tilde{A}_8$ or $\tilde{A}_9$, and $\dim \phi = 8$;*
(v) *$p = 5$, $G = \tilde{A}_5$, and $\dim \phi = 2$;*
(vi) *$p = 5$, $G = \tilde{A}_5$, and $\dim \phi = 4$;*
(vii) *$p = 5$, $G = 3 \cdot A_6$ or $3 \cdot A_7$, and $\dim \phi = 3$;*
(viii) *$p = 7$, $G = 3 \cdot A_7$ or $6 \cdot A_7$, and $\dim \phi = 6$.*

Moreover, in all the cases above, except (iv)*, the Jordan normal form of $\phi(g)$ has a single block, and in case* (iv) *it has two blocks of size* 4.

Remark. The representations ϕ appearing in (ii)–(v) are basic.

Lemma 3.2. *Let $G = A_n$ or $\tilde{A}_n$, with $n \geq 2p > 6$, and let $g \in G$ be an element of order p. If $p = 5$, suppose additionally that $\pi(g)$ is a 5-cycle. If $p = 7$ suppose additionally that either $G = A_n$ or $\pi(g)$ is a 7-cycle. If ϕ is a faithful irreducible representation of G with $\deg \phi(g) < p$, then $p = 5$, $G = \tilde{A}_n$, and ϕ is basic.*

Proof. We may assume that $\pi(g)$ is a product of cycles of the form:

$$\pi(g) = (1, 2, \ldots, p)(p + 1, \ldots, 2p) \cdots .$$

Recall that for $m < n$, A_m is assumed to be embedded into A_n as acting on the first m letters, unless otherwise stated. Define a subgroup H of G by requiring that (1) $H \supset Z(G)$; (2) $\pi(H) \cong A_7$ if $p = 5$; $\pi(H) \cong A_8$ if $p = 7$ and $G = \tilde{A}_n$; $\pi(H) \cong A_p$ otherwise.

Set $X = \langle g, H\rangle$. Then we have $H \cong X/O_p(X)$ and $g = hg_1$, where $h = (1, 2, \ldots, p) \in H$ and $g_1 \in O_p(X)$. Let τ be a nontrivial composition factor of $\phi|X$. Then $\tau(O_p(X)) = \mathrm{Id}$; so we can also consider τ as a representation of H. We have $\tau(g) = \tau(h)$. In view of Lemma 2.1, $\deg \tau(g) < p$.

If $Z(G) = \{1\}$, then $Z(H) = \{1\}$, and so $\tau = \varepsilon_n$, thanks to Lemma 3.1. By induction on n it follows from Corollary 2.4 that $\phi = \varepsilon_n$. The result now follows from Lemma 2.2.

Finally, let $|Z(G)| = 2$. By Lemma 3.1, $p = 5$ and τ is basic. So Proposition 2.7 implies that ϕ is basic. □

Lemma 3.3. *Let $G = \tilde{A}_n$ or A_n, and let $g \in G$ be an element of order $p > 3$ such that $\pi(g)$ has k nontrivial cycles. If $\deg \phi(g) < p$ for some faithful irreducible representation ϕ of G, then $k < 3$.*

Proof. Suppose $k \geq 3$. We may assume that

$$\pi(g) = (1, 2, \ldots, p)(p + 1, p + 2, \ldots, 2p)(2p + 1, 2p + 2, \ldots, 3p) \ldots .$$

Let A be the elementary abelian 3-subgroup of A_n of order 3^p generated by the commuting 3-cycles $(j, p + j, 2p + j)$ for $1 \leq j \leq p$. If $G = \tilde{A}_n$, let $B = \pi^{-1}(A)$. If $G = A_n$, take $B = A$. In both cases B is abelian of order prime to p, and $g \in N_G(B) \setminus C_G(B)$. Now we apply Lemma 2.8. □

Lemma 3.4. *Let $G = \tilde{A}_n$ or A_n, and let $g \in G$ be an element of order $p = 5$ or 7. If $\deg \phi(g) < p$ for some faithful irreducible representation ϕ of G, then $\pi(g)$ is a p-cycle.*

Proof. In view of Lemma 3.3, we may assume that

$$\pi(g) = (1, 2, \ldots, p)(p + 1, p + 2, \ldots, 2p).$$

Set $h_{ij} = (i, i + p)(j, j + p) \in A_n$ for $1 \leq i < j \leq p$. The subgroup H generated by the h_{ij} is abelian of order 2^{p-1}. If $G = A_n$, we may apply Lemma 2.8, since $g \in N_G(H) \setminus C_G(H)$. Now, let $G = \tilde{A}_n$.

Assume first that $p = 7$. Observe that H can be considered as an $\mathbb{F}_2\langle\pi(g)\rangle$-module via conjugation, and $\langle\pi(g)\rangle$ is a cyclic group of order p. Then the dimension of H over $\mathbb{F}_2$ is 6. Hence $\langle\pi(g)\rangle$ has an irreducible constituent M on H of dimension 3. In other words, $\pi(g)$ normalizes M, and $[\pi(g), M] \neq 1$. Let $L = \pi^{-1}(M)$. Then $|L| = 16$; hence it is not extraspecial. Now it is easy to deduce, using conjugation with g, that L is abelian. Since $g \in N_G(L) \setminus C_G(L)$, the result follows from Lemma 2.8.

Finally, let $p = 5$. Then g is contained in a group X isomorphic to the central product of two copies of $\tilde{A}_5$. Let τ be an irreducible constituent of the restriction ϕ to X. Then $\tau = \tau_1 \otimes \tau_2$ where τ_1 and τ_2 are faithful representations of the respective copies of $\tilde{A}_5$. In view of Lemma 3.1 and [5, Chapter VIII, Theorem 2.7], $\deg \tau(g) < 5$ only if $\dim \tau_1 = \dim \tau_2 = 2$. This means that every irreducible constituent of the restriction of ϕ to the naturally embedded $\tilde{A}_5$ is basic. By Proposition 2.7, ϕ is basic. Then $\deg \phi(g) = 5$ by [8, Lemma 3.12]. □

Proof of Theorem 1.1. For $p = 3$, see Chermak [3], and for $n < 2p$, see Lemma 3.1. Let $p > 3$ and $n \geq 2p$. Then the "only-if" part follows from Lemmas 3.2–3.4. For the "if" part it remains to show that $d := \deg \phi(g) = 4$ for ϕ basic spin, $p = 5$, and $\pi(g)$ a 5-cycle. Restricting to a natural subgroup $\tilde{A}_7$ containing g and using Lemma 3.1, we see that $d \geq 4$. On the other hand, for complex representations of $\tilde{A}_n$ a theorem similar to Theorem 1.1 has been proved in [13]. In particular, if $g \in \tilde{A}_n$ is a 5-cycle, then $\deg \beta(g) = 4$ for complex basic spin representations β. Since ϕ is a constituent of a reduction of β modulo 5, we have $d \leq 4$. □ □

Now we prove Theorem 1.2. The result is contained in Lemmas 3.5–3.11.

Lemma 3.5. *Theorem 1.2 is true for $n = 5$.*

Proof. Since A_5 has no elements of order 4 we may assume that $G = S_5$. Then G has two nontrivial irreducible representations, both of dimension 4; see [6, Tables]. One of them is ε_5, for which $\varepsilon_5 \oplus 1_{S_5} = \pi$, where π is the natural permutation representation of dimension 5. Clearly, $\mathrm{Jord}\, \pi(g) = \mathrm{diag}(J_4, J_1)$; so $\mathrm{Jord}\, \varepsilon_n(g) = J_4$. Another irreducible representation of G corresponds to the partition $(3, 2)$, and so it is reducible on A_5, thanks to [1] or [9]. Therefore, $\mathrm{Jord}\, \phi(g^2) = \mathrm{diag}(J_2, J_2)$ whence $\mathrm{Jord}\, \phi(g) = J_4$. □

Lemma 3.6. *Let $n \geq 5$, $G \in \{A_n,\ S_n,\ 3 \cdot A_6,\ 3 \cdot S_6,\ 3 \cdot A_7,\ 3 \cdot S_7\}$, and let $g \in G$ be an element of order 4 fixing at least one point of the natural permutation set. Then $\deg \phi(g) = 4$ for any faithful irreducible representation ϕ of G.*

Proof. We may assume that g transitively permutes $1, 2, 3, 4$ and fixes 5. Let $H := \mathrm{Alt}\{1, 2, 3, 4, 5\}$, and let $\hat{H}$ be the preimage of H in G. Set $X := \langle g, \hat{H} \rangle$. Since H contains no element of order 4, the restriction homomorphism $h : X \to \mathrm{Sym}\{1, 2, 3, 4, 5\} \cong S_5$ is surjective. Let $K = \ker h$. Clearly, K is central in X. Since S_5 has no non-split central extension with center of order 3, we have $X \cong Z(G) \times Y$ for some subgroup Y with $g \in Y$. Let τ be a composition factor of $\phi|Y$ with $\dim \tau > 1$. Then $\tau(Y) \cong S_5$. By Lemma 3.5, $\deg \tau(g) = 4$; hence $\deg \phi(g) = 4$ in view of Lemma 2.1. □

Lemma 3.7. *Theorem 1.2 is true for $G = A_6$ and S_6.*

Proof. For $g \in S_6 \setminus A_6$ this follows from Lemma 3.6. So we may assume that $G = A_6$. We use [9]. Irreducible FG-modules of dimension 8 are projective. So the Jordan form of g on each of these modules is $\mathrm{diag}(J_4, J_4)$. Other nontrivial irreducible FG-modules are of dimension 4. Since $A_6 \subseteq S_6 \cong Sp(4, 2)$, one of them is the natural $Sp(4, 2)$-module V restricted to A_6. Since the Jordan form of a unipotent element of $Sp(4, 2)$ does not have a block of size 3, the theorem is true for the natural representation. The second FG-module of dimension 4 is obtained from V by twisting with the outer automorphism σ of $S_6 = Sp(4, 2)$. Since A_6 has only one conjugacy class of elements of order 4, $\sigma(g)$ is conjugate to g in A_6; so $\operatorname{Jord}\sigma(g) = \operatorname{Jord} g$. $\square$

Lemma 3.8. *Let $n \geq 6$, $G = A_n$ or S_n, and let $g \in G$ be an element of order 4 having a 2-cycle in its cycle type. Then $\deg \phi(g) = 4$ for any faithful irreducible representation ϕ of G.*

Proof. Clearly g normalizes a subgroup $H \cong A_6$ fixing $n-6$ points such that g has a 2- and 4-cycle on the remaining 6 points. Then $g = g_1 g_2$ where $g_1 \in H$, $g_2 \in C_G(H)$. Set $X = \langle g, H\rangle = \langle g_2, H\rangle$, and let τ be a nontrivial composition factor of $\phi|X$. Since $X/O_2(X) \cong H$, Lemma 3.7 gives $\deg \tau(g) = 4$. So by Lemma 2.1, $\deg \phi(g) = 4$. $\square$

Lemma 3.9. *Theorem 1.2 is true for $n = 8$.*

Proof. In view of Lemmas 3.8 and 3.6, we may assume that the cycle type of g is $(4,4)$ and $G = A_8 \cong SL(4,2)$. Note that the group A_8 has 2 conjugacy classes of elements of order 4, corresponding to cycle types $(4,2)$ and $(4,4)$, and only the first one meets the subgroup A_6. The group $SL(4,2)$ has 2 conjugacy classes of elements of order 4, with Jordan forms J_4 and $\operatorname{diag}(J_3, J_1)$, and the second one does not meet $Sp(4,2)$. Since $A_6 \cong Sp(4,2)'$, we conclude that the class $(4,4)$ corresponds to the class $\operatorname{diag}(J_3, J_1)$. So g belongs to the intermediate subgroup $H \cong SL(3,2)$.

Let τ be an irreducible representation of H. Then τ is a restriction of a rational representation of $\bar{H}$, the algebraic group of type A_2. The irreducible representations of $\bar{H}$ are labelled by their highest weights $a_1\omega_1 + a_2\omega_2$, where a_1, a_2 are nonnegative integers and ω_1, ω_2 are the fundamental weights. It is well known that τ is a restriction of one of the four irreducible representations of $\bar{H}$ labelled by 0, ω_1, ω_2, or $\omega_1 + \omega_2$. The last one corresponds to the Steinberg module, whose restriction to H is projective, and so all Jordan blocks of g are of size 4. Two other representations are the natural and its dual. So the Jordan form of g on both of them is $\operatorname{diag}(J_3, J_1)$. Finally, corresponding to the zero highest weight we have the trivial representation.

Now, let $\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$ be the highest weight of ϕ. By a theorem of Smith [10] (also proved independently by R. Dipper), the restriction $\phi|H$ contains a direct summand τ with highest weight $a_1\omega_1 + a_2\omega_2$. From the previous paragraph, we may assume that at least one of a_1, a_2 is zero. By duality, the same is true for a_2, a_3. So we are left with the cases $\lambda \in \{\omega_1, \omega_2, \omega_3, \omega_1 + \omega_3\}$. The last one is the adjoint representation of G. Clearly, its restriction to H contains a composition factor isomorphic to the adjoint representation of H. Since the last representation is projective, it is a direct summand. Hence this case is ruled out. The cases $\lambda = \omega_1, \omega_3$ are obvious. Finally, the module corresponding to $\lambda = \omega_2$ is the exterior square of the natural module. So its restriction to H is a direct sum of the natural and dual natural modules; hence the Jordan blocks of $\phi(g)$ are of size 3. $\square$

Lemma 3.10. *Let $G = A_n$ or S_n, and let $g \in G$ be an element of order 4 containing at least three 4-cycles. Then $\deg \phi(g) = 4$ for any faithful irreducible representation ϕ of G.*

Proof. We may assume that

$$g = (1,2,3,4)(5,6,7,8)(9,10,11,12)\ldots.$$

Set $h_j := (j, j+4, j+8)$ for $j = 1, 2, 3, 4$, and $H := \langle h_1, h_2, h_3, h_4\rangle$. Then H is an abelian 3-group and $g \in N_G(H)$. Moreover, $g^2 \notin C_G(H)$; so the result follows from Lemma 2.8. $\square$

Lemma 3.11. *Theorem* 1.2 *is true for* $G = 3 \cdot A_6$ *and* $3 \cdot A_7$.

Proof. For $G = 3 \cdot A_7$ see Lemma 3.6. Let $G = 3 \cdot A_6$. Then $\dim \phi = 3$ or 9; see [9]. In the former case, $\deg \phi(g) = 3$, since $\deg \phi(g) < 3$ implies $\phi(g)^2 = 1$, which is false. Let $\dim \phi = 9$. Observe that g^2 normalizes a cyclic group $\langle c\rangle$ of order 5. Set $X := \langle g^2, c\rangle$. Since $g^2cg^{-2} = c^{-1}$ and the multiplicity of every eigenvalue $\alpha \neq 1$ of $\phi(c)$ is 2 (see [9]), it follows that $\phi|X$ has four composition factors of dimension 2 and one composition factor of dimension 1. Therefore $\operatorname{Jord} g^2 = \operatorname{diag}(J_2, J_2, J_2, J_2, J_1)$, whence $\operatorname{Jord} g = \operatorname{diag}(J_4, J_4, J_1)$. $\square$

References

[1] D. Benson, *Spin modules for symmetric groups*, J. London Math. Soc. (2) **38** (1988), 250–262. MR **89k:**20020

[2] B. Huppert and N. Blackburn, *Finite groups II*, Grundlehren der mathematischen Wissenschaften, Band 242, Springer-Verlag, Berlin, 1982. MR **84i:**20001a

[3] A. Chermak, *Quadratic Pairs*, preprint, 2001.

[4] Ch. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, John Wiley and Sons, New York-London, 1962. MR **26:**2519

[5] W. Feit, *The representation theory of finite groups*, North-Holland, Amsterdam, 1982. MR **83g:**20001

[6] G. D. James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Mathematics, Vol. 682, Springer-Verlag, Berlin-New York, 1978. MR **80g:**20019

[7] A. S. Kleshchev, *Branching rules for modular representations of symmetric groups, II*, J. reine angew. Math. **459** (1995), 163–212. MR **96m:**20019b

[8] A. S. Kondratiev and A. E. Zalesski, *Linear groups of degree at most* 27 *over residue rings modulo* p^k, J. Algebra **240** (2001), 120–142. MR **2002c:**20076

[9] C. Jansen, K. Lux, R. A. Parker, and R. A. Wilson, *An atlas of Brauer characters*, The Clarendon Press, Oxford University Press, New York, 1995. MR **96k:**20016

[10] S. D. Smith, *Irreducible modules and parabolic subgroups*, J. Algebra **75** (1982), 286–289. MR **83g:**20043

[11] D. Wales, *Some projective representations of* S_n, J. Algebra **61** (1979), 37–57. MR **81f:**20015

[12] A. E. Zalesskiĭ, *Minimal polynomials and eigenvalues of* p*-elements in representations of quasi-simple groups with a cyclic Sylow* p*-subgroup*, J. London Math. Soc. (2) **59** (1999), 845–866. MR **2001a:**20018

[13] A. E. Zalesskiĭ, *Eigenvalues of prime-order elements in projective representations of alternating groups*, Vestsi Akad. Navuk Belarusi, ser. Fiz.-Mat-Inform. (1996), no. 3, 41–43 (in Russian). MR **98b:**20017

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