MINIMAL POLYNOMIALS OF ELEMENTS OF ORDER $p$
IN $p$-MODULAR PROJECTIVE REPRESENTATIONS
OF ALTERNATING GROUPS

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Abstract. Let $F$ be an algebraically closed field of characteristic $p > 0$ and let $G$ be a quasi-simple group with $G/Z(G) \cong A_n$. We describe the minimal polynomials of elements of order $p$ in irreducible representations of $G$ over $F$. If $p = 2$, we determine the minimal polynomials of elements of order 4 in 2-modular irreducible representations of $A_n$, $S_n$, $3 \cdot A_6$, $3 \cdot S_6$, $3 \cdot A_7$, and $3 \cdot S_7$.

1. Introduction

Throughout the paper, $F$ is an algebraically closed field of characteristic $p > 0$ and all representations are $F$-representations unless otherwise stated. Let $A_n$ and $S_n$ denote the alternating and symmetric groups on $n$ letters. We always assume that $n \geq 5$. Let $G$ be a quasi-simple group with $G/Z(G) \cong A_n$, and let

$$\pi : G \to A_n$$

be the natural projection. Thus $G$ is one of the following groups: $A_n$, $\tilde{A}_n := 2 \cdot A_n$, $k \cdot A_6$, or $k \cdot A_7$ for $k = 3, 6$.

Our goal is to determine the minimal polynomials of the elements $g \in G$ of order $p$ in the irreducible representations of $G$. Minimal polynomials of such elements are always of the form $(x - 1)^d$ for some $d \leq p$, and we determine all configurations where $d < p$.

Theorem 1.1. Let $G$ be a quasi-simple group with $G/Z(G) \cong A_n$, let $g \in G \setminus Z(G)$ be an element of order $p$, and let $\phi$ be a faithful irreducible representation of $G$ over $F$. Then the degree $d$ of the minimal polynomial of $\phi(g)$ is less than $p$ if and only if one of the following happens:

(i) $\pi(g)$ is a product of two 3-cycles, $G = \tilde{A}_6$, $p = 3$, and $\phi$ is a basic spin representation of dimension 2.

(ii) $\pi(g)$ is a $p$-cycle and one of the following holds:

(a) $G = A_p$, and $\phi$ is the “natural” representation of dimension $p - 2$;
(b) $G = \tilde{A}_n$, $p = 3$ or 5, and $\phi$ is a basic spin representation;
(c) $G = 3 \cdot A_7$ or $6 \cdot A_7$, $p = 7$, and $\dim \phi = 6$;
(d) $G = \tilde{A}_7$, $p = 7$, and $\dim \phi = 4$;
(e) $G = \tilde{A}_5$, $p = 5$, and $\dim \phi = 4$;

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(f) \( G = 3 \cdot A_6 \) or \( 3 \cdot A_7 \), \( p = 5 \), and \( \dim \phi = 3 \).

Moreover, \( d = p - 1 \) in the case (ii)(b) above, and \( d = \dim \phi \) in the remaining exceptional cases.

In particular, we see that there are two “reasons” for the minimal polynomial of an element of order \( p \) to have degree less than \( p \) in an irreducible representation of \( G \). One is trivial—the dimension of our representation might be less than \( p \). The other is less obvious—\( p = 3 \) or \( 5 \) and the representation is a basic spin representation (these representations are known to be a source of many counterexamples and are pretty well-understood). We note that the degrees of the basic representations of \( \hat{A}_n \) in prime characteristic may differ from those in zero characteristic; see Lemma 2.6 below.

In the proofs we only have to deal with the case \( p > 3 \), since the case \( p = 3 \) of Theorem 1.1 has recently been settled by Chermak [3].

Obviously, the case \( p = 2 \) is trivial for elements of order 2. However, a version of the question for \( g \in G \) of order 4 is of essential interest. Of course, when \( p = 2 \) we do not need to deal with two-fold coverings. However, the case \( G = S_n \) does not automatically reduce to \( A_n \) since \( g \) may not belong to \( A_n \). So we consider \( S_n \) as well.

**Theorem 1.2.** Let \( p = 2 \), \( n \geq 5 \), \( G \in \{ A_n, S_n, 3 \cdot A_6, 3 \cdot S_6, 3 \cdot A_7, 3 \cdot S_7 \} \), \( g \in G \) be an element of order 4, and let \( \phi \) be a faithful irreducible representation of \( G \) over \( F \). Then the degree \( d \) of the minimal polynomial of \( \phi(g) \) is less than 4 if and only if \( d = 3 \) and one of the following happens:

(a) \( g \) is of cycle type \((4, 2)\), and either \( G \cong 3 \cdot A_6 \), \( \dim \phi = 3 \) or \( G \cong 3 \cdot S_6 \), \( \dim \phi = 6 \);

(b) \( G \cong A_8 \cong SL(4, 2) \), \( g \) is of cycle type \((4, 4)\), and \( \phi \) is either the natural representation of \( SL(4, 2) \), or its dual, or its exterior square;

(c) \( G \cong S_8 \), \( g \) is of cycle type \((4, 4)\), and \( \dim \phi = 8 \) or 6.

2. Preliminaries

If \( M \) is a matrix, we denote by \( \deg M \) the degree of the minimal polynomial of \( M \) and by \( \text{Jord} M \) the Jordan normal form of \( M \) (defined up to the ordering of Jordan blocks). The Jordan block of size \( k \) with eigenvalue 1 is denoted by \( J_k \). The symbol \( \text{diag}(a_1, \ldots, a_k) \) denotes the block-diagonal matrix with square matrices \( a_1, \ldots, a_k \) along the diagonal.

If \( G \) is any group, we denote by \( 1_G \) the trivial \( FG \)-module (or the corresponding representation). If \( M \) is an \( FG \)-module (resp. \( \phi : G \to GL(M) \) is a representation of \( G \)), and \( H < G \) is a subgroup, then \( M|H \) (resp. \( \phi|H \)) stands for the restriction of \( M \) (resp. \( \phi \)) to \( H \).

We record the following obvious fact.

**Lemma 2.1.** Let \( G \) be a finite group and \( g \in G \). If \( \rho \) and \( \phi \) are representations of \( G \) such that \( \rho \) is a subfactor of \( \phi \), then \( \deg \rho(g) \leq \deg \phi(g) \).

Let \( m < n \). Throughout the paper we will often consider \( S_m \) as a subgroup of \( S_n \), \( A_m \) as a subgroup of \( A_n \), etc. Unless otherwise stated, the embeddings are assumed to be natural, i.e., the subgroup acts on the first \( m \) letters.

Now, let \( G = A_n \) or \( S_n \). We will refer to the nontrivial composition factor of the natural \( n \)-dimensional permutation \( FG \)-module as the natural irreducible module.
and denote it by $E_n$. Denote by $\varepsilon_n$ the corresponding representation. We have $\dim \varepsilon_n = n - 2$ if $p | n$ and $\dim \varepsilon_n = n - 1$ otherwise.

**Lemma 2.2.** Let $G = S_n$ or $A_n$ and let $g \in G$ be an element of order $p$. Then the degree $d$ of the minimal polynomial of $\varepsilon_n(g)$ is $p$, unless $n = p$, in which case $d = p - 2$.

**Proof.** An easy explicit calculation (see, e.g., [8, Lemmas 2.1 and 2.2]). \qed

Let $1_{S_n}$ be the sign module over $FS_n$, and set $E_n^- = E_n \otimes 1_{S_n}$. Define

$$E_n := \{1_{S_n}, 1_{S_n}, E_n, E_n^\sim\}. $$

If $\lambda$ is a $p$-regular partition, $D^\lambda$ denotes the irreducible $FS_n$-module corresponding to $\lambda$; see [6]. The following is a useful inductive characterization of the $FS_n$-modules from $E_n$.

**Proposition 2.3.** Let $n \geq 6$ and let $D$ be an irreducible $FS_n$-module. Suppose that all composition factors of the restriction $D|S_{n-1}$ belong to $E_{n-1}$. Then $D \in E_n$, unless $n = 6, p = 3$ and $D \in \{D^{(4,2)}, D^{(2^2,1^2)}\}$, or $n = 6, p = 5$ and $D \in \{D^{(4,1^2)}, D^{(3,1^3)}\}$.

**Proof.** By tensoring with $1_{S_n}$, if necessary, we may assume that $1_{S_{n-1}}$ or $E_{n-1}$ occurs in the socle of $D|S_{n-1}$. Then it follows from [7, Theorem 0.5] that either $D \in E_n$ or $D \in \{D^{(n-2,2)}, D^{(n-2,1^2)}\}$. However, by [7, Theorem 0.4(ii)], $D^{(n-2,2)}|S_{n-1}$ contains $D^{(n-3,2)}$ as a composition factor, and $D^{(n-3,2)} \notin E_{n-1}$ unless $n = 6$ and $p = 3$. Similarly, $D^{(n-2,1^2)}|S_{n-1}$ contains $D^{(n-3,1^2)}$ as a composition factor and $D^{(n-3,1^2)} \notin E_{n-1}$ unless $n = 6$ and $p = 5$. \qed

**Corollary 2.4.** Let $n \geq 7$, and let $V$ be an irreducible $FA_n$-module such that all composition factors of the restriction $V|A_{n-1}$ belong to $\{1_{A_n}, E_{n-1}\}$. Then $V \in \{1_{A_n}, E_n\}$.

**Proof.** Follows from Clifford theory and Proposition 2.3. \qed

We need the following result of Benson in characteristic 2:

**Lemma 2.5 (11).** Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a 2-regular partition. Then $D^\lambda|A_n$ splits as a direct sum of two non-equivalent irreducible $FA_n$-modules if and only if for all $j$ with $\lambda_{2j} > 0$ we have $\lambda_{2j-1} = 1$ or 2, and $\lambda_{2j-1} + \lambda_{2j} \equiv 2 \text{ (mod 4)}$. Otherwise, $D^\lambda|A_n$ is irreducible.

Let $S_n$ denote a (nontrivial) two-fold central cover of $S_n$. Of course, $\hat{A}_n$ is a subgroup in $S_n$ of index 2. The group $S_n$ has (one or two) remarkable complex representations called basic (spin) representations. These can be characterized as its faithful complex representations of minimal degree and can be constructed using Clifford algebras. A basic spin representation can also be defined as an irreducible representation of $S_n$ whose character is labelled by the partition $(n)$ in Schur’s parametrization of irreducible characters. The degree of a basic representation of $S_n$ is $2^{(n-1)/2}$ if $n$ is odd, and $2^{(n-2)/2}$ if $n$ is even. On restriction to $A_n$, basic representations remain irreducible if $n$ is even and split as a direct sum of two non-equivalent irreducibles if $n$ is odd. In both cases the corresponding complex representations of $\hat{A}_n$ are also called basic.
Finally, for both $\tilde{S}_n$ and $\tilde{A}_n$, every irreducible constituent of Brauer reduction of a basic representation modulo $p$ is called a (modular) basic representation. Dimensions of modular basic representations of $\tilde{S}_n$ have been determined by Wales [11]. For $p > 2$, these are the same as for complex representations, unless $p$ divides $n$, in which case they are twice as small. Moreover, in [11, Table III], Wales provides complete information concerning tensoring basic modular representations with sign, from which the dimensions of basic modular representations of $\tilde{A}_n$ also follow, at least for $p > 2$. If $p = 2$, one can use Benson [1]. To summarize, we have:

**Lemma 2.6.** Let $d_n(p)$ be the dimension of a modular basic representation of $\tilde{A}_n$.

(i) Let $p > 2$ and $p | n$. Then $d_n(p) = 2^{(n-3)/2}$ if $n$ is odd, and $2^{(n-2)/2}$ if $n$ is even.

(ii) Let $p > 2$ and $p | n$. Then $d_n(p) = 2^{(n-3)/2}$ if $n$ is odd, and $2^{(n-4)/2}$ if $n$ is even.

(iii) Let $p = 2$. Then $d_n(2) = 2^{(n-3)/2}$ if $n$ is odd, $2^{(n-2)/2}$ if $n \equiv 2 \pmod{4}$, and $2^{(n-4)/2}$ if $n \equiv 0 \pmod{4}$.

We cite another result of Wales for future reference:

**Proposition 2.7.** Let $n > 5$ and let $\phi$ be a faithful irreducible $r$ of $\tilde{A}_n$. Then $\phi$ is basic if and only if all composition factors of $\phi|\tilde{A}_{n-1}$ are basic.

**Proof.** For $\tilde{S}_n$, a similar result is contained in the proof of [11, Theorem 8.1]. Then Clifford theory implies the result for $\tilde{A}_n$. \hfill \Box

Finally, we record a lemma of G. Higman which is often used below.

**Lemma 2.8** ([2, Ch. IX, Theorem 1.10]). Let $G \subset GL(n, F)$ be a finite subgroup with abelian normal subgroup $A$ of order coprime to $p$. Let $g \in G$ be an element of order $p^k$ such that $g^{p^{k+1}} \not\in C_G(A)$. Then $\deg g = p^k$.

3. Main results

The following result of the second author provides us with an induction base for future arguments:

**Lemma 3.1** ([12, Lemma 2.12]). Let $n < 2p$, let $G$ be a quasi-simple group with $G/Z(G) \cong A_n$, and let $g \in G$ be an element with $g^p \in Z(G)$. Suppose that $\phi$ is a faithful irreducible representation of $G$ such that $\deg \phi(g) < p$. Then one of the following holds:

(i) $Z(G) = 1$, $n = p$, and $\phi = \varepsilon_n$ with $\dim \phi = p - 2$;

(ii) $p = 3$, $G = \tilde{A}_5$, and $\dim \phi = 2$;

(iii) either $p = 5$, $G \cong \tilde{A}_6$, or $p = 5, 7, G \cong \tilde{A}_7$, and in both cases $\dim \phi = 4$;

(iv) $p = 5$, $G = \tilde{A}_8$ or $\tilde{A}_9$, and $\dim \phi = 8$;

(v) $p = 5$, $G = \tilde{A}_5$, and $\dim \phi = 2$;

(vi) $p = 5$, $G = \tilde{A}_5$, and $\dim \phi = 4$;

(vii) $p = 5$, $G = 3 \cdot A_6$ or $3 \cdot A_7$, and $\dim \phi = 3$;

(viii) $p = 7$, $G = 3 \cdot A_7$ or $6 \cdot A_7$, and $\dim \phi = 6$.

Moreover, in all the cases above, except (iv), the Jordan normal form of $\phi(g)$ has a single block, and in case (iv) it has two blocks of size 4.

**Remark.** The representations $\phi$ appearing in (ii)–(v) are basic.
Lemma 3.2. Let $G = A_n$ or $\tilde{A}_n$, with $n \geq 2p > 6$, and let $g \in G$ be an element of order $p$. If $p = 5$, suppose additionally that $\pi(g)$ is a $5$-cycle. If $p = 7$ suppose additionally that either $G = A_n$ or $\pi(g)$ is a $7$-cycle. If $\phi$ is a faithful irreducible representation of $G$ with $\deg \phi(g) < p$, then $p = 5$, $G = \tilde{A}_n$, and $\phi$ is basic.

Proof. We may assume that $\pi(g)$ is a product of cycles of the form:

$$\pi(g) = (1, 2, \ldots, p)(p + 1, \ldots, 2p) \cdots$$

Recall that for $m < n$, $A_m$ is assumed to be embedded into $A_n$ as acting on the first $m$ letters, unless otherwise stated. Define a subgroup $H$ of $G$ by requiring that $H \supseteq Z(G)$; (2) $\pi(H) \cong A_5$ if $p = 5$; $\pi(H) \cong A_8$ if $p = 7$ and $G = \tilde{A}_n$; $\pi(H) \cong A_p$ otherwise.

Set $X = \langle g, H \rangle$. Then we have $H \cong X/O_p(X)$ and $g = h_1g_1$, where $h = (1, 2, \ldots, p) \in H$ and $g_1 \in O_p(X)$. Let $\tau$ be a nontrivial composition factor of $\phi[X]$. Then $\tau(O_p(X)) = \text{Id}$; so we can also consider $\tau$ as a representation of $H$. We have $\tau(g) = \tau(h)$. In view of Lemma 2.1 $\deg \tau(g) < p$.

If $Z(G) = \{1\}$, then $Z(H) = \{1\}$, and so $\tau = \varepsilon_n$, thanks to Lemma 3.1. By induction on $n$ it follows from Corollary 2.4 that $\phi = \varepsilon_n$. The result now follows from Lemma 2.8.

Finally, let $|Z(G)| = 2$. By Lemma 3.1 $p = 5$ and $\tau$ is basic. So Proposition 2.7 implies that $\phi$ is basic. \qed

Lemma 3.3. Let $G = \tilde{A}_n$ or $A_n$, and let $g \in G$ be an element of order $p > 3$ such that $\pi(g)$ has $k$ nontrivial cycles. If $\deg \phi(g) < p$ for some faithful irreducible representation $\phi$ of $G$, then $k < 3$.

Proof. Suppose $k \geq 3$. We may assume that

$$\pi(g) = (1, 2, \ldots, p)(p + 1, p + 2, \ldots, 2p)(2p + 1, 2p + 2, \ldots, 3p) \cdots$$

Let $A$ be the elementary abelian $3$-subgroup of $A_n$ of order $3^p$ generated by the commuting $3$-cycles $(j, p + j, 2p + j)$ for $1 \leq j \leq p$. If $G = \tilde{A}_n$, let $B = \pi^{-1}(A)$. If $G = A_n$, take $B = A$. In both cases $B$ is abelian of order prime to $p$, and $g \in N_G(B) \setminus C_G(B)$. Now we apply Lemma 2.8. \qed

Lemma 3.4. Let $G = \tilde{A}_n$ or $A_n$, and let $g \in G$ be an element of order $p = 5$ or $7$. If $\deg \phi(g) < p$ for some faithful irreducible representation $\phi$ of $G$, then $\pi(g)$ is a $p$-cycle.

Proof. In view of Lemma 3.3 we may assume that

$$\pi(g) = (1, 2, \ldots, p)(p + 1, p + 2, \ldots, 2p).$$

Set $h_{ij} = (i, i + p)(j, j + p) \in A_n$ for $1 \leq i < j \leq p$. The subgroup $H$ generated by the $h_{ij}$ is abelian of order $2p^{-1}$. If $G = A_n$, we may apply Lemma 2.8 since $g \in N_G(H) \setminus C_G(H)$. Now, let $G = \tilde{A}_n$.

Assume first that $p = 7$. Observe that $H$ can be considered as an $\mathbb{F}_2 \langle \pi(g) \rangle$-module via conjugation, and $\langle \pi(g) \rangle$ is a cyclic group of order $p$. Then the dimension of $H$ over $\mathbb{F}_2$ is $6$. Hence $\langle \pi(g) \rangle$ has an irreducible constituent $M$ on $H$ of dimension $3$. In other words, $\pi(g)$ normalizes $M$, and $[\pi(g), M] \neq 1$. Let $L = \pi^{-1}(M)$. Then $|L| = 16$; hence it is not extraspecial. Now it is easy to deduce, using conjugation with $g$, that $L$ is abelian. Since $g \in N_G(L) \setminus C_G(L)$, the result follows from Lemma 2.8.
Finally, let \( p = 5 \). Then \( g \) is contained in a group \( X \) isomorphic to the central product of two copies of \( A_5 \). Let \( \tau \) be an irreducible constituent of the restriction \( \phi \) to \( X \). Then \( \tau = \tau_1 \otimes \tau_2 \) where \( \tau_1 \) and \( \tau_2 \) are faithful representations of the respective copies of \( A_5 \). In view of Lemma 3.1 and [5, Chapter VIII, Theorem 2.7], \( \deg \tau(g) < 5 \) only if \( \dim \tau_1 = \dim \tau_2 = 2 \). This means that every irreducible constituent of the restriction of \( \phi \) to the naturally \( A_5 \)-basic representation is basic. By Proposition 7.7, \( \phi \) is basic. Then \( \deg \phi(g) = 5 \) by [8, Lemma 3.12].

**Proof of Theorem 1.4.** For \( p = 3 \), see Chermak [3], and for \( n < 2p \), see Lemma 3.1. Let \( p > 3 \) and \( n \geq 2p \). Then the “only-if” part follows from Lemmas 3.2, 3.4. For the “if” part it remains to show that \( d := \deg \phi(g) = 4 \) for \( \phi \) basic spin, \( p = 5 \), and \( \pi(g) \) a 5-cycle. Restricting to a natural subgroup \( t \) containing \( g \) and using Lemma 3.1, we see that \( d \geq 4 \). On the other hand, for complex representations of \( A_n \), a theorem similar to Theorem 1.2 has been proved in [13]. In particular, if \( g \in A_n \) is a 5-cycle, then \( \deg \beta(g) = 4 \) for complex basic spin representations \( \beta \). Since \( \phi \) is a constituent of a reduction of \( \beta \) modulo 5, we have \( d \leq 4 \). 

Now we prove Theorem 1.2. The result is contained in Lemmas 3.5, 3.11.

**Lemma 3.5.** Theorem 1.2 is true for \( n = 5 \).

**Proof.** Since \( A_5 \) has no elements of order 4 we may assume that \( G = S_5 \). Then \( G \) has two nontrivial irreducible representations, both of dimension 4; see [5, Tables]. One of them is \( \varepsilon_5 \), for which \( \varepsilon_5 \oplus 1_{S_5} = \pi \), where \( \pi \) is the natural permutation representation of dimension 5. Clearly, \( \text{Jord } \pi(g) = \text{diag}(J_4, J_1) \); so \( \text{Jord } \varepsilon_5(g) = J_4 \). Another irreducible representation of \( G \) corresponds to the partition \((3,2)\), and so it is reducible on \( A_5 \), thanks to [1] or [2]. Therefore, \( \text{Jord } \phi(g^2) = \text{diag}(J_2, J_2) \) whence \( \text{Jord } \phi(g) = J_4 \).

**Lemma 3.6.** Let \( n \geq 5 \), \( G \in \{ A_n, S_n, 3 \cdot A_6, 3 \cdot S_6, 3 \cdot A_7, 3 \cdot S_7 \} \), and let \( g \in G \) be an element of order 4 fixing at least one point of the natural permutation set. Then \( \deg \phi(g) = 4 \) for any faithful irreducible representation \( \phi \) of \( G \).

**Proof.** We may assume that \( g \) transitively permutes \( 1,2,3,4 \) and fixes 5. Let \( H := \text{Alt}\{1,2,3,4,5\} \), and let \( H \) be the preimage of \( H \) in \( G \). Set \( X := \langle g, H \rangle \). Since \( H \) contains no element of order 4, the restriction homomorphism \( h : X \to \text{Sym}\{1,2,3,4,5\} \cong S_5 \) is surjective. Let \( K = \ker h \). Clearly, \( K \) is central in \( X \). Since \( S_5 \) has no non-split central extension with center of order 3, we have \( X \cong Z(G) \times Y \) for some subgroup \( Y \) with \( g \in Y \). Let \( \tau \) be a composition factor of \( \phi Y \) with \( \dim \tau > 1 \). Then \( \tau(Y) \cong S_5 \). By Lemma 3.6, \( \deg \tau(g) = 4 \); hence \( \deg \phi(g) = 4 \) in view of Lemma 2.1.

**Lemma 3.7.** Theorem 1.2 is true for \( G = A_6 \) and \( S_6 \).

**Proof.** For \( g \in S_6 \setminus A_6 \) this follows from Lemma 3.6. So we may assume that \( G = A_6 \). We use [9]. Irreducible \( FG \)-modules of dimension 8 are projective. So the Jordan form of \( g \) on each of these modules is \( \text{diag}(J_4, J_4) \). Other nontrivial irreducible \( FG \)-modules are of dimension 4. Since \( A_6 \subseteq S_6 \cong Sp(4,2) \), one of them is the natural \( Sp(4,2) \)-module \( V \) restricted to \( A_6 \). Since the Jordan form of a unipotent element of \( Sp(4,2) \) does not have a block of size 3, the theorem is true for the natural representation. The second \( FG \)-module of dimension 4 is obtained from \( V \) by twisting with the outer automorphism \( \sigma = Sp(4,2) \). Since \( A_6 \) has
Lemma 3.8. Let \( n \geq 6, G = A_n \) or \( S_n \), and let \( g \in G \) be an element of order 4 having a 2-cycle in its cycle type. Then \( \deg \phi(g) = 4 \) for any faithful irreducible representation \( \phi \) of \( G \).

Proof. Clearly \( g \) normalizes a subgroup \( H \cong A_6 \) fixing \( n - 6 \) points such that \( g \) has a 2- and 4-cycle on the remaining 6 points. Then \( g = g_1g_2 \) where \( g_1, g_2 \in H \), and let \( \tau \) be a nontrivial composition factor of \( \phi(X) \). Since \( X/O_2(X) \cong H \), Lemma 3.7 gives \( \deg \tau(g) = 4 \). So by Lemma 2.1 \( \deg \phi(g) = 4 \).

Lemma 3.9. Theorem 1.2 is true for \( n = 8 \).

Proof. In view of Lemmas 3.8 and 3.6, we may assume that the cycle type of \( g \) is \( (4, 4) \) and \( G = A_8 \cong SL(4, 2) \). Note that the group \( A_8 \) has 2 conjugacy classes of elements of order 4, corresponding to cycle types \( (4, 4) \) and \( (4, 4) \), and only the first one meets the subgroup \( A_6 \). The group \( SL(4, 2) \) has 2 conjugacy classes of elements of order 4, with Jordan forms \( J_4 \) and \( \text{diag}(J_3, J_1) \), and the second one does not meet \( Sp(4, 2) \). Since \( A_6 \cong Sp(4, 2)' \), we conclude that the class \( (4, 4) \) corresponds to the class \( \text{diag}(J_3, J_1) \). So \( g \) belongs to the intermediate subgroup \( H \cong SL(3, 2) \).

Let \( \tau \) be an irreducible representation of \( H \). Then \( \tau \) is a restriction of a rational representation of \( \hat{H} \), the algebraic group of type \( A_2 \). The irreducible representations of \( \hat{H} \) are labelled by their highest weights \( a_1\omega_1 + a_2\omega_2 \), where \( a_1, a_2 \) are nonnegative integers and \( \omega_1, \omega_2 \) are the fundamental weights. It is well known that \( \tau \) is a restriction of one of the four irreducible representations of \( \hat{H} \) labelled by \( 0, \omega_1, \omega_2, \omega_1 + \omega_2 \). The last one corresponds to the Steinberg module, whose restriction to \( H \) is projective, and so all Jordan blocks of \( g \) are of size 4. Two other representations are the natural and its dual. So the Jordan form of \( g \) on both of them is \( \text{diag}(J_3, J_1) \). Finally, corresponding to the zero highest weight we have the trivial representation.

Now, let \( \lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3 \) be the highest weight of \( \phi \). By a theorem of Smith [10] (also proved independently by R. Dipper), the restriction \( \phi|H \) contains a direct summand \( \tau \) with highest weight \( a_1\omega_1 + a_2\omega_2 \). From the previous paragraph, we may assume that at least one of \( a_1, a_2 \) is zero. By duality, the same is true for \( a_2, a_3 \). So we are left with the cases \( \lambda \in \{\omega_1, \omega_2, \omega_3, \omega_1 + \omega_3\} \). The last one is the adjoint representation of \( G \). Clearly, its restriction to \( H \) contains a composition factor isomorphic to the adjoint representation of \( H \). Since the last representation is projective, it is a direct summand. Hence this case is ruled out. The cases \( \lambda = \omega_1, \omega_3 \) are obvious. Finally, the module corresponding to \( \lambda = \omega_2 \) is the exterior square of the natural module. So its restriction to \( H \) is a direct sum of the natural and dual natural modules; hence the Jordan blocks of \( \phi(g) \) are of size 3.

Lemma 3.10. Let \( G = A_n \) or \( S_n \), and let \( g \in G \) be an element of order 4 containing at least three 4-cycles. Then \( \deg \phi(g) = 4 \) for any faithful irreducible representation \( \phi \) of \( G \).

Proof. We may assume that
\[
g = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12) \ldots
\]
Set \( h_j := (j, j + 4, j + 8) \) for \( j = 1, 2, 3, 4 \), and \( H := \langle h_1, h_2, h_3, h_4 \rangle \). Then \( H \) is an abelian 3-group and \( g \in N_G(H) \). Moreover, \( g^2 \not\in C_G(H) \); so the result follows from Lemma 2.8.

**Lemma 3.11.** Theorem 1.2 is true for \( G = 3 \cdot A_6 \) and \( 3 \cdot A_7 \).

**Proof.** For \( G = 3 \cdot A_7 \) see Lemma 3.6. Let \( G = 3 \cdot A_6 \). Then \( \dim \phi = 3 \) or 9; see [9]. In the former case, \( \deg \phi(g) = 3 \), since \( \deg \phi(g) < 3 \) implies \( \phi(g)^2 = 1 \), which is false. Let \( \dim \phi = 9 \). Observe that \( g^2 \) normalizes a cyclic group \( (c) \) of order 5. Set \( X := (g^2, c) \). Since \( g^2 c g^{-2} = c^{-1} \) and the multiplicity of every eigenvalue \( \alpha \neq 1 \) of \( \phi(c) \) is 2 (see [9]), it follows that \( \phi(X) \) has four composition factors of dimension 2 and one composition factor of dimension 1. Therefore Jord \( g^2 = \text{diag}(J_2, J_2, J_2, J_2, J_1) \), whence Jord \( g = \text{diag}(J_4, J_4, J_1) \).

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