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INTEGRABILITY OF THE CONTINUUM WAVELET KERNEL

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Abstract. We state and prove sufficient conditions for the absolute integrability of the inverse of the continuum wavelet kernel.

1. Introduction

In 1984 Grossman and Morlet [GM] formulated the continuous wavelet transform, rediscovering the Calderón reproducing formula [C]. The L^2 theory of the continuous wavelet transform is now well established and available in several books [D], [I], [KL], [M]. The history of Calderón's formula and other useful information is found in [FJW]. In addition to the L^2 convergence, one can also expect norm convergence in the spaces $C_0(\mathbf{R})$ and $L^p(\mathbf{R})$ for $1 \leq p < \infty$, in case the relevant kernel is integrable.

The purpose of this note is to formulate sufficient conditions for the integrablility of the relevant wavelet kernel. This will be satisfied in case i) the kernel is nonnegative, or ii) a certain condition of logarithmic integrability is satisfied. As soon as we have proved integrability of the wavelet kernel, then it is automatic that we have norm convergence in the Banach spaces $L^p(\mathbf{R})$, $1 \leq p < \infty$ and the space $C_0(\mathbf{R})$.

We give a self-contained treatment of the wavelet transform and its partial inverse. In particular, we demonstrate that the partial inverse transform can be defined in terms of a single cutoff parameter $\epsilon \downarrow 0$, corresponding to small scales. This is in contrast to the three-parameter cutoff that appears in [D] or [I]. A previous study of this problem [S] was carried out in case the basic continuum wavelet is integrable, but not necessarily square-integrable. This excludes the interesting example of the Shannon wavelet.

We also remark that, in the case of orthonormal wavelet series expansions, the projection operators that define the small-scale partial sums of the wavelet series may fail to converge in $L^1(\mathbf{R})$ or $C_0(\mathbf{R})$. This is demonstrated in detail for the Shannon wavelet in [HW], p. 229, where the relevant wavelet kernel is identical to the Dirichlet kernel for Fourier inversion on the line. For the continuum analogue of the Shannon wavelet, we show (Example 4) that the kernel is integrable, so that the small-scale partial sum operators are defined by convolution with an integrable kernel.

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2. Notation

The L^2 inner product of f, g is denoted $(f,g) = \int_{\mathbf{R}} f(x)\bar{g}(x)\,dx$, with $||f||_2 = \sqrt{(f,f)}$. The Fourier transform of $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ is defined by $\hat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-2\pi i\xi x}\,dx$. This map extends to $L^2(\mathbf{R})$ as a unitary mapping, in particular, $(f,g) = (\hat{f},\hat{g})$ for all $f, g \in L^2(\mathbf{R})$. The convolution of $f, g \in L^2(\mathbf{R})$ is the absolutely convergent integral $(f * g)(x) = \int_{\mathbf{R}} f(y)g(x-y)\,dy$, a continuous function that vanishes when $|x| \to \infty$. The reflection/conjugate of f is defined by $\tilde{f}(x) = \bar{f}(-x)$. The two-parameter dilate-translate of $\psi \in L^2(\mathbf{R})$ is defined by

$$\psi_{ab}(x) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right) \tag{1}$$

where $b \in \mathbf{R}, 0 \neq a \in \mathbf{R}$. Note that $||\psi_{ab}||_2 = ||\psi_{a0}||_2 = ||\psi||_2$, for all $b \in \mathbf{R}, 0 \neq a \in \mathbf{R}$.

3. The continuous wavelet transform

Definition 3.1. $\psi \in L^2(\mathbf{R})$ is a *normalized continuum wavelet* if $||\psi||_2 = 1$ and the Fourier transform satisfies

$$C_\psi := \int_{\mathbf{R}} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\,d\xi < \infty. \tag{2}$$

If, in addition, $\psi \in L^1(\mathbf{R})$, then a normalized continuum wavelet must also satisfy $\hat{\psi}(0) = \int_{\mathbf{R}} \psi = 0$, since $\hat{\psi}$ is continuous and $\hat{\psi}(0) \neq 0$ would contradict the convergence of the integral (2).

By rescaling the spatial coordinate, we may assume that both $||\psi||_2 = 1$ and $C_\psi = 1$. Indeed, for any $\psi \in L^2(\mathbf{R})$ with $C_\psi < \infty$, we can define

$$\hat{\psi}_{\text{new}}(\xi) = \frac{1}{\sqrt{C_\psi}}\hat{\psi}\left(\frac{\xi||\psi||_2}{C_\psi}\right)$$

and check that $||\psi_{\text{new}}||_2 = 1$ and $C_{\psi_{\text{new}}} = 1$. This will be referred to as a *renormalized continuum wavelet.*

Definition 3.2. The *wavelet transform* of $f \in L^2(\mathbf{R})$ is defined by the absolutely convergent integral

$$W_\psi f(a,b) = \int_{\mathbf{R}} f(y)\bar{\psi}_{ab}(y)\,dy = (f * \tilde{\psi}_{a0})(b). \tag{3}$$

The L^2 norm is written

$$N(a) = \left(\int_{\mathbf{R}} |W_\psi f(a,b)|^2 db\right)^{\frac{1}{2}} = ||W_\psi f(a,\cdot)||_2.$$

Lemma 3.1. *Let $\psi \in L^2(\mathbf{R})$ be a renormalized continuum wavelet and $f \in L^2(\mathbf{R})$. Then*

(i) $|W_\psi f(a,b)| \leq ||f||_2$.

(ii) $b \to W_\psi f(a,b)$ is continuous with $\lim_{|b|\to\infty} W_\psi f(a,b) = 0, \forall a \in \mathbf{R} \setminus \{0\}$.

(iii) For almost all $a \in \mathbf{R}, b \to W_\psi f(a,b)$ is $L^2(\mathbf{R})$ and the norm $N(a)$ satisfies

$$\int_{\mathbf{R}} \frac{N(a)^2}{|a|^2}\,da = \int_{\mathbf{R}} \left(\int_{\mathbf{R}} |W_\psi f(a,b)|^2\,db\right)\frac{da}{|a|^2} = ||f||_2^2. \tag{4}$$

Proof. The Fourier transform of $\tilde{\psi}_{a0}$ is $\sqrt{|a|}\bar{\hat{\psi}}(a\xi)$. From Parseval's identity we can write

$$\begin{aligned} W_\psi f(a,b) &= (f * \tilde{\psi}_{a0})(b) \\ &= \int_{\mathbf{R}} \hat{f}(\xi)\hat{\tilde{\psi}}_{a0}(\xi)e^{-2\pi i b\xi}\, d\xi \\ &= \int_{\mathbf{R}} \hat{f}(\xi)\sqrt{|a|}\bar{\hat{\psi}}(a\xi)e^{-2\pi i b\xi}\, d\xi. \end{aligned}$$

Property (i) follows from Cauchy-Schwarz and Parseval's identity. Since the product $\hat{f}\hat{\tilde{\psi}}$ is integrable, property (ii) follows from the properties of L^1 Fourier transforms, especially the Riemann-Lebesgue lemma. To prove (iii), we first assume that $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Then $\hat{f} \in L^\infty(\mathbf{R})$ and we can use Parseval to compute

$$\begin{aligned} ||W_\psi f||_2^2 &= \int_{\mathbf{R}} |W_\psi f(a,b)|^2 db \\ &= \int_{\mathbf{R}} |\hat{f}(\xi)|^2 |a| |\hat{\psi}(a\xi)|^2\, d\xi \\ &< \infty. \end{aligned}$$

In particular, $b \to W_\psi f(a,b) \in L^2(\mathbf{R})$ for *every* $a \in \mathbf{R} \setminus \{0\}$. Now

$$\begin{aligned} \int_{\mathbf{R}} ||W_\psi f||_2^2 \frac{da}{|a|^2} &= \int_R \frac{da}{|a|^2}\left(\int_{\mathbf{R}} |\hat{f}(\xi)|^2 |a| |\hat{\psi}(a\xi)|^2\, d\xi\right) \\ &= \int_{\mathbf{R}} |\hat{f}(\xi)|^2 \left(\int_{\mathbf{R}} \frac{|\hat{\psi}(a\xi)|^2\, da}{|a|}\right) \\ &= ||f||_2^2 \end{aligned}$$

where we have used the Fubini theorem to interchange the order of integration, together with another application of Parseval's identity. This proves (4), in case $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$; in particular, $a \to ||W_\psi f||^2$ is finite everywhere.

Now if $f \in L^2(\mathbf{R})$, we take a sequence $f_n \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ with $||f - f_n||_2 \to 0$. On the one hand, by Cauchy-Schwarz we have the pointwise bound

$$|W_\psi f(a,b) - W_\psi f_n(a,b)| \le ||f - f_n||_2 \to 0,$$

so that $W_\psi f_n$ converges uniformly to $W_\psi f$. On the other hand, applying (4) to $f_m - f_n$, we see that $W_\psi f_n$ is a Cauchy sequence in the Hilbert space $L^2(\mathbf{R}^2, da\, db/|a|^2)$. Hence there exists a limit F in this space, for which

$$\int_{\mathbf{R}}\int_{\mathbf{R}} |F(a,b)|^2 \frac{da\, db}{|a|^2} = \lim_n \int_{\mathbf{R}}\int_{\mathbf{R}} |W_\psi f_n(a,b)|^2 \frac{da\, db}{|a|^2} = \lim_n ||f_n||_2^2 = ||f||_2^2.$$

Now take a subsequence that converges almost everywhere. Along this subsequence we also have the uniform convergence to $W_\psi f$. Hence we conclude that a.e. $F = W_\psi f$. But $||f_n||_2 \to ||f||_2$. Taking the limit $n \to \infty$ in (4), we conclude that (4) holds for f. In particular, $\int_{\mathbf{R}} |W_\psi f(a,b)|^2\, db < \infty$ for almost all $a \in \mathbf{R}$, and the proof is complete. $\square$

The partial inverse transform is defined for $\epsilon > 0$ by

$$S_\epsilon f(x) = \int_{|a|>\epsilon} \left(\int_{\mathbf{R}} W_\psi f(a,b) \psi_{ab}(x)\, db \right) \frac{da}{|a|^2} = \int_{|a|>\epsilon} (W_\psi f) * \psi_{a0} \frac{da}{|a|^2}. \tag{5}$$

Theorem 3.2. *Let $\psi \in L^2(\mathbf{R})$ be a renormalized continuum wavelet. Let $\epsilon > 0$, $f \in L^2(\mathbf{R})$ and $x \in \mathbf{R}$. Then the integral* (5) *converges absolutely and we have the pointwise bound*

$$|S_\epsilon f(x)| \le \sqrt{\frac{2}{\epsilon}}\, ||f||_2. \tag{6}$$

Furthermore, $S_\epsilon f \in L^2(\mathbf{R})$ and $||S_\epsilon f - f||_2 \to 0$ when $\epsilon \to 0$.

Proof. From Cauchy-Schwarz, we have the pointwise bound

$$|(W_\psi f) * \psi_{a0}(x)| \le ||W_\psi f(a,\cdot)||_2 = N(a).$$

Thus

$$\begin{aligned} |S_\epsilon f(x)| &\le \int_{|a|>\epsilon} \frac{N(a)}{|a|^2} da \\ &\le \left(\int_{|a|>\epsilon} \frac{|N(a)|^2}{|a|^2} da \right)^{\frac{1}{2}} \left(\int_{|a|>\epsilon} \frac{da}{|a|^2} \right)^{\frac{1}{2}} \\ &\le ||f||_2 \sqrt{\frac{2}{\epsilon}} \end{aligned}$$

where we have used the isometry property (4) in the last line.

To prove that $S_\epsilon f \in L^2(\mathbf{R})$, first assume that $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Then from the properties of convolution operators on the real line, we have

$$||(W_\psi f) * \psi_{a0}||_2 \le ||W_\psi f||_1 = ||\psi_{a0} * f||_1 \le ||f||_1.$$

Applying the generalized Minkowski inequality to (5) we have

$$||S_\epsilon f||_2 \le ||f||_1 \int_{|a|>\epsilon} \frac{da}{|a|^2} < \infty$$

for each $\epsilon > 0$. To obtain the desired estimate, we multiply (5) by $g \in L^2(\mathbf{R})$ to obtain

$$(S_\epsilon f, g) = \int_{|a|>\epsilon} \int_{\mathbf{R}} W_\psi f(a,b) \bar{W}_\psi g(a,b) \frac{db\, da}{|a|^2} \tag{7}$$

and

$$\begin{aligned} ||S_\epsilon f||_2 &= \sup_{g \ne 0} \frac{|(S_\epsilon f, g)|}{||g||_2} \\ &\le \left(\int_{|a|>\epsilon} |W_\psi f|^2 \frac{db\, da}{|a|^2} \right)^{\frac{1}{2}} \\ &\le ||f||_2. \end{aligned} \tag{8}$$

If $f \in L^2(\mathbf{R})$, let $f_n \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ with $||f - f_n||_2 \to 0$. Then (8) shows that $||S_\epsilon f_n - S_\epsilon f_m||_2 \le ||f_n - f_m||_2 \to 0$ when $m, n \to \infty$. Hence $S_\epsilon f_n$ converges in

$L^2(\mathbf{R})$; in particular, a subsequence converges pointwise a.e. But the pointwise bound (6) shows that $|S_\epsilon f(x) - S_\epsilon f_n(x)| \le \sqrt{2/\epsilon}\,||f - f_n||_2 \to 0$, so that $S_\epsilon f_n(x)$ converges to $S_\epsilon f(x)$ uniformly and in $L^2(\mathbf{R})$ when $n \to \infty$ with ϵ fixed. In particular, $S_\epsilon f \in L^2(\mathbf{R})$ with $||S_\epsilon f||_2 \le ||f||_2$.

Finally, to prove the L^2 convergence when $\epsilon \to 0$, we use the L^2 isometry (4) to write

$$\begin{aligned}
(f,g) &= \int_{\mathbf{R}}\int_{\mathbf{R}} W_\psi f\, \bar{W}_\psi g \frac{db\,da}{|a|^2},\\
(f - S_\epsilon f, g) &= \int_{|a|<\epsilon}\left(\int_{\mathbf{R}} W_\psi f\, \bar{W}_\psi g\, db\right)\frac{da}{|a|^2},\\
|(f - S_\epsilon f, g)| &\le \left(\int_{|a|<\epsilon}\int_{\mathbf{R}} |W_\psi f|^2 \frac{db\,da}{|a|^2}\right)^{\frac{1}{2}} ||g||_2,\\
||f - S_\epsilon f||_2 &= \sup_{g\neq 0}\frac{|(f - S_\epsilon f, g)|}{||g||_2}\\
&\le \left(\int_{|a|<\epsilon}\int_{\mathbf{R}} |W_\psi f|^2 \frac{db\,da}{|a|^2}\right)^{\frac{1}{2}},
\end{aligned}$$

which tends to zero by the dominated convergence theorem, completing the proof. □

4. Kernel of the inverse transform

In order to study the partial inverse transform we write

$$S_\epsilon f(x) = \int_{\mathbf{R}} K_\epsilon(x,y) f(y)\, dy. \tag{9}$$

This can be reduced to a more explicit form by making the two-dimensional change of variable

$$\alpha = (x-b)/a, \qquad \beta = (y-b)/a$$

whose Jacobian is $\partial(\alpha,\beta)/\partial(a,b) = |x-y|/|a||^3$ for $x \neq y$, $a \neq 0$. This maps horizontal strips centered on the a-axis to 45 degree strips centered on the line $\alpha = \beta$, so that

$$\begin{aligned}
K_\epsilon(x,y) &= \frac{1}{|x-y|}\int_{\mathbf{R}}\int_{\alpha-\frac{|x-y|}{\epsilon}}^{\alpha+\frac{|x-y|}{\epsilon}} \psi(\alpha)\bar{\psi}(\beta) d\alpha d\beta\\
&= \frac{1}{|x-y|}\int_{-\frac{|x-y|}{\epsilon}}^{\frac{|x-y|}{\epsilon}}\left(\int_{\mathbf{R}} \psi(\alpha)\bar{\psi}(\alpha - z) d\alpha\right) dz,
\end{aligned}$$

which can be abbreviated as

$$K_\epsilon(x,y) = \epsilon^{-1} K((x-y)/\epsilon), \qquad K(x) := \frac{1}{|x|}\int_{-|x|}^{|x|} (\psi * \tilde{\psi})(z)\, dz. \tag{10}$$

K is a bounded continuous function that tends to zero at infinity. Its Fourier transform, in the sense of distributions, is

$$\begin{aligned} \hat{K}(\xi) &= \int_{\mathbf{R}} K(x) e^{-2\pi i x\xi}\, dx \\ &= \int_{|a|>1} \frac{|\hat{\psi}(a\xi)|^2}{|a|}\, da. \end{aligned} \tag{11}$$

5. Some computable examples

We illustrate the computation of the wavelet kernel by means of some examples. In some of these examples an additional normalization is necessary to ensure that both $||\psi||_2 = 1$ and $C_\psi = 1$.

Example 1. The Haar wavelet is defined by $\psi(x) = 1_{[0,\frac{1}{2})}(x) - 1_{[\frac{1}{2},1)}(x)$. Direct computation using (10) gives the nonnegative kernel

$$\begin{aligned} K(x) &= 2 - 3|x| \qquad \left(0 \le |x| \le \frac{1}{2}\right) \\ &= \frac{(1-|x|)^2}{|x|} \qquad \left(\frac{1}{2} \le |x| \le 1\right) \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Example 2. A Gaussian wavelet is defined by $\psi(x) = Cxe^{-\pi x^2}$. The constant C is chosen so that $C_\psi = 1$. The Fourier transform is $\hat{\psi}(\xi) = -iC\xi e^{-\pi\xi^2}$. The kernel is obtained from (11) as

$$\hat{K}(\xi) = 2\pi \int_{|a|>1} a\xi^2 e^{-2\pi a^2\xi^2}\, da = e^{-2\pi\xi^2},$$

which is the Fourier transform of a Gaussian density; thus $K(x) = 2^{-1/2}e^{-\pi x^2/2}$. In this case we have $K(x) \ge 0$.

Example 3. A Mexican hat wavelet is defined by $\hat{\psi}(\xi) = C\xi^2 e^{-\pi\xi^2}$. Direct computation of the Fourier transform shows that

$$K(x) = \frac{1 + 2\pi - 2\pi x^2}{\sqrt{2}} e^{-\pi x^2/2}.$$

This provides an example where K is integrable but changes sign.

Example 4. A continuum "Shannon wavelet" is defined by

$$\hat{\psi}(\xi) = 1_{[\frac{1}{4},\frac{1}{2}](|\xi|)}, \qquad \psi(t) = \frac{\sin \pi t - \sin(\pi t/2)}{\pi t} \notin L^1(\mathbf{R}).$$

We have $\psi * \tilde{\psi} = \psi = \tilde{\psi}$ and

$$\begin{aligned} x\,K(x) &= \int_{-x}^{x} (\psi * \tilde{\psi})(t)\,dt = 2\int_0^x \frac{\sin \pi t}{\pi t}\,dt - 2\int_0^x \frac{\sin(\pi t/2)}{\pi t}\,dt \\ &= 2\int_0^x \frac{\sin \pi t}{\pi t}\,dt - 2\int_0^{x/2} \frac{\sin \pi t}{\pi t}\,dt \\ &= 2\int_{x/2}^x \frac{\sin \pi t}{\pi t} \\ &= O\left(\frac{1}{|x|}\right), \quad |x| \to \infty. \end{aligned}$$

Hence $K(x) = O(|x|^{-2}), |x| \to \infty$; thus $K \in L^1(\mathbf{R})$, but $\psi \notin L^1(\mathbf{R})$. The same conclusions apply to the continuum wavelet defined by $\hat{\psi} = 1_S$, where S is a finite union of intervals with $0 \notin \bar{S}$.

6. General properties of the wavelet kernel

In order to discuss the general properties of the wavelet kernel, we work with the Fourier transform of the kernel

$$\Psi(\xi) = \hat{K}(\xi) = \int_{|\nu| \geq |\xi|} \frac{|\hat{\psi}(\nu)|^2}{|\nu|}\,d\nu. \tag{12}$$

Proposition 6.1. *Let ψ be a renormalized continuum wavelet. Then $\Psi \in L^1(\mathbf{R})$, Ψ is continuous with $\Psi(0) = 1$ and $\Psi(\xi) \to 0$ when $|\xi| \to \infty$. In particular, K is continuous and tends to zero at infinity.*

Proof.

$$\begin{aligned} \int_{\mathbf{R}} |\Psi(\xi)|\,d\xi &= \int_{\mathbf{R}} \left(\int_{|\nu| \geq |\xi|} \frac{|\hat{\psi}(\nu)|^2}{|\nu|}\,d\nu \right) d\xi \\ &= \int_{\mathbf{R}} \frac{|\hat{\psi}(\nu)|^2}{|\nu|} \left(\int_{|\xi| < \nu} d\xi \right) d\nu \\ &= 2\int_{\mathbf{R}} |\hat{\psi}(\nu)|^2\,d\nu, \end{aligned}$$

which proves that $\Psi \in L^1(\mathbf{R})$; (12) shows that Ψ is continuous with $\Psi(0) = 1$. To prove the behavior for large $|\xi|$, note that $|\hat{\psi}|^2 \in L^1(\mathbf{R})$, so that the dominated convergence theorem shows that $\Psi(\xi) \to 0$ when $|\xi| \to \infty$. It follows that we can recover K by a pointwise Fourier inversion

$$K(x) = \int_{\mathbf{R}} \Psi(\xi) e^{2\pi i x \xi}\,d\xi;$$

in particular, K is a continuous function that tends to zero at infinity, by the Riemann-Lebesgue lemma. □

The following theorems give sufficient conditions for integrability of the wavelet kernel.

Theorem 6.2. *Suppose that ψ is a renormalized continuum wavelet for which the associated wavelet kernel is nonnegative: $K(x) \geq 0$ for $x \in \mathbf{R}$. Then $\int_{\mathbf{R}} K(x)\,dx =$*

1*; in particular,* $K \in L^1(\mathbf{R})$*. Hence for any bounded uniformly continuous* f*, we have* $||S_\epsilon f - f||_\infty \to 0$ *when* $\epsilon \to 0$ *. If* $f \in L^p(\mathbf{R}), 1 \le p < \infty$*, then* $||S_\epsilon f - f||_p \to 0$*.*

Proof. It is a classical fact that a nonnegative function with a bounded distributional Fourier transform is integrable. We can also argue directly, as follows: applying Parseval's identity to the Fejér kernel, we have

$$\int_{-M}^{M} \left(1 - \frac{|x|}{M}\right) K(x)\, dx = M \int_{\mathbf{R}} \left(\frac{\sin M\pi\xi}{M\pi\xi}\right)^2 \Psi(\xi)\, d\xi.$$

But Ψ is bounded and continuous at $\xi = 0$ with $\Psi(0) = 1$. Fatou's lemma and the properties of the Fejér kernel then yield

$$\int_R K(x)\, dx \le \lim_M \int_{-M}^{M} \left(1 - \frac{|x|}{M}\right) K(x)\, dx = 1.$$

The equality $\int_{\mathbf{R}} K(x)\, dx = 1$ follows from another application of the dominated convergence theorem. □

Theorem 6.3. *Suppose that the renormalized continuum wavelet satisfies the condition that* $\int_{|x|\ge 1} \log|x|\, |\psi * \tilde{\psi}(x)|\, dx < \infty$*. Then* $K \in L^1(\mathbf{R})$ *and* $\int_{\mathbf{R}} K(x)\, dx = 1$*. Hence for any bounded uniformly continuous* f*, we have* $||S_\epsilon f - f||_\infty \to 0$ *when* $\epsilon \to 0$ *. If* $f \in L^p(\mathbf{R}), 1 \le p < \infty$*, then* $||S_\epsilon f - f||_p \to 0$*.*

Proof. The integrability condition implies that $\psi * \tilde{\psi} \in L^1(\mathbf{R})$, in particular, that $\psi \in L^1(\mathbf{R})$, by Fubini's theorem. Now $\psi \in L^1(\mathbf{R})$ implies that $\int_{\mathbf{R}} (\psi * \tilde{\psi}) = \int_{\mathbf{R}} \psi \int_{\mathbf{R}} \tilde{\psi} = 0$. We use this to write the equivalent formulas

$$K(x) = \frac{1}{|x|} \int_{-|x|}^{|x|} (\psi * \tilde{\psi})(z)\, dz \tag{13}$$

$$= -\frac{1}{|x|} \int_{|z|\ge|x|} (\psi * \tilde{\psi})(z)\, dz. \tag{14}$$

Since K is continuous, we have that $\int_{|x|\le 1} |K(x)|\, dx < \infty$.

On the other hand, (14) gives

$$\int_{|x|\ge 1} K(x)\, dx \le \int_{|z|\ge 1} |(\psi * \tilde{\psi})|(z) \int_1^{|z|} \frac{dx}{|x|} = \int_{|z|\ge 1} \log|z|\, |(\psi * \tilde{\psi})|(z)\, dz < \infty,$$

which proves that $K \in L^1(\mathbf{R})$. To check the normalization, we use the absolutely convergent integral

$$\Psi(\xi) = \hat{K}(\xi) = \int_{|\nu|\ge|\xi|} \frac{|\hat{\psi}(\nu)|^2}{|\nu|} d\nu$$

from which it follows that $\int_{\mathbf{R}} K = \Psi(0) = 1$, as required. □

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