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A GENERALIZATION OF A RESULT OF KAZHDAN AND LUSZTIG

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Abstract. Kazhdan and Lusztig showed that every topologically nilpotent, regular semisimple orbit in the Lie algebra of a simple, split group over the field $\mathbb{C}((t))$ is, in some sense, close to a regular nilpotent orbit. We generalize this result to a setting that includes most quasisplit p-adic groups.

1. Introduction

Suppose G is the group of $\mathbb{C}((t))$-rational points of a simple, split, algebraic $\mathbb{C}((t))$-group. In [7, Corollary 4.1], Kazhdan and Lusztig show that if Z is a topologically nilpotent, regular semisimple element of the Lie algebra of G and x is a special vertex in the Bruhat-Tits building of G, then there exists $g \in G$ such that the image of ${}^gZ$ ($= \mathrm{Ad}(g)Z$) in the complex Lie algebra associated to x is regular nilpotent. We generalize this result to a setting that includes most quasisplit groups over p-adic fields.

Motivation. Suppose k is a p-adic field, and G is the group of k-rational points of a connected reductive k-quasisplit group.

Our motivation for considering this problem came from harmonic analysis on p-adic groups; we were interested in what role the Bruhat-Tits building might play in stability questions. For example, we learned from lectures of Kottwitz that if S denotes a Kostant section [9, §2.4] in $\mathfrak{g}$, then the map $G \times S \longrightarrow \mathfrak{g}$ given by $(g, Z) \longmapsto {}^gZ$ is a submersion. This implies that for a fixed regular nilpotent element X in $\mathfrak{g}$ there is a neighborhood $U \subset \mathfrak{g}$ of X with the following property: every stable regular semisimple orbit that intersects U nontrivially contains a unique G-orbit that intersects U nontrivially. The main result of this paper is the determination of a natural neighborhood of X that has this property with respect to all regular semisimple topologically nilpotent elements.

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This paper. Let k denote a complete field with nontrivial discrete valuation and residue field $\mathfrak{f}$. We suppose that $\mathfrak{f}$ is perfect. All extensions of k that we consider will lie in a fixed algebraic closure $\overline{k}$ of k. Let K denote the maximal unramified extension of k.

Let $\mathbf{G}$ denote a connected, reductive group with Lie algebra $\mathbf{\mathfrak{g}}$. Let $G = \mathbf{G}(k)$ and $\mathfrak{g} = \mathbf{\mathfrak{g}}(k)$. Let $\mathfrak{g}^{\text{r.s.s.}}$ denote the set of regular, semisimple elements in $\mathfrak{g}$. For any algebraic extension E/k of finite ramification index, and any facet F in the Bruhat-Tits building $\mathcal{B}(\mathbf{G}, E)$ of $\mathbf{G}(E)$, one can define a parahoric subgroup $\mathbf{G}(E)_F$ of $\mathbf{G}(E)$ [5]. Let $\mathbf{G}(E)_F^+$ denote the pro-unipotent radical of $\mathbf{G}(E)_F$. As in [2], we define lattices $\mathbf{\mathfrak{g}}(E)_F$ and $\mathbf{\mathfrak{g}}(E)_F^+$ in $\mathbf{\mathfrak{g}}(E)$; for any $x \in \mathcal{B}(\mathbf{G}, E)$, we define $\mathbf{\mathfrak{g}}(E)_x$ and $\mathfrak{g}(E)_x^+$ to be $\mathfrak{g}(E)_F$ and $\mathfrak{g}(E)_F^+$, respectively, where F is the unique facet in $\mathcal{B}(\mathbf{G}, E)$ containing x; and we define the $\mathbf{G}(E)$-invariant set $\mathbf{\mathfrak{g}}(E)_{0^+}$ to be $\bigcup_F \mathbf{\mathfrak{g}}(E)_F^+$, where the union is taken over all facets F in $\mathcal{B}(\mathbf{G}, E)$. When $E = k$, we denote the building by $\mathcal{B}(G)$, the reduced building by $\mathcal{B}^{\text{red}}(G)$, and the above objects by G_F, G_F^+, $\mathfrak{g}_F$, $\mathfrak{g}_F^+$, $\mathfrak{g}_x$, $\mathfrak{g}_x^+$, and $\mathfrak{g}_{0^+}$. This last object is the set of *topologically nilpotent* elements in $\mathfrak{g}$. We can identify G_F/G_F^+ with the group of $\mathfrak{f}$-points of a connected reductive $\mathfrak{f}$-group G_F. Let $\mathsf{L}_F := \text{Lie}(\mathsf{G}_F)$. We identify $\mathsf{L}_F(\mathfrak{f})$ with $\mathfrak{g}_F/\mathfrak{g}_F^+$.

Suppose $\mathbf{G}$ is k-quasisplit and let $X \in \mathfrak{g}$ be regular nilpotent (see §3). Suppose that we can complete X to an $\mathfrak{sl}_2(k)$-triple (Y, H, X). Under the hypotheses of Corollary 4.5 we can find a unique minimal facet $F \subset \mathcal{B}(G)$ such that the image of F in $\mathcal{B}^{\text{red}}(G)$ is a special vertex and $Y, H, X \in \mathfrak{g}_F$. The main result of this paper is:

Proposition 1. *Suppose that all of the hypotheses of §2 and of Corollary 4.5 are valid, that $\mathbf{G}$, X, and F are as above, and that $Z \in \mathfrak{g}^{\text{r.s.s.}}$. Then $Z \in \mathfrak{g}_{0^+}$ if and only if there is some $g \in \mathbf{G}(\overline{k})$ such that ${}^gZ \in X + \mathfrak{g}_F^+$. Moreover, if $g' \in \mathbf{G}(\overline{k})$ is such that ${}^{g'}Z \in X + \mathfrak{g}_F^+$, then ${}^{g'}Z = {}^{\ell g}Z$ for some $\ell \in G_F^+$.*

In other words, the coset $X + \mathfrak{g}_F^+$ picks out a unique G-orbit in every topologically nilpotent, regular semisimple, stable orbit in $\mathfrak{g}$. Note that when $k = \mathbb{C}(\!(t)\!)$ and $\mathbf{G}$ is split and simple, we recover Corollary 4.1 of [7].

To prove Proposition 1, we will need some additional notation. Given a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$, we have the torus $S = \mathbf{S}(k)$ in G and the corresponding apartment $\mathcal{A}(S) = \mathcal{A}(\mathbf{S}, k)$ in $\mathcal{B}(G)$. For any subgroup $\mathbf{H} \subset \mathbf{G}$, let $C_{\mathbf{G}}(\mathbf{H})$ denote the centralizer of $\mathbf{H}$ in $\mathbf{G}$. For $g \in \mathbf{G}$, let ${}^g\mathbf{H} = g\mathbf{H}g^{-1}$. For $Y \in \mathfrak{g}$, we denote the centralizer of Y in $\mathfrak{g}$ by $C_{\mathfrak{g}}(Y)$. For $Z \in \mathfrak{g}$, let $\mathbf{O}_Z$ denote the $\mathbf{G}$-orbit of Z in $\mathfrak{g}$. Recall that for every algebraic extension E/k,

$$\mathbf{O}_Z(E) = {}^{\mathbf{G}(\overline{k})}Z \cap \mathbf{\mathfrak{g}}(E) = \bigsqcup_{Z'} {}^{\mathbf{G}(E)}Z',$$

where $Z' \in \mathbf{\mathfrak{g}}(E)$ ranges over the elements of $\mathbf{O}_Z(E)$ up to $\mathbf{G}(E)$-conjugacy.

2. Hypotheses

In this section, we list various properties which we require. Under some restrictions on $\mathbf{G}$ and k, all of these hypotheses are valid. In particular, they are all true when $\mathfrak{f}$ has characteristic zero.

The first hypothesis is valid whenever the characteristic of $\mathfrak{f}$ is either zero or larger than some constant that can be determined by looking at the absolute root datum of $\mathbf{G}$. For more information, see [6].

Hypothesis 1. Let $X \in \mathfrak{g}$ be regular nilpotent. We can complete X to an $\mathfrak{sl}_2(k)$-triple (Y, H, X), produce a maximal k-split torus $\mathbf{S}$ in $\mathbf{G}$, and find a point $x \in \mathcal{A}(\mathbf{S}, k)$, such that $H \in \mathrm{Lie}(\mathbf{S})(k)$, $Y, H, X \in \mathfrak{g}_x$, and, for all finite extensions E/k,

$$^{\mathbf{G}(E)_x^+}(X + C_{\mathfrak{g}(E)_x^+}(Y)) = X + \mathfrak{g}(E)_x^+.$$

When $G = \mathbf{SL}_2(\mathbb{Q}_2)$, this hypothesis fails, and so does Proposition 1.

The remaining hypotheses are valid whenever k has characteristic zero.

Hypothesis 2. Let X and $\mathbf{S}$ be as in Hypothesis 1. Let $\mathbf{Z} = C_{\mathbf{G}}(\mathbf{S})$. (Since $\mathbf{G}$ is k-quasisplit, $\mathbf{Z}$ is a maximal k-torus.) For any algebraic extension E/k over which $\mathbf{Z}$ splits, if $Z \in \mathrm{Lie}(\mathbf{Z})(E)$ is regular semisimple, then $X + Z$ is $\mathbf{G}(E)$-conjugate to Z.

Hypothesis 3. Suppose $Z \in \mathfrak{g}^{\text{r.s.s.}}$. For all $g \in \mathbf{G}(\overline{k})$ such that ${}^gZ \in \mathfrak{g}(K)$, there exists $g' \in \mathbf{G}(K)$ such that ${}^{g'}Z = {}^gZ$.

When the characteristic of k is not a "torsion" prime for $\mathbf{G}$, then the centralizer of Z in $\mathbf{G}$ is connected [15]. So this hypothesis follows immediately from Theorem III.2.3.1′ of [11]. (See also Remark 1 in loc. cit.)

Hypothesis 4. Let X and Y be as in Hypothesis 1. For any algebraic extension E/k, if $Z \in \mathfrak{g}(E)^{\text{r.s.s.}}$, then the set $(X + C_{\mathfrak{g}(E)}(Y)) \cap \mathbf{O}_Z(E)$ consists of one element.

This last hypothesis asserts the existence of a Kostant section (see, for example, [8, Theorem 0.10] and [9, §2.4 and §4.3]).

3. Some notation and results in a general setting

In this section only, let k be any field.

General definitions. The term *Levi subgroup* will mean a rational Levi factor of a rational parabolic subgroup; a *Levi subalgebra* means the Lie algebra of a Levi subgroup.

If L is a Levi subgroup of G, let (L) denote the set of all subgroups of G that are G-conjugate to L. If M is another Levi subgroup of G, we write $(L) \leq (M)$ provided that ${}^gL \subseteq M$ for some $g \in G$.

We call an element of the Lie algebra of a reductive group *distinguished* provided that it is nilpotent and does not lie in any proper Levi subalgebra. Similarly, an orbit in such a Lie algebra is said to be *distinguished* if some (hence any) element of it is distinguished.

For any k-group $\mathbf{H}$, let $\mathbf{X}_*(\mathbf{H})$ denote the set of one-parameter subgroups of $\mathbf{H}$, and let $\mathbf{X}_*^k(\mathbf{H})$ denote the subset of k-rational elements.

If $\mathbf{H}$ is connected and reductive, then $\lambda \in \mathbf{X}_*^k(\mathbf{H})$ determines a rational parabolic subgroup $\mathbf{P}_\lambda$ with rational Levi decomposition $\mathbf{P}_\lambda = \mathbf{M}_\lambda \mathbf{N}_\lambda$. Specifically, $\mathbf{P}_\lambda$ (resp. $\mathbf{M}_\lambda$, $\mathbf{N}_\lambda$) consists of those elements $g \in \mathbf{H}$ such that $\lim_{t \to 0} {}^{\lambda(t)}g$ exists (resp. $= g$, $= 1$). Note that $\mathbf{M}_\lambda = C_{\mathbf{H}}(\lambda)$, which is connected (by [13, Theorem 6.4.7]).

We will call an element $u \in \mathbf{H}(k)$ *unipotent* if there is some $\lambda \in \mathbf{X}_*^k(\mathbf{H})$ such that $\lim_{t \to 0} {}^{\lambda(t)}u = 1$. Similarly, we will call an element $X \in \mathrm{Lie}(\mathbf{H})(k)$ *nilpotent* if there is some $\lambda \in \mathbf{X}_*^k(\mathbf{H})$ such that $\lim_{t\to 0} {}^{\lambda(t)}X = 0$. The terms "unipotent" and "nilpotent" are sometimes given other definitions. See §2.5 of [2] for a discussion.

Comments on regular nilpotent elements. In this subsection, we discuss some results concerning regular nilpotent elements in $\mathfrak{g}$. Undoubtedly, these results are well known to the experts, but we could not find a reference.

Suppose $\mathbf{B} \subset \mathbf{G}$ is a rational Borel subgroup and $\mathbf{S} \subset \mathbf{G}$ is a maximal k-split torus contained in $\mathbf{B}$. Then we have associated sets $\Phi^+(\mathbf{S},\mathbf{B},\mathbf{G})$ and $\Delta(\mathbf{S},\mathbf{B},\mathbf{G})$ of positive and simple roots, respectively. Fixing an order on $\Phi^+(\mathbf{S},\mathbf{B},\mathbf{G})$, we may write each u in the unipotent radical uniquely in the form $u = \prod_{\alpha\in\Phi^+(\mathbf{S},\mathbf{B},\mathbf{G})} u_\alpha$, where each u_α belongs to the root group corresponding to α. For any such element u, u_α will always denote the factor of u associated to α. No statement that we make concerning u_α will depend on the ordering of the roots. Similarly, for any X in the Lie algebra of the unipotent radical of $\mathbf{B}$, let X_α denote the projection of X onto the α-eigenspace in $\mathfrak{g}$.

Lemma 3.1. *Suppose* $\mathbf{G}$ *is a k-quasisplit group and $u \in G$ is unipotent. If u is regular, then there is a unique rational Borel subgroup* $\mathbf{B} \subset \mathbf{G}$ *such that* $B = \mathbf{B}(k)$ *contains u. Moreover, if* $\mathbf{B}$ *is a rational Borel subgroup such that $u \in B$, then u is regular if and only if $u_\alpha \neq 1$ for all* $\alpha \in \Delta(\mathbf{S},\mathbf{B},\mathbf{G})$*, where* $\mathbf{S}$ *is any maximal k-split torus in* $\mathbf{B}$.

Proof. Since u belongs to the group of k-rational points of the derived group of $\mathbf{G}$, it is enough to assume that $\mathbf{G}$ is semisimple. From [14, Lemma 3.2 and Theorem 3.3], u is regular if and only if it is contained in exactly one Borel subgroup $\mathbf{B}$ of $\mathbf{G}$.

Suppose u is regular. Let $\mathbf{B}$ denote the unique Borel subgroup of $\mathbf{G}$ that contains u. By our definition of unipotent, u is contained in the unipotent radical of some rational parabolic subgroup $\mathbf{P}$. Since $\mathbf{G}$ is k-quasisplit, there exists a rational Borel subgroup $\mathbf{B}' \subset \mathbf{P}$ such that $u \in \mathbf{B}'(k)$. By uniqueness, $\mathbf{B} = \mathbf{B}'$.

We now consider the final statement of the lemma. Suppose that $\mathbf{B}$ is a rational Borel subgroup of $\mathbf{G}$ such that $u \in \mathbf{B}(k)$. Let $\mathbf{S}$ be a maximal k-split torus of $\mathbf{G}$ in $\mathbf{B}$ and let $\mathbf{T} = C_{\mathbf{G}}(\mathbf{S})$. If $\mathbf{U}$ denotes the unipotent radical of $\mathbf{B}$, then $\mathbf{B} = \mathbf{TU}$ is a rational Levi factorization of $\mathbf{B}$ and $\mathbf{T}$ is a maximal k-torus in $\mathbf{G}$. From Lemma 3.2 of [14], $u_\beta \neq 1$ for all simple $\beta \in \Delta(\mathbf{T},\mathbf{B},\mathbf{G})$ if and only if u is regular. Let E be a Galois splitting field for $\mathbf{G}$ over k. Then each $\alpha \in \Delta(\mathbf{S},\mathbf{B},\mathbf{G})$ corresponds to a $\mathrm{Gal}(E/k)$-orbit in $\Delta(\mathbf{T},\mathbf{B},\mathbf{G})$. Thus, u is regular if and only if $u_\alpha \neq 1$ for all $\alpha \in \Delta(\mathbf{S},\mathbf{B},\mathbf{G})$. □

To transfer the above result to the Lie algebra, we will need to assume the following hypothesis.

Hypothesis E. There is a $\mathbf{G}$-equivariant k-isomorphism from the unipotent variety in $\mathbf{G}$ to the nilpotent variety in $\mathfrak{g}$.

It follows from work of Springer [12] that Hypothesis E holds under certain mild restrictions on $\mathbf{G}$ and k.

Corollary 3.2. *Suppose* $\mathbf{G}$ *is a k-quasisplit group, $X \in \mathfrak{g}$ is nilpotent, and Hypothesis E is true for* $\mathbf{G}$ *and k. If X is regular, then there is a unique rational Borel subgroup* $\mathbf{B} \subset \mathbf{G}$ *such that the nilradical of the Lie algebra of* $B = \mathbf{B}(k)$ *contains X. Moreover, if* $\mathbf{B}$ *is a rational Borel subgroup such that the nilradical of the Lie algebra of* $\mathbf{B}$ *contains X, then X is regular if and only if $X_\alpha \neq 0$ for all* $\alpha \in \Delta(\mathbf{S},\mathbf{B},\mathbf{G})$*, where* $\mathbf{S}$ *is any maximal k-split torus in* $\mathbf{B}$. □

Corollary 3.3. *Suppose X is a regular nilpotent element in $\mathfrak{g}$ and $\lambda \in \mathbf{X}_*^k(\mathbf{G})$ is a one-parameter subgroup such that $\lim_{t\to 0} {}^{\lambda(t)}X = 0$. If Hypothesis E is true for $\mathbf{G}$ and k, then $\lambda \in \mathbf{X}_*^k(\mathbf{B})$, where $\mathbf{B}$ is the unique rational Borel subgroup containing X in its Lie algebra. In particular, $C_{\mathbf{G}}(\lambda)$ is a maximal k-torus.*

Proof. As usual, λ determines a rational parabolic subgroup $\mathbf{P}_\lambda$ with rational Levi decomposition $\mathbf{P}_\lambda = \mathbf{M}_\lambda \mathbf{N}_\lambda$, where $\mathbf{M}_\lambda = C_{\mathbf{G}}(\lambda)$. Since $\lim_{t\to 0} {}^{\lambda(t)}X = 0$, we must have $X \in \mathrm{Lie}(\mathbf{N}_\lambda)$. Since there is a unique rational Borel subgroup containing X in its Lie algebra, it follows that $\mathbf{P}_\lambda$ is this Borel subgroup. □

4. Some comments on the parametrization of nilpotent orbits

We now return to our assumption that k is complete with respect to a nontrivial discrete valuation and has perfect residue field $\mathfrak{f}$.

Remark 4.1. The hypotheses of [6, §4.2] are enough to guarantee that Hypothesis E holds for $\mathbf{G}$ and k and also for G_F and $\mathfrak{f}$ for every facet F.

Some notation. Let

$$I^d := \left\{ (F, e) \,\middle|\, F \text{ is a facet in } \mathcal{B}(G), \text{ and } e \in \mathsf{L}_F(\mathfrak{f}) \text{ is distinguished} \right\}.$$

Under some hypotheses on k and $\mathbf{G}$ (see [6, §4.2]), to each pair $(F, e) \in I^d$ we can associate a nilpotent orbit $\mathcal{O}(F, e)$ in $\mathfrak{g}$ such that $\mathcal{O}(F, e)$ is the unique nilpotent orbit of minimal dimension that intersects e nontrivially [6, Lemma 5.3.3].

For any $\mathfrak{sl}_2(k)$-triple (Y, H, X) in $\mathfrak{g}$, we have the set

$$\mathcal{B}(Y, H, X) := \left\{ x \in \mathcal{B}(G) \,\middle|\, Y, H, X \in \mathfrak{g}_x \right\}.$$

This set is closed, convex, nonempty, and a union of facets [6, §5.1].

Suppose $\mathbf{S}$ is a maximal k-split torus in $\mathbf{G}$. Following [10], we associate to any facet $F \subset \mathcal{A}(\mathbf{S}, k)$ the Levi subgroup $M(F, \mathbf{S})$ that is generated by $C_{\mathbf{G}}(\mathbf{S})(k)$ and the root groups $U_{\dot\psi}(k)$, where $\dot\psi$ is the gradient of an affine root ψ of $\mathbf{G}$ with respect to $\mathbf{S}$, k, and a nontrivial discrete valuation of k such that the restriction of ψ to F is constant. Since $(M(F, \mathbf{S}))$ does not depend on $\mathbf{S}$, we may write (M_F) instead.

Nilpotent orbits and Levi subalgebras. In this subsection, we associate to $(F, e) \in I^d$ a unique conjugacy class (namely, (M_F)) of Levi subgroups that are minimal with respect to the property that $\mathcal{O}(F, e)$ intersects the Lie algebra of some (hence any) element of this class nontrivially (compare to §5 of [3]). This answers a question of D. Kazhdan.

Proposition 2. *Suppose the hypotheses of* [6, §4.2] *hold. Suppose that $(F, e) \in I^d$ and that L is a Levi subgroup of G. If $\mathcal{O}(F, e) \cap \mathrm{Lie}(L) \neq \emptyset$, then $(M_F) \leq (L)$. Moreover, for every maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$ with $F \subset \mathcal{A}(\mathbf{S}, k)$, we have $\mathcal{O}(F, e) \cap \mathrm{Lie}(M(F, \mathbf{S})) \neq \emptyset$.*

Proof. The last claim follows immediately from [6, Corollary 4.3.2 and Lemma 5.3.3(2)].

Suppose L is a Levi subgroup of G for which $\mathcal{O}(F, e) \cap \mathrm{Lie}(L) \neq \emptyset$. From [6, Lemma 5.3.3], we can produce an $\mathfrak{sl}_2(k)$-triple (Y, H, X) in $\mathfrak{g}$ such that $Y, H, X \in \mathfrak{g}_F$ and $X \in \mathcal{O}(F, e)$. From [6, §5.5], F is a maximal facet in $\mathcal{B}(Y, H, X)$. Since $\mathcal{O}(F, e) \cap \mathrm{Lie}(L) \neq \emptyset$, without loss of generality, we assume that $Y, H, X \in \mathrm{Lie}(L)$.

From the last paragraph of the proof of Theorem 5.6.1 of [6], there exists (F', e') in the analogue of I^d for L such that $Y, H, X \in \mathrm{Lie}(L)_{F'}$ and $X \in \mathcal{O}(F', e')$. Note that F' is a facet in $\mathcal{B}(L)$. If F'' is maximal among those facets of $\mathcal{B}(G)$ that lie in F', then $(M_{F''}) \leq (L)$.

On the other hand, $F'' \subset \mathcal{B}(Y,H,X)$ and F is a maximal facet in $\mathcal{B}(Y,H,X)$; so $(M_F) \leq (M_{F''})$. □

Remark 4.2. With suitable changes, the above result remains valid in the context of generalized r-facets.

Some consequences.

Corollary 4.3. *Suppose the hypotheses of* [6, §4.2] *hold. Suppose* $(F,e) \in I^d$*. The orbit* $\mathcal{O}(F,e)$ *is distinguished if and only if* F *is a minimal facet in* $\mathcal{B}(G)$. □

Corollary 4.4. *Suppose the hypotheses of* [6, §4.2] *hold. If* $X \in \mathfrak{g}$ *is a distinguished element and* (Y,H,X) *is an* $\mathfrak{sl}_2(k)$*-triple completing* X*, then there exists a unique point* $x \in \mathcal{B}^{\mathrm{red}}(G)$ *such that* $Y,H,X \in \mathfrak{g}_x$*. Moreover,* x *is a vertex.*

Proof. Let F be a maximal facet in $\mathcal{B}(Y,H,X)$ and let (f,h,e) denote the $\mathfrak{sl}_2(\mathfrak{f})$-triple in $\mathsf{L}_F(\mathfrak{f})$ that is the image of (Y,H,X). From [6, §5.5], we have that $(F,e) \in I^d$. From [6, Lemma 5.3.3(2)], $X \in \mathcal{O}(F,e)$. From Corollary 4.3, F is a minimal facet in $\mathcal{B}(G)$. □

Corollary 4.5. *Suppose* $\mathbf{G}$ *is* k*-quasisplit and the hypotheses of* [6, §4.2] *hold. Suppose* $(F,e) \in I^d$*. The orbit* $\mathcal{O}(F,e)$ *is regular if and only if* $e \in \mathsf{L}_F(\mathfrak{f})$ *is regular and the image of* F *in* $\mathcal{B}^{\mathrm{red}}(G)$ *is a special vertex for which (a choice of) the simple* $\mathfrak{f}$*-roots of* G_F *may be naturally identified with (a choice of) the simple* k*-roots of* $\mathbf{G}$.

Proof. Suppose $\mathcal{O}(F,e)$ is regular. Then $\mathcal{O}(F,e)$ is distinguished. From Corollary 4.3, we may assume that F is a minimal facet in $\mathcal{B}(G)$. Let (f,h,e) be an $\mathfrak{sl}_2(\mathfrak{f})$-triple completing e, and let (Y,H,X) be an $\mathfrak{sl}_2(k)$-triple lifting (f,h,e) (see [6, 5.3.3(1)]). From [6, 5.3.3(2)], $X \in \mathcal{O}(F,e)$. Let λ be the one-parameter subgroup adapted (see Definition 4.5.6 of [6]) to (Y,H,X). Let $\mathbf{M} = C_{\mathbf{G}}(\lambda)$, and $M = \mathbf{M}(k)$. From [6, Corollary 4.5.9] we have $F \subset \mathcal{B}(M)$. From Corollary 3.2, Corollary 3.3, and Remark 4.1, X lies in the Lie algebra of a unique rational Borel subgroup $\mathbf{B}$ of $\mathbf{G}$, and $\mathbf{M}$, a Levi factor of $\mathbf{B}$, is a maximal k-torus of $\mathbf{G}$. Let B denote the Borel $\mathfrak{f}$-subgroup of G_F corresponding to the image of $\mathbf{B}(k) \cap \mathsf{G}_F$ in $\mathsf{G}_F(\mathfrak{f})$. Note that e belongs to the Lie algebra of $\mathsf{B}(\mathfrak{f})$.

Let $\mathbf{S}$ denote the maximal k-split torus in $\mathbf{M}$ and let S denote the corresponding maximal $\mathfrak{f}$-split torus in G_F. Let $\mathfrak{g}(2)$ denote the 2-eigenspace for the action of λ on $\mathfrak{g}$. From [6, Corollary 4.3.2 and Lemma 5.3.3(2)], $e \cap \mathfrak{g}(2) \subset \mathcal{O}(F,e)$. Hence, any element of $e \cap \mathfrak{g}(2) = X + (\mathfrak{g}_F^+ \cap \mathfrak{g}(2)) \subset \mathrm{Lie}(\mathbf{B})(k)$ is regular. Thus, from Corollary 3.2 we must have that for all $Z \in e \cap \mathfrak{g}(2)$ and for all $\alpha \in \Delta(\mathbf{S},\mathbf{B},\mathbf{G})$, $Z_\alpha \neq 0$. This implies that every such α, considered as a character of S, must be a root in G_F. Thus, we have an embedding of $\Delta(\mathbf{S},\mathbf{B},\mathbf{G})$ into $\Phi^+(\mathsf{S},\mathsf{B},\mathsf{G}_F)$, and by comparing dimensions we see that $\Delta(\mathbf{S},\mathbf{B},\mathbf{G})$ can be identified with $\Delta(\mathsf{S},\mathsf{B},\mathsf{G}_F)$. In particular, the image of F in $\mathcal{B}^{\mathrm{red}}(G)$ is special. Thus, $e_\alpha \neq 0$ for all $\alpha \in \Delta(\mathsf{S},\mathsf{B},\mathsf{G}_F)$; so e is regular.

To prove the converse, suppose that the image of F in $\mathcal{B}^{\mathrm{red}}(G)$ is a special vertex, and $e \in \mathsf{L}_F(\mathfrak{f})$ is regular. Let (e,h,f), (Y,H,X), λ, and $\mathbf{M}$ be as above. From Corollary 4.3, X is distinguished. Pick a maximal k-torus $\mathbf{T} \subset \mathbf{M}$ so that $\mathbf{T}$ contains a maximal k-split torus $\mathbf{S}$ with $F \subset \mathcal{A}(S) \subset \mathcal{B}(M)$. Since $\mathbf{T} \subset \mathbf{M}$, we have $\lambda \in \mathbf{X}_*^k(\mathbf{T})$. Thus, there exists a rational Borel subgroup $\mathbf{B}$ with Levi factorization $\mathbf{B} = \mathbf{TU}$ such that $X \in \mathrm{Lie}(\mathbf{U})$. Let B, S, and U denote the objects in G_F corresponding to $\mathbf{B}$, $\mathbf{S}$, and $\mathbf{U}$, respectively. Note that $e \in \mathrm{Lie}(\mathsf{U})(\mathfrak{f})$. Since for all $\alpha \in \Delta(\mathsf{S}, \mathsf{B}, \mathsf{G}_F)$ we have $e_\alpha \neq 0$, we have that for all $\alpha \in \Delta(\mathbf{S}, \mathbf{B}, \mathbf{G})$, $X_\alpha \neq 0$. Hence, X is regular. □

5. Proof of Proposition 1

Produce an $\mathfrak{sl}_2(k)$-triple (Y, H, X) completing X, a maximal k-split torus $\mathbf{S}$, and a point $x \in \mathcal{A}(\mathbf{S}, k)$ as in Hypothesis 1. From Corollary 4.5, the image of x in $\mathcal{B}^{\mathrm{red}}(G)$ is special. Let $\mathbf{Z}$ denote the maximal k-split torus $C_{\mathbf{G}}(\mathbf{S})$. Fix $Z \in \mathfrak{g}^{\mathrm{r.s.s.}}$. From Hypotheses 1 and 4, it is enough to prove that

$$Z \in \mathfrak{g}_{0^+} \iff \mathbf{O}_Z(k) \cap (X + C_{\mathfrak{g}_x^+}(Y)) \neq \emptyset.$$

"$\Longleftarrow$": Suppose there exists $g \in \mathbf{G}(\overline{k})$ such that ${}^gZ \in X + C_{\mathfrak{g}_x^+}(Y)$. (From [2, Corollary 3.2.6], this latter set is contained in $\mathfrak{g}_{0^+}$.) From Hypothesis 3, we may assume $g \in \mathbf{G}(K)$. From [1, Lemma 2.2.5], $\mathfrak{g}_{0^+} = (\mathfrak{g}(K)_{0^+})^{\mathrm{Gal}(K/k)}$, the set of $\mathrm{Gal}(K/k)$-fixed points in $\mathfrak{g}(K)_{0^+}$. Therefore, $Z \in ({}^{g^{-1}}\mathfrak{g}(K)_{0^+})^{\mathrm{Gal}(K/k)} = \mathfrak{g}_{0^+}$.

"$\Longrightarrow$": Suppose $Z \in \mathfrak{g}_{0^+}$. Let E/k be a finite extension over which $\mathbf{Z}$ splits and for which there exists $g \in \mathbf{G}(E)$ such that ${}^gZ \in \mathrm{Lie}(\mathbf{Z})(E)$. Since $Z \in \mathfrak{g}_{0^+}$, we must have that $Z \in \mathfrak{g}_y^+$ for some $y \in \mathcal{B}(G)$. Thus, $Z \in \mathfrak{g}(E)_y^+ \subset \mathfrak{g}(E)_{0^+}$; so ${}^gZ \in \mathfrak{g}(E)_{0^+}$. From [2, Theorem 3.1.2(2)] or [16, Lemma 8.2], ${}^gZ \in \mathrm{Lie}(\mathbf{Z})(E)_{0^+} \subset \mathfrak{g}(E)_x^+$.

From Hypothesis 2, there exists $h \in \mathbf{G}(E)$ such that ${}^{hg}Z = {}^gZ + X$. From Hypothesis 1, there exists $\ell \in \mathbf{G}(E)_x^+$ such that ${}^{\ell hg}Z \in X + C_{\mathfrak{g}(E)_x^+}(Y)$. Since $\mathbf{O}_Z(k)$ (resp. $\mathbf{O}_Z(E)$) intersects $X + C_{\mathfrak{g}}(Y)$ (resp. $X + C_{\mathfrak{g}(E)}(Y)$) exactly once (from Hypothesis 4), we conclude that ${}^{\ell hg}Z \in X + C_{\mathfrak{g}_x^+}(Y) \subset \mathfrak{g}$. Therefore, $\mathbf{O}_Z(k) \cap (X + C_{\mathfrak{g}_x^+}(Y)) \neq \emptyset$.

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