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STRONG LIMIT-POINT CLASSIFICATION OF SINGULAR HAMILTONIAN EXPRESSIONS

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Abstract. Strong limit-point criteria for singular Hamiltonian differential expressions with complex coefficients are obtained. The main results are extensions of the previous results due to Everitt, Giertz, and Weidmann for scalar differential expressions.

1. Introduction

In this paper, we are concerned with the deficiency index problem for the singular Hamiltonian differential expression

$$Lz := Jz' - Q(t)z = \lambda P(t)z, \quad t \geq 0, \tag{1.1}$$

where λ is a complex parameter, $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, I is the $n \times n$ identity matrix,

$$Q(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix}, \quad P(t) = \begin{pmatrix} W(t) & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.2}$$

A, B, C, W are locally integrable $n \times n$ matrix-valued functions on $[0, \infty)$, B, C, and W are Hermitian, i.e. $B^* = B$, $C^* = C$, and $W^* = W$ with $W > 0$. Here B^* is the conjugate transpose of B and inequalities of Hermitian matrices are in the positive, non-negative sense.

Let $z = (x^T, u^T)^T$, where x and u are n-dimensional complex-valued column vectors. We will sometimes, however, write $z = (x, u)$ instead of $z = (x^T, u^T)^T$ whenever there is no danger of confusion. The weight function $W(t)$ can be reduced to the identity by the transformation $z \mapsto z_1 = (x_1, u_1)$,

$$x_1 = W^{1/2}x, u_1 = W^{-1/2}u. \tag{1.3}$$

Thus, without loss of generality, in what follows we always suppose that $W(t) = I$.

A solution z is said to be square integrable and denoted as $z \in L^2_P([0, \infty), \mathbf{C}^{2n})$ (or $z \in L^2_P$, for simplicity in notation), if

$$\int_0^\infty z^* P z < \infty.$$

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Since $P(t)$ is singular, we take it for granted that L^2_P is the quotient space. The differential operator L generated by the Hamiltonian system (1.1) is formally self-adjoint. It can be shown (cf. [3, Lemma XIII.2.9]) that, with the maximal domain

$$D(L) = \{z \in AC_{\text{loc}}[0,\infty) \cap L^2_P : \exists f \in L^2_P \text{ such that } Lz = Pf\}, \tag{1.4}$$

L is densely defined in L^2_P under the controlability condition that the system

$$u' = -A^* u, \quad 0 = Bu$$

has only the trivial solution $u = 0$.

Let N_+ and N_- denote the numbers of linearly independent solutions of (1.1) in $L^2[0,\infty)$ with $\lambda = \nu + i\mu$ for $\mu > 0$ and $\mu < 0$, respectively. The numbers N_+ and N_- are known to be independent of the value of λ in the respective upper and lower complex plane (see [2, Chapter XII, Theorem 4.1.19]), and are called the deficiency indices of L in the corresponding half plane. Moreover, they satisfy $n \le N_+, N_- \le 2n$. The deficiency indices of a differential operator are crucial in the investigation of its spectra since the deficiency indices determine the number of linearly independent self-adjoint boundary conditions that one needs to get a self-adjoint extension of a minimal operator (see [2, Chapter XIII]).

We say that L is in the *limit-point-n case* ($LP(n)$, for short) if $N_+ = N_- = n$, and in the *limit-circle case* if $N_+ = N_- = 2n$. Let $[z,w](t)$ be the bilinear form, associated with L, given by

$$[z,w](t) = z^*(t)Jw(t), z = (x^T, u^T)^T, w = (y^T, v^T)^T \in D(L).$$

It is known (see [6], [1] for scalar cases and [10], [14], [15] for system cases) that L is in the limit-point-n case if and only if

$$\lim_{t\to+\infty} [z,w](t) = 0 \quad \forall z, w \in D(L). \tag{1.5}$$

The further classification of the limit-point case into strong and weak limit-point cases was clearly made by Everitt, Giertz, and Weidmann [6], [7] in the deficiency index problem of higher-order scalar differential equations. But strong limit-point case for differential operators was studied even earlier (see [4], [5]). Besides, some results on the bounds of spectra and the presence of pure point spectra of symmetric differential expressions are in the strong limit-point case (cf. [16, Theorems 10.3.3–3.5]). In this direction, see also [8], [13].

For the Hamiltonian system (1.1), we say that L is in the *strong limit-point-n case* ($SLP(n)$, for short) if

$$\lim_{t\to\infty} x^* v = 0, \quad \forall (x,u), (y,v) \in D(L). \tag{1.6}$$

We say that L is in the *weak limit-point-n case* if (1.5) holds but (1.6) does not. Clearly, $SLP(n)$ implies $LP(n)$ and, if $n = 1$, our definition of the strong limit-point-n case coincides with that of the strong limit-point case given in [6], [7].

The spectral theory for singular Hamiltonian systems has been considerably developed. The reader is referred to [1], [10], [14], [17] for the theory of the Weyl-Titchmarsh $M(\lambda)$ function and to [2], [11], [15] for the limit-point classification. However, there has been little literature on the strong limit-point case for Hamiltonian systems.

In this paper, we will establish criteria of strong limit-point-n case for the operator L generated by the singular Hamiltonian system (1.1). We will first prepare

two lemmas in Section 2 and give the main results in Section 3. Some examples will be given to illustrate our criteria.

2. Lemmas

The following result will play an important role in this paper.

Lemma 1. *The operator L is in the strong limit-point-n case if and only if*

$$\lim_{t\to\infty} x^*u = 0, \quad \forall (x^T, u^T)^T \in D(L). \tag{2.1}$$

Proof. The necessity is obvious. Suppose (2.1) holds and take any pair (x, u), $(y, v) \in D(L)$. Set

$$(z_1, w_1) = (x, u) + (y, v), (z_2, w_2) = (x, u) + i(y, v).$$

Then (2.1) implies that, as $t \to \infty$,

$$z_1^* w_1 = (x + y)^*(u + v) = x^*u + y^*v + x^*v + y^*u \to 0,$$
$$z_2^* w_2 = (x^* - iy^*)(u + iv) = x^*u + y^*v + i(x^*v - y^*u) \to 0,$$

and hence,

$$\lim_{t\to\infty} (x^*v + y^*u) = 0, \quad \lim_{t\to\infty} (x^*v - y^*u) = 0. \tag{2.2}$$

Thus, (2.2) implies $\lim_{t\to\infty} x^*v = 0$ which means $L \in SLP(n)$. □

Lemma 2. *For every $\lambda_0 \in \mathbf{R}$, if $L_{\lambda_0} = L - \lambda_0 P$ with the domain*

$$D(L_{\lambda_0}) = \{z \in AC_{\mathrm{loc}}[0, \infty) \cap L_P^2 : \exists f \in L_P^2 \text{ such that } L_{\lambda_0} z = Pf\}, \tag{2.3}$$

then $D(L) = D(L_{\lambda_0})$ and L is in the strong limit-point-n case if and only if L_{λ_0} is.

Proof. $D(L) = D(L_{\lambda_0})$ follows from the definitions (1.4) and (2.3). Notice that the strong limit-point-n classification of L is completely determined by its domain $D(L)$. Thus, $L \in SLP(n)$ if and only if $L_{\lambda_0} \in SLP(n)$. □

Consider Hermitian $n \times n$ matrices. All eigenvalues of a Hermitian matrix A are real numbers. We will denote by $\lambda_{\min}(A)$ the smallest eigenvalue of A and by $\lambda_{\max}(A)$ the largest. If $A > 0$, the roots of $\det(B - \lambda A) = 0$ are called the eigenvalues of B with respect to A (see [12]). Thus, if in addition $B \geq 0$, the eigenvalues of AB or BA are merely the eigenvalues of B with respect to A^{-1}, and hence, are all nonnegative.

3. Main results

In [4], Everitt, Giertz, and McLeod gave a sufficient condition for the second-order differential expression

$$\tau f := -(p(t)f')' + q(t)f, \quad t \in [0, \infty),$$

to be in the strong limit-point case.

Theorem EGM. *Suppose p and q are real-valued such that $p \in AC_{\mathrm{loc}}[0, \infty)$, $p(t) > 0$ and q is locally integrable on $[0, \infty)$. If additionally, q is essentially bounded below on $[0, \infty)$, then τ is in the strong limit-point case.*

For the Hamiltonian system (1.1), we have

Theorem 1. *Suppose that $C(t)$ is bounded below and $B(t) \geq 0$. Let $b(t) = \lambda_{\min}(B(t))$. If*

$$\int_0^\infty \sqrt{b(s)}ds = \infty, \tag{3.1}$$

then L is in the strong limit-point-n case.

Proof. Let $(x, u) \in D(L)$. By Lemma 1, we need only to prove that (2.1) holds. In view of (1.4), there exists an $f \in L^2([0, \infty), \mathbf{C}^n)$ such that

$$x' = Ax + Bu, \quad u' = Cx - A^*u + f, \tag{3.2}$$

and hence,

$$(x^*u)' = u^*Bu + x^*Cx + x^*f.$$

Let $h(t) = x^*(t)u(t)$, $\varphi(t) = x^*(t)x(t)$, and $\psi(t) = u^*(t)u(t)$. Then

$$h(t) = h(0) + \int_0^t u^*Buds + \int_0^t x^*Cxds + \int_0^t x^*fds. \tag{3.3}$$

Since $C(t)$ is bounded below, by Lemma 2, we may assume that $C(t) \geq 0$, or otherwise replace $C(t)$ with $C(t) + \lambda_0 I$. Since $B(t) \geq 0$ and $C(t) \geq 0$, and x and f are square integrable, (3.3) implies that $\lim_{t\to\infty} h(t) \leq +\infty$ exists. Suppose $\lim_{t\to\infty} |h(t)| \geq 2l > 0$. Then there exists an $N > 0$ such that

$$|h(t)| \geq l \tag{3.4}$$

and $\varphi(t) > 0$ for $t \geq N$. Set $H(t) = \mathrm{Re}(h(t))$. By (3.3), one sees that

$$H(t) \geq K + \int_0^t b(s)\psi(s)ds \tag{3.5}$$

for some real number K. Since

$$H(t)^2 \leq |h(t)|^2 = |x^*u|^2 \leq u^*ux^*x = \psi(t)\varphi(t) \tag{3.6}$$

and (3.4), we get that, for all $t \geq N$,

$$\int_N^t b(s)\psi(s)ds \geq \int_N^t \frac{b(s)|h(s)|^2}{\varphi(s)}ds \geq l^2 \int_N^t \frac{b(s)}{\varphi(s)}ds. \tag{3.7}$$

By Schwartz inequality and (3.1),

$$\int_N^t \frac{b(s)}{\varphi(s)}ds \geq \frac{[\int_N^t \sqrt{b(s)}ds]^2}{\int_N^t \varphi(s)ds} \to \infty \ \text{ as } \ t \to \infty. \tag{3.8}$$

From (3.5), (3.7), and (3.8), we have that $H(t) \to \infty$ as $t \to \infty$, and hence, there is an $N_1 > N$ such that for all $t \geq N_1$,

$$H(t) = H(N_1) + \mathrm{Re}\int_{N_1}^t x^*fds + \int_{N_1}^t (u^*Bu + x^*Cx)ds \geq \int_{N_1}^t b\psi ds. \tag{3.9}$$

Now, set $G(t) = \int_N^t b(s)\psi(s)ds$. In view of (3.6) and (3.9), we have $G'(t) \geq b(t)G^2(t)/\varphi(t)$, i.e.

$$b(t)/\varphi(t) \leq (-1/G(t))', \ t > N_1.$$

Integrating from $N_1 + 1$ to $t > N_1 + 1$ gives

$$\int_{N_1+1}^t \frac{b(s)}{\varphi(s)}ds \leq \frac{1}{G(N_1+1)} - \frac{1}{G(t)} < \infty, \tag{3.10}$$

which contradicts (3.8). □

Similarly, we have the following counterpart of Theorem 1.

Theorem 2. *Suppose that $C(t)$ is bounded above and $B(t) \leq 0$ and let $b(t) = \lambda_{\max}(B(t))$. If $\int_0^\infty \sqrt{|b(s)|}ds = \infty$, then L is in the strong limit-point-n case.*

Remark. We would like to mention here that in Theorem 1, $B(t)$ is allowed to be singular at some points in $[0, \infty)$ and there are not any restrictions on $A(t)$. The divergence condition on the smallest eigenvalue of $B(t)$ in magnitude eliminates the influence of $A(t)$. The following example shows that if $\int_0^\infty \sqrt{b(s)}ds < \infty$, the validity of Theorem 1 critically depends on $A(t)$.

Example 1. Let $n = 1$ and let a and b be real constants. The Hamiltonian system

$$x' = ax + e^{-bt}u, \quad u' = e^{bt}x - au \tag{3.11}$$

has two linearly independent solutions $(x_1(t), u_1(t))$ and $(x_2(t), u_2(t))$, where $x_1(t) = \exp[(\sqrt{(2a+b)^2+4} - b)t/2]$ and $x_2(t) = \exp[(-\sqrt{(2a+b)^2+4} - b)t/2]$. Therefore, (3.11) is in $LP(1)$ if and only if either $b \leq 0$ or $b > 0$ and $a^2 + ab + 1 \geq 0$. For instance, if $b = 4$, then $b(t) = e^{-4t}$ and (3.1) does not hold. In this case, we see that (3.11) is in the limit-circle case if $-2 - \sqrt{3} < a < -2 + \sqrt{3}$ but in the limit-point-1 case otherwise.

In the case where (3.1) does not hold, we may impose stronger restrictions on $C(t)$ to guarantee the strong limit-point-n case.

Theorem 3. *Suppose that $B(t) \geq 0$ and $C(t)$ is bounded below. Let $b(t) = \lambda_{\min}(B(t))$ and $q(t) = \lambda_{\min}(C(t))$. If there exist a real number q_0 and an $\varepsilon \in (0, 1)$ such that $q(t) + q_0 \geq 0$ and*

$$\int_0^\infty \sqrt{b(t)} \exp\left((1-\varepsilon)\int_0^t \sqrt{b(s)(q(s)+q_0)}ds\right) dt = \infty, \tag{3.12}$$

then L is in the strong limit-point-n case.

Proof. By Lemma 2, we may assume that $q_0 = 0$ and $q(t) \geq 1$. For every $(x, u) \in D(L)$, we will prove that (2.1) holds. Let $h(t) = x^*(t)u(t)$, $\varphi(t) = x^*(t)x(t)$, and $\psi(t) = u^*(t)u(t)$. Let $H(t) = \mathrm{Re}\,(h(t))$.

Since $B(t) \geq 0$ and $C(t) \geq 0$, from (3.3) we know that $\lim_{t\to\infty} h(t)$ exists (may be infinity). We claim that $\lim_{t\to\infty} |h(t)| = 0$, for otherwise, there exist $l > 0$ and $N > 0$ such that (3.4) holds for $t \geq N$. It follows from (3.6) that

$$\begin{aligned}\int_N^t u^*Buds + \int_N^t x^*Cxds &\geq \int_N^t b\psi ds + \int_N^t q\varphi ds \\ &\geq 2\int_N^t \sqrt{bq}|h|ds \geq 2l\int_N^t \sqrt{bq}ds.\end{aligned}$$

We notice that $\lim_{t\to\infty} \int_N^t \sqrt{b(s)q(s)}ds = \infty$ follows from (3.12) and $q(t) \geq 1$. Since x and f are square integrable, we have as $t \to \infty$,

$$H(t) = H(N) + \mathrm{Re}\int_N^t x^*fds + \int_N^t u^*Buds + \int_N^t x^*Cxds \to \infty. \tag{3.13}$$

By (3.13), we may choose $N > 0$ larger, if necessary, such that

$$\begin{aligned} H(t) &\geq 1 + \int_N^t b\psi ds + \int_N^t q\varphi ds \\ &\geq 1 + 2\int_N^t \sqrt{bq}|h|ds \geq 1 + 2\int_N^t \sqrt{bq}Hds. \end{aligned} \tag{3.14}$$

By Gronwall inequality,

$$H(t) \geq \exp\left[2\int_N^t \sqrt{b(s)q(s)}ds\right], \quad t \geq N. \tag{3.15}$$

Using (3.6), (3.14), and (3.15) we get, for $t \geq N$,

$$b(t)\psi(t) \geq b(t)\frac{H(t)^2}{\varphi(t)} \geq b(t)\left[\int_N^t b(s)\psi(s)ds\right]^{1+\varepsilon}\frac{H(t)^{1-\varepsilon}}{\varphi(t)}.$$

Set $G(t) = \int_N^t b(s)\psi(s)ds$. One sees that

$$\left(-G^{-\varepsilon}(t)\right)'/\varepsilon \geq b(t)H(t)^{1-\varepsilon}/\varphi(t).$$

Now,

$$\begin{aligned} \int_N^t \sqrt{b(s)H(s)^{1-\varepsilon}}ds &\leq \int_N^t \varphi(s)ds \int_N^t b(s)H(s)^{1-\varepsilon}\varphi(s)^{-1}ds \\ &\leq \frac{1}{\varepsilon}\int_N^t \varphi(s)ds\left(\frac{1}{G(N)^\varepsilon} - \frac{1}{G(t)^\varepsilon}\right) < \infty, \end{aligned}$$

which contradicts (3.12) and (3.15). □

In Example 1, we know (3.11) is in the limit-circle case if $a = -1$ and $b > 2$. But, if $b < 2$, we may choose $\varepsilon < 1 - (b/2)$ so that (3.12) holds, namely,

$$\int_0^\infty \sqrt{b(t)}\exp\left((1-\varepsilon)\int_0^t \sqrt{bq}ds\right)dt = \int_0^\infty e^{(2-b-2\varepsilon)t/2}dt = \infty,$$

and hence, (3.11) is in the strong limit-point-1 case for every $a \in \mathbf{R}$.

Theorem 4. *Suppose that $C(t) > q_0 I$ and $B(t) \geq 0$. Set*

$$b(t) = \lambda_{\min}(B(t)),\ \overline{q}(t) = \lambda_{\max}(C(t) - q_0 I),\ \overline{b}(t) = \lambda_{\min}[B(t)(C(t) - q_0 I)].$$

If there exists an $\varepsilon \in (0,1)$ such that either $C(t)$ is differentiable and

$$\int_0^\infty \sqrt{\overline{b}(t)/\overline{q}(t)}\exp\left((1-\varepsilon)\int_0^t \sqrt{\overline{b}(s)}ds\right)dt = \infty \tag{3.16}$$

or $B(t)$ is differentiable, invertible, and

$$\int_0^\infty \sqrt{b(t)}\exp\left((1-\varepsilon)\int_0^t \sqrt{\overline{b}(s)}ds\right)dt = \infty, \tag{3.17}$$

then L is in the strong limit-point-n case.

Proof. As before, we may assume $q_0 = 0$. From the remark in the last paragraph of Section 2, $\overline{b}(t) \geq 0$ since $B(t) \geq 0$ and $C(t) > 0$. If $C(t)$ is differentiable and (3.16) holds, then we let $T(t) = C^{-1/2}(t)$ and $w = (y, v)$ such that

$$x = Ty, \quad u = T^{-1}v. \tag{3.18}$$

Then (1.1) is transformed into

$$L_0 w := Jw' - Q_0 w = \lambda P_0 w,$$

where

$$Q_0(t) = \begin{pmatrix} -I & A_0^*(t) \\ A_0(t) & B_0(t) \end{pmatrix}, \quad P_0(t) = \begin{pmatrix} C^{-1}(t) & 0 \\ 0 & 0 \end{pmatrix}$$

with $A_0 = T^{-1}(AT - T')$ and $B_0 = T^{-1}BT^{-1}$.

Since $B_0 = T^{-1}BCT$, we know that $\lambda_{\min}(B_0(t)) = \lambda_{\min}[B(t)C(t)] = \overline{b}(t)$. It is easy to see that $(x, u) \in D(L)$ if and only if (y, v) is in the set

$$D(L_0) := \{w \in AC_{\text{loc}}[0, \infty) \cap L^2_{P_0} : \exists f \in L^2_{P_0} \text{ such that } L_0 w = P_0 f\}.$$

Since $x^* u = (Ty)^* T^{-1} v = y^* v$, we need only to prove $\lim_{t\to\infty} y^* v = 0$ for every $(y, v) \in D(L_0)$.

Set $h(t) = y^*(t)v(t)$, $H(t) = \text{Re } h(t)$, $\varphi(t) = y^*(t)y(t)$, and $\psi(t) = v^*(t)v(t)$. We can then prove that $\lim_{t\to\infty} h(t)$ exists (may be infinity). Suppose $\lim_{t\to\infty} h(t) \neq 0$. Let $G(t) = \int_N^t \overline{b}(s)\psi(s)ds$ and notice that

$$\varphi(t) = y^*(t)T(t)C(t)T(t)y(t) \leq \overline{q}(t)y^*(t)C^{-1}(t)y(t) =: \overline{\varphi}(t).$$

Now, following the steps in the proof of Theorem 1, replacing $b(t)$ with $\overline{b}(t)$ and $\varphi(t)$ with $\overline{\varphi}(t)$, we can obtain a desired contradiction.

If $B(t)$ is differentiable and invertible, then we let $T = B^{1/2}$ in the transformation (3.18). The rest of the proof is similar and hence omitted. □

Example 2. Let $n = 2$. Let $A(t)$ be arbitrary and

$$B(t) = \begin{pmatrix} 2e^{-2t} & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad C(t) = \begin{pmatrix} 3e^{2t} & 0 \\ 0 & 2e^{t} \end{pmatrix}.$$

Then $b(t) = 2e^{-2t}$, $q(t) = 2e^t$, $\overline{q}(t) = 3e^{2t}$, and $\overline{b}(t) \equiv 2$. Since

$$\int_0^\infty \sqrt{b(t)} \exp\left(\int_0^t \sqrt{bq}ds\right) dt = \sqrt{2}\int_0^\infty e^{-t} \exp\left(2\int_0^t e^{-s/2}ds\right) dt < \infty,$$

Theorem 3 does not apply. But it is easy to verify that (3.16) and (3.17) are satisfied. So, this is the strong limit-point-2 case by Theorem 4.

References

1. F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964. MR **31:**416
2. S. L. Clark, *A criterion for absolute continuity of the continuous spectrum of a Hamilton system*, J. Math. Anal. Appl. **151** (1990), 108–128. MR **91i:**34097
3. N. Dunford and J. T. Schwartz, *Linear operators, Part II*, Spectral theory: Self-adjoint operators in Hilbert space, Interscience, New York, 1963. MR **32:**6181
4. W. N. Everitt, *On the limit-point classification of second-order differential operators*, J. London Math. Soc. **41** (1966), 531–534. MR **34:**410
5. W. N. Everitt and M. Giertz, *On some properties of the domains of powers of certain differential operators*, Proc. London Math. Soc. (3) **24** (1972), 756–768. MR **46:**2492
6. W. N. Everitt, M. Giertz and J. B. McLeod, *On the strong and weak limit-point classification of second-order differential expressions*, Proc. London Math. Soc. (3) **29** (1974), 142–158. MR **50:**13701
7. W. N. Everitt, M. Giertz, and J. Weidmann, *Some remarks on a separation and limit-point criterion of second-order, ordinary differential expressions*, Math. Ann. **200** (1973), 335–346. MR **48:**4393

8. W. N. Everitt, D. B. Hinton, and J. S. W. Wong, *On the strong limit-n classification of linear ordinary differential expressions of order* $2n$, Proc. London Math. Soc. (3) **29** (1974), 351–367. MR **53:**13708
9. E. Hille, *Lectures on Ordinary Differential Equations*, Addison-Wesley, London, 1969. MR **40:**2939
10. D. B. Hinton and J. K. Shaw, *On boundary value problems for Hamiltonian systems with two singular points*, SIAM. J. Math. Anal. **15** (1984), 272–286. MR **87a:**34021
11. D. B. Hinton and J. K. Shaw, *Absolutely continuous spectra of Dirac systems with long range, short range and oscillating potentials*, Quart. J. Math. Oxford Ser. (2) **36** (1985), 183–213. MR **87f:**34024
12. R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985. MR **87e:**15001
13. R. M. Kauffman, T. T. Read, and A. Zettl, *The deficiency index problem for powers of ordinary differential expressions*, Lecture Notes in Math., vol. 621, Springer-Verlag, Berlin, 1977. MR **58:**1370
14. A. M. Krall, $M(\lambda)$ *theory for singular Hamiltonian systems with one singular point*, SIAM J. Math. Anal. **20** (1989), 664–700. MR **91c:**34036
15. A. M. Krall, *A limit-point criterion for linear Hamiltonian systems*, Applicable Analysis **61** (1996), 115–119. MR **99a:**34066
16. V. K. Kumar, *On the strong limit-point classification of fourth-order differential expressions with complex coefficients*, J. London Math. Soc. (2) **12** (1976), 287–298.
17. P. W. Walker, *A vector-matrix formulation for formally symmetric ordinary differential equations with applications to solutions of integrable square*, J. London Math. Soc. (2) **9** (1974/75), 151–159. MR **51:**6021

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