A NOTE ON COMMUTATIVITY UP TO A FACTOR OF BOUNDED OPERATORS

JIAN YANG AND HONG-KE DU

(Communicated by Joseph A. Ball)

Abstract. In this note, we explore commutativity up to a factor $AB = \lambda BA$ for bounded operators $A$ and $B$ in a complex Hilbert space. Conditions on possible values of the factor $\lambda$ are formulated and shown to depend on spectral properties of the operators. Commutativity up to a unitary factor is considered. In some cases, we obtain some properties of the solution space of the operator equation $AX = \lambda XA$ and explore the structures of $A$ and $B$ that satisfy $AB = \lambda BA$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. A quantum effect is an operator $A$ on a complex Hilbert space that satisfies $0 \leq A \leq I$. The sequential product of quantum effects $A$ and $B$ is defined by $A \circ B = A^{1/2}BA^{1/2}$. We also obtain properties of the sequential product.

1. Introduction

Commutation relations between selfadjoint operators in a complex Hilbert space play an important role in the interpretation of quantum mechanical observables and analysis of their spectra. For related works refer to [2], [3], [4], [6], [8], [9] and [12]. Accordingly, such relations have been extensively studied in the mathematical literature (see, for example, the classic study of C. R. Putnam in [11]). An interesting, related aspect concerns the commutativity up to a factor for a pair of operators. Certain forms of non-commutativity can be conveniently phrased in this way. In [1], J. A. Brooke, P. Busch and D. B. Pearson gave some examples to illustrate this. A quantum effect is a yes-no measurement. An effect is represented by an operator $A$ on a Hilbert space that satisfies $0 \leq A \leq I$. A sharp effect is represented by a selfadjoint projection operator on a Hilbert space. The sequential product of quantum effects $A$ and $B$ is defined by $A \circ B = A^{1/2}BA^{1/2}$. Sequential measurements are very important in quantum mechanics. For detailed works refer to [8], [9] and [12]. Let $H$ be a complex Hilbert space, $B(H)$ be the Banach algebra of bounded linear operators on $H$, $\mathcal{E}(H)$ be the set of quantum effects on $H$, $P(H)$ be the set of sharp effects on $H$, $I$ be the identity operator on some Hilbert space and $M_{n \times m}$ be the set of $n \times m$ matrices. For an operator $A \in B(H)$, denote by $N(A)$, $R(A)$, $\sigma(A)$, $r(A)$ the null space, the range, the spectra and the spectral radius of $A$, respectively; $\dim N(A)$ denotes the dimension of $N(A)$. Recall that, for...
\(A, B \in B(H)\), \(A\) and \(B\) commute up to a factor means that \(AB = \lambda BA\), for some 
\(\lambda \in \mathbb{C} \setminus \{0\}\), and \(A\) and \(B\) commute up to a unitary factor means that \(AB = UBA\), 
where \(U\) is a unitary operator in \(B(H)\). For each \(0 < A \in B(H), B \in B(H)\), we 
also define \(A \circ B = A^\dagger BA^\dagger\). The main results shown by J. A. Brooke, P. Busch 
and D. B. Pearson in [1] are the following two theorems.

**Theorem 1.1** ([1]). Let \(A, B \in B(H)\) such that \(AB = \lambda BA \neq 0, \lambda \in \mathbb{C}\). Then 
(i) if \(A\) or \(B\) is selfadjoint, then \(\lambda \in \mathbb{R}\);
(ii) if both \(A\) and \(B\) are selfadjoint, then \(\lambda \in \{-1, 1\}\); and 
(iii) if \(A\) and \(B\) are selfadjoint and one of them is positive, then \(\lambda = 1\).

**Theorem 1.2** ([1]). Let \(A, B \in B(H)\) be selfadjoint operators. The following 
statements are equivalent.
(i) \(AB = UBA\) for some unitary operator \(U\).
(ii) \(AB^2 = B^2A\) and \(BA^2 = A^2B\).

In [9] S. Gudder and G. Nagy gave the following result on sequential measurement.

**Theorem 1.3** ([9]). For \(A, B \in \varepsilon(H)\), if \(A \circ B \in P(H)\), then \(AB = BA\).

They put forward an open problem: if \(A, B \in \varepsilon(H)\) with \(\dim H = \infty\) and \(A \circ B \succeq B\),
does \(AB = BA = B\) hold?

In this paper, firstly we give simple proofs of the theorems above and generalizations of them. Secondly, we show a further relation between the spectra of \(AB\) and the factor \(\lambda\). In the case that \(H\) is finite dimensional, we obtain a property of the solution space of the operator equation \(AX = \lambda XA\). Also, if \(A\) has finite rank and is normal, we explore the structure of \(A\) and \(B\) which commute up to a factor.

Thirdly, we give a generalization of Theorem 1.3 in [9] with proof different from [9]
and answer the open question raised by S. Gudder and G. Nagy. This question was 
also independently answered by A. Gheondea and S. Gudder in [7].

We will use repeatedly the Fuglede-Putnam Theorem.

**Theorem 1.4** (Fuglede-Putnam Theorem ([11])). If \(N\) and \(M\) are normal 
operators on \(H\) and \(K\), and \(B : K \to H\) is an operator such that \(NB = BM\); then 
\(N^*B = BM^*\).

2. Pairs commuting up to a factor

**Theorem 2.1.** Let \(A, B \in B(H)\) such that \(AB = \lambda BA \neq 0, \lambda \in \mathbb{C}\). Then 
(i) if \(A\) or \(B\) is selfadjoint, then \(\lambda \in \mathbb{R}\);
(ii) if either \(A\) or \(B\) is selfadjoint and the other is normal, then \(\lambda \in \{-1, 1\}\); and 
(iii) if both \(A\) and \(B\) are normal, then \(|\lambda| = 1\).

**Proof.** (i) Suppose that \(A\) is selfadjoint; then \(\lambda A\) is normal. By the Fuglede-Putnam 
Theorem, we have \(AB = \overline{\lambda}BA\). Hence \((\lambda - \overline{\lambda}) = 0\). This implies \(\lambda \in \mathbb{R}\).

(ii) Suppose that \(A\) is normal and \(B\) is selfadjoint. By the Fuglede-Putnam 
Theorem, we have \(A^*B = \overline{\lambda}BA^*\). Then \(AA^*B = \overline{\lambda}ABA^* = |\lambda|^2BAA^*\). From (i) 
and (iii) of Theorem 1.1, we have \(\lambda \in \mathbb{R}\) and \(|\lambda|^2 = 1\). Hence \(\lambda \in \{-1, 1\}\).

(iii) From \(AB = \lambda BA\), we have \(B^*A^* = \overline{\lambda}A^*B^*\). By the Fuglede-Putnam Theorem, 
we get \(A^*B = \overline{\lambda}BA^*\) and \(BA^* = \lambda A^*B\). Hence \(A^*B = |\lambda|^2A^*B\). Since \(AB \neq 0\) 
and \(A\) is normal, we can get \(A^*B \neq 0\). Therefore \(|\lambda| = 1\). \(\square\)
In fact, with similar deduction, we can generalize (i) of Theorem 2.1 in the following way:

**Corollary 2.2.** Suppose $A, B, C \in B(H)$ and $A, C$ are selfadjoint operators with $AB = \lambda BC \neq 0$, $\lambda \in \mathbb{C}$; then $\lambda \in \mathbb{R}$.

The following result is a generalization of Theorem 1.2 in [1].

**Theorem 2.3.** Let $A, B \in B(H)$. Then the following statements are equivalent.

(i) $ABB^* = B^*BA$ and $BAA^* = A^*AB$.

(ii) Both $AB$ and $BA$ are normal.

(iii) There exist unitary operators $U$ and $V$ in $B(H)$ such that $AB = U^*B^*A^*$ and $BA = V^*A^*B^*$.

**Proof.** The equivalence of (i) and (ii) was proved by S. Gudder and G. Nagy in [8].

(ii) $\Rightarrow$ (iii) By the fact that an operator $N \in B(H)$ is normal if and only if there exists a unitary operator $U$ such that $N = U^*N^*$, the proof is trivial.

(iii) $\Rightarrow$ (ii) Observe that $B^*A^* = AB^*$, and so $AB = UAB^*$. Hence $AB$ commutes with $U$ and $U^*$, and similarly for $B^*A^*$. Thus we get $ABB^* = U^*B^*A^*U^*AB = B^*A^*U^*UAB = B^*A^*AB$. Hence $AB$ is normal. Similarly, from $BA = V^*A^*B^*$, we can get $BA$ is normal. □

If both $A$ and $B$ are selfadjoint, then Theorem 2.3 becomes Proposition 4.1 and Proposition 4.2 in [1].

For given $0 \neq A \in B(H)$, let

$$S^n_A = \{B \in B(H) : \sigma(AB) \text{ has exactly } n \text{ distinct nonzero values}\}.$$ 

Then we have

**Theorem 2.4.** For each $0 < n < \infty$ and $B \in S^n_A$, if $AB = \lambda BA$ for some $\lambda \in \mathbb{C}$, then $\lambda^n = 1$.

**Proof.** Suppose $B \in S^n_A$ with $AB = \lambda BA$. Let $\sigma(AB) \setminus \{0\} = \{\alpha_1, \ldots, \alpha_n\}$. Then $\{\alpha_1, \ldots, \alpha_n\} = \{\lambda\alpha_1, \ldots, \lambda\alpha_n\}$, since $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$. Clearly, for each $\alpha_i \in \sigma(AB) \setminus \{0\}$, there exist $1 \leq i \leq n$ and $\alpha_j \in \sigma(AB) \setminus \{0\}$, $1 < j \leq p$, with $\alpha_i \neq \alpha_j, 1 < k \leq l \leq p$, such that $\alpha_i = \lambda\alpha_k = \cdots = \lambda^{p-1}\alpha_p = \lambda^p\alpha_i$. If $\alpha_{i_1} = \lambda\alpha_{i_2} = \cdots = \lambda^{p-1}\alpha_p = \lambda^p\alpha_{i_1}$ and $\alpha_{j_1} = \lambda\alpha_{j_2} = \cdots = \lambda^{q-1}\alpha_q = \lambda^q\alpha_{j_1}$, for distinct $i_1$ and $j_1$, then $p = q$ and $\alpha_i \neq \alpha_{i_1}, 1 < k \leq p$. In fact, we assume that $p < q$. Then $\lambda^p = \lambda^q = 1$ and $\lambda^{q-p} = 1$; hence $\alpha_{j_1} = \alpha_{j_{1-p+1}}$. It is a contradiction. Then we have $\frac{\pi}{p} \in \mathbb{N}$. Hence $\lambda^n = 1$. □

The next result gives some restrictions that must be satisfied by a pair of operators commuting up to a scalar factor.

**Theorem 2.5.** Let $A, B \in B(H)$ such that $AB = \lambda BA \neq 0$, for some $\lambda \in \mathbb{C}$. Then

(i) $AB$ is bounded below if and only if both $A$ and $B$ are bounded below.

(ii) if $A$ is normal and $R(B)$ is dense, then $AB$ is not nilpotent.

**Proof.** (i) If $A$ and $B$ are bounded below, clearly, we have that $AB$ is bounded below. Conversely, if $AB$ is bounded below, then $B$ is bounded below. We also have that $A$ is bounded below, since $AB = \lambda BA$.

(ii) If $AB = \lambda BA$ and $A$ is normal, then $N(A)$ and $R(A)$ are reducing subspaces of $A$ and $B$. Thus $A$ and $B$ have forms $A = \text{diag}(A_1, 0)$ and $B = \text{diag}(B_1, B_2)$, with respect to $H = N(A)^\perp \oplus N(A)$. Since $AB = \text{diag}(A_1 B_1, 0)$, we have that $AB$ is
nilpotent if and only if $A_1B_1$ is nilpotent. Hence, without loss of generality, we may assume that $A$ is injective. Then $R(A)$ is dense, since $A$ is normal and injective. If there exists $n \in \mathbb{N}$ such that $(AB)^n = 0$, then $(AB)^{n-1} = 0$, since $R(A)$ and $R(B)$ are dense. By the same deduction, we can get $AB = 0$. It is a contradiction. □

Next, we turn to the study of properties of pairs of operators commuting up to a scalar factor with one of the operators having finite rank.

As in the proof of Theorem 2.5, if $AB = \lambda BA$ and $A$ is normal, then $N(A)$ and $R(A)$ are reducing subspaces of $A$ and $B$. Thus $A$ and $B$ have forms $A = \text{diag}(A_1, 0)$ and $B = \text{diag}(B_1, B_2)$, with respect to $H = N(A) \perp \oplus N(A)$. Suppose that $A$ has finite rank and $\{a_1, \cdots , a_m\}$ are the distinct nonzero eigenvalues of $A$. Then $A = \sum_{i=1}^m a_i P_i$, where $P_i$ is the orthogonal projection from $H$ onto $N(A - a_i I)$, for $1 \leq i \leq m$ and $P_i P_j = P_j P_i = 0, i \neq j$. Denote $H_i = P_i H$. Then $A_1$ and $B_1$ have operator matrix forms $A_1 = \text{diag}(a_1 I, \cdots , a_m I)$ and $B_1 = (B_{ij})_{m \times m}$, with respect to the space decomposition $N(A) \perp = \bigoplus_{i=1}^m H_i$, respectively. With the symbols above, we have the following three results.

**Proposition 2.6.** Suppose that $A \in B(H)$ has finite rank and is normal. If $\sigma(A) \cap \sigma(\lambda A)$ has no nonzero element, then there does not exist nonzero $B \in B(H)$ such that $AB = \lambda BA \neq 0$. If $\sigma(A) \cap \sigma(\lambda A)$ has nonzero elements, then there exists $B \in B(H)$ such that $AB = \lambda BA \neq 0$ if and only if either $m = 1, \lambda = 1$ or $m > 1, \lambda \in \mathbb{C} \setminus \{0\}$.

**Proof.** If $A = 0$, it is trivial. If $A \neq 0$, then we have $AB = \lambda BA \neq 0$ if and only if $A_1B_1 = \lambda B_1 A_1 \neq 0$, since $AB = \text{diag}(A_1B_1, 0)$. Hence, without loss of generality, we may assume that $A$ is injective. Also, $AB = \lambda BA$ if and only if $a_i B_{ij} = \lambda a_j B_{ij}$, for $1 \leq i, j \leq m$. If $\sigma(A) \cap \sigma(\lambda A)$ has no nonzero element, then $a_i \neq \lambda a_j$, for $i, j$. Hence there does not exist $B_{ij}$ such that $a_i B_{ij} = \lambda a_j B_{ij} \neq 0$, for $i, j$. Clearly, if $m = 1$ and $\lambda = 1$, then we have $AB = \lambda BA \neq 0$, for each $0 \neq B \in B(H)$. If $m > 1$ and $\sigma(A) \cap \sigma(\lambda A)$ has nonzero elements, then there exists $1 \leq i, j \leq m$ such that $a_i = \lambda a_j$. Hence for each $B$ with $B_{ij} \neq 0$ and $B_{st} = 0, s \neq i, t \neq j$, we have $AB = \lambda BA \neq 0$. On the other hand, if $AB = \lambda BA \neq 0$ and $m = 1$, then we have $\lambda = 1$, since $\sigma(A) \cap \sigma(\lambda A)$ has nonzero element $a_1$. □

If, with respect to a suitable decomposition of the space, $B$ has a form $B = \bigoplus_{i=1}^t \bigoplus_{j=1}^t M_{ij}^{(i)} \oplus B_2$, then we call it a standard form, where the $k \times k$ block operator matrix $J_k^{(i)}$ and the $j_i \times j_i$ block operator matrix $M_{j_i}^{(i)}$ are defined by

\[
J_k^{(i)} = \begin{pmatrix}
\begin{array}{cccc}
0 & B_{12}^{(i)} & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
B_{k1}^{(i)} & \cdots & B_{(k-1)k}^{(i)} & 0 \\
0 & \cdots & 0 & 0 \\
\end{array}
\end{pmatrix},
M_{j_i}^{(i)} = \begin{pmatrix}
\begin{array}{cccc}
0 & B_{12}^{(i)} & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & B_{(j_i-1)j_i}^{(i)} & 0 \\
0 & \cdots & 0 & 0 \\
\end{array}
\end{pmatrix},
\]

$s, t \in \mathbb{N} \cup \{0\}, k, j_i \in \mathbb{N}$ and $B_{(p)}_{(j_i)}$ are exactly the nonzero entries of $(B_{ij})_{m \times m}$. In this case, if there exist $a^{(i)} \in \sigma(A) \setminus \{0\}, 1 \leq i \leq s + t$ with $a^{(p)} \neq a^{(i)}$ for $1 \leq p \neq i \leq s + t$, such that $A$ has a form $A = \bigoplus_{i=1}^s I_{k}^{(i)} \bigoplus_{i=s+1}^{s+t} N_{j_i}^{(i)} \oplus A'$, $\lambda^k = 1$ and $j_i \leq k$ for $s + 1 \leq i \leq s + t$, where $A'$ is a diagonal operator and the $k \times k$ block...
operator matrix $I_{k_i}^{(i)}$ and the $j_i \times j_i$ block operator matrix $N_{j_i}^{(i)}$ are defined by

$$I_{k_i}^{(i)} = \begin{pmatrix} a^{(i)}I & \lambda a^{(i)}I & \cdots & \lambda^{k_i-1}a^{(i)}I \\ \end{pmatrix}, \quad N_{j_i}^{(i)} = \begin{pmatrix} a^{(i)}I & \lambda a^{(i)}I & \cdots & \lambda^{j_i-1}a^{(i)}I \\ \end{pmatrix},$$

respectively, then we say $A$ and $B$ are compatible.

**Theorem 2.7.** Suppose that $0 \neq A \in B(H)$ has finite rank and is normal. Then $AB = \lambda BA$ if and only if one of the following conditions holds:

(i) there exists a $B_{ii} \neq 0$, $1 \leq i \leq m$, $\lambda = 1$ and $B = \text{diag}(B_{11}, \cdots, B_{mm}, B_2)$;

(ii) $B_{ii} = 0$, for $1 \leq i \leq m$, and $A$ and $B$ are compatible.

**Proof.** It is easy to show that (i) and (ii) imply $AB = \lambda BA$, respectively.

Conversely, from $AB = \lambda BA$, we have $a_{ij} B_{ij} = \lambda a_{ij} B_{ij}$. If there exists a $B_{ii} \neq 0$, for some $1 \leq i \leq m$, then $\lambda = 1$ and $AB = BA$. Since $(A - a_{ij})B = B(A - a_{ij})$, by the Fuglede-Putnam Theorem, $H_i$ is a reducing subspace of $B$. Then $B_{ij} = 0$, $i \neq j$. Hence $B = \text{diag}(B_{11}, \cdots, B_{mm})$. If $B_{ii} = 0$ for $1 \leq i \leq m$, then $AB = \lambda BA$ implies that $a_{ij} B_{ij} = \lambda a_{ij} B_{ij}$. Since $a_i \neq a_j$, $i \neq j$, we obtain that there is at most one nonzero entry in each row and column of $(B_{ij})_{m \times m}$. It is easy to see that with respect to a suitable decomposition of the space, $B$ has a form $B = \bigoplus_{i=1}^s I_{k_i}^{(i)} \bigoplus_{i=s+1}^{s+t} M_{j_i}^{(i)} \oplus B_2$, where

$$I_{k_i}^{(i)} = \begin{pmatrix} 0 & B_{12}^{(i)} & 0 & \cdots & B_{k_i-1}^{(i)} & 0 \\ B_{12}^{(i)} & 0 & \cdots & B_{k_i-1}^{(i)} & 0 & \cdots & B_{(k_i-1)j_i}^{(i)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{12}^{(i)} & 0 & \cdots & B_{k_i-1}^{(i)} & 0 & \cdots & B_{(k_i-1)j_i}^{(i)} \\ \end{pmatrix}, \quad M_{j_i}^{(i)} = \begin{pmatrix} 0 & B_{12}^{(i)} & 0 & \cdots & B_{(k_i-1)j_i}^{(i)} \\ 0 & 0 & \cdots & B_{(k_i-1)j_i}^{(i)} & 0 & \cdots & B_{(k_i-1)j_i}^{(i)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{(k_i-1)j_i}^{(i)} & 0 & \cdots & B_{(k_i-1)j_i}^{(i)} \\ \end{pmatrix},$$

$s, t \in \mathbb{N} \cup \{0\}$, $k_i, j_i \in \mathbb{N}$ and $B_{l, \cdots}^{(i)}$ are exactly the nonzero entries of $(B_{ij})_{m \times m}$. Also, simple computation, we can get that there exists $a^{(i)} \in \sigma(A) \setminus \{0\}, 1 \leq i \leq s + t$ with $a^{(p)} \neq a^{(l)}$ for $1 \leq p \neq l \leq s + t$ such that $A$ has a form $A = \bigoplus_{i=1}^s I_{k_i}^{(i)} \bigoplus_{i=s+1}^{s+t} N_{j_i}^{(i)} \oplus A'$, with respect to the same decomposition of the space, where $A'$ is a diagonal operator and the $k_i \times k_i$ block operator matrix $I_{k_i}^{(i)}$ and the $j_i \times j_i$ block operator matrix $N_{j_i}^{(i)}$ are defined by

$$I_{k_i}^{(i)} = \begin{pmatrix} a^{(i)}I & \lambda a^{(i)}I & \cdots & \lambda^{k_i-1}a^{(i)}I \\ \end{pmatrix}, \quad N_{j_i}^{(i)} = \begin{pmatrix} a^{(i)}I & \lambda a^{(i)}I & \cdots & \lambda^{j_i-1}a^{(i)}I \\ \end{pmatrix}. $$

We will show that if $s > 1$, then $k_1 = \cdots = k_s$. We assume that $k_1 < k_j, 1 \leq i, j \leq s$; then $\lambda^{k_j} = 1$ and $\lambda^{k_i} = 1$. Hence $\lambda^{k_j-k_i} = 1$. Therefore $a^{(i)} = \lambda^{k_j-k_i}a^{(i)}$. It is a contradiction with $a_i \neq a_j, i \neq j$. Hence $k_1 = \cdots = k_s$. Let $k = k_1$; then, clearly we have $\lambda^k = 1$ and the $j_i \leq k$, for $s + 1 \leq i \leq s + t$. That is to say, $A$ and $B$ are compatible.

**Remark.** If $A$ is compact and normal, a result similar to Theorem 2.7 can be obtained by the same deduction. Its representation is more complex, and so it is omitted.
For given $0 \neq A \in B(H)$, let $S_A^\lambda = \{ B \in B(H) : AB = \lambda BA \}$. Then $S_A^\lambda$ is a closed subspace of $B(H)$. If $\lambda^n = 1$, for some $n \in \mathbb{N}$, then for each $B \in S_A^\lambda$, we have $B^{n+1} \in S_A^\lambda$. In fact, $AB^{n+1} = \lambda^{n+1}B^{n+1}A = \lambda B^{n+1}A$. If $H$ is finite dimensional, by Theorem 4.4.6 in [10], we have $S_A^\lambda \neq \{0\}$ if and only if $\sigma(A) \cap \sigma(\lambda A) \neq \phi$.

**Proposition 2.8.** Suppose that $H$ is finite dimensional and $0 \neq A \in B(H)$ is normal. Then

$$\dim S_A^\lambda = (\dim N(A))^2 + \sum_{1 \leq i, j \leq n} \dim H_i \dim H_j \psi(i, j),$$

where $\psi(i, j) = \begin{cases} 1 & a_i = \lambda a_j \\ 0 & a_i \neq \lambda a_j. \end{cases}$

**Proof.** Suppose that $A$ is normal. Then $AB = \lambda BA$ if and only if $A_1 B_1 = \lambda B_1 A_1$. Hence $\dim S_A^\lambda = \dim S_A^\lambda + (\dim N(A))^2$. Next, we assume that $A$ is injective. We will show that $\dim S_A^\lambda = \sum_{1 \leq i, j \leq m} \dim H_i \dim H_j \psi(i, j)$. In this case, $AB = \lambda BA$ if and only if $a_i B_{ij} = \lambda a_j B_{ij}$, for $1 \leq i, j \leq m$. If $a_i \neq \lambda a_j$, i.e. $\psi(i, j) = 0$, for some $1 \leq i, j \leq m$, then $AB = \lambda BA$ implies $B_{ij}$ must be zero. If $a_i = \lambda a_j$, i.e. $\psi(i, j) = 1$, for some $1 \leq i, j \leq m$, then for each $B_{ij} \in M_{\dim H_i \times \dim H_j}$, we have $a_i B_{ij} = \lambda a_j B_{ij}$. Thus $\dim S_A^\lambda = \sum_{1 \leq i, j \leq m} \dim H_i \dim H_j \psi(i, j).$ \hfill \qed

3. **Sequential quantum measure**

We first prove a lemma.

**Lemma 3.1.** Suppose that $A$ is an injective positive operator and $B \in B(H)$ with dense range. If $A \circ B \in P(H)$, then $AB = BA = I$.

**Proof.** Since $R(A)$ is dense in $H$, we have that $R(A^{1/2})$ is dense. That is, $\overline{R(A^{1/2})} = H$. Hence $R(BA^{1/2}) = H$. This implies that $R(A^{1/2}BA^{1/2}) = H$. By the assumption $A \circ B \in P(H)$, we have $A^{1/2}BA^{1/2} = I$. Therefore $R(A^{1/2}) = H$. This shows that $A$ is invertible. Hence $B = A^{-1}$, i.e. $AB = BA = I$. \hfill \qed

If $A$ is positive and $B \in B(H)$, then $A$ and $B$ have operator matrices $A = \text{diag}(A_1, 0)$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ with respect to the space decomposition $H = \overline{R(A)} \oplus N(A)$, respectively. Since

$$(A_1^{1/2} \ 0) \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} (A_1^{1/2} \ 0) = \begin{pmatrix} A_1^{1/2} B_{11} A_1^{1/2} & 0 \\ 0 & 0 \end{pmatrix},$$

$A \circ B \in P(H)$ is equivalent to $A_1 \circ B_{11} \in P(\overline{R(A)})$. With the symbols above we have

**Theorem 3.2.** Suppose that $A \in B(H)$ is positive and that $B \in B(H)$ is selfadjoint. Then $N(A)$ is an invariant subspace of $B$, $N(B|_{\overline{R(A)}})$ is an invariant subspace of $A$, and $A \circ B \in P(H)$ if and only if $AB = BA$ and $A|_{\overline{R(B|_{\overline{R(A)}})}} \circ B|_{\overline{R(B|_{\overline{R(A)}})}} \in P(\overline{R(B|_{\overline{R(A)}})})$.

**Proof.** Suppose that $A \in B(H)$ is positive, $B \in B(H)$ is selfadjoint, $N(A)$ is an invariant subspace of $B$, $N(B|_{\overline{R(A)}})$ is an invariant subspace of $A$ and $A \circ B \in P(H)$. Then we have $A = \text{diag}(A_1, 0)$, $B = \text{diag}(B_{11}, B_{22})$ and $A_1 \circ B_{11} \in$
$P(\overline{R(A)})$, where $A_1 = A|_{R(A)}$, $B_{11} = B|_{R(A)}$, and $B_{22} = B|_{N(A)}$. Hence $B_{11}A_1B_{11} = B_{11}$. If we consider the space decomposition $\overline{R(A)} = N(B_{11}) \oplus \overline{R(B_{11})}$, we have $A_1 = \text{diag}(A_{11}, A_{22})$ and $B_{11} = \text{diag}(0, B_{11}')$. Hence $B_{11}A_1|_{\overline{R(B_{11})}} = I_{\overline{R(B_{11})}}$ and $B_{11}'A_2B_{11}' = B_{11}'$. Then we have $A_{22}B_{11}' \in P(\overline{R(B_{11})})$. By Lemma 3.1, we have $A_{22}B_{11}' = I_{\overline{R(B_{11})}}$. Hence $A_1B_{11} = \begin{pmatrix} 0 & 0 \\ 0 & A_{22}B_{11}' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = B_{11}A_1$. Therefore $AB = \text{diag}(0, I, 0) = BA$ and $A|_{\overline{R(B_{11})}} \circ B|_{\overline{R(B_{11})}} \in P(\overline{R(B_{11})})$.

On the other hand, suppose that $AB = BA$ and $A|_{\overline{R(B_{11})}} \circ B|_{\overline{R(B_{11})}} \in P(\overline{R(B_{11})})$. With respect to $H = \overline{R(A)} \oplus N(A)$, $A$ and $B$ have operator matrices $A = \text{diag}(A_1, 0)$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{pmatrix}$, respectively. By the assumption, we have $AB = \begin{pmatrix} A_1B_{11} & A_1B_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_{11}A_1 & B_{12}'A_1 \\ 0 & 0 \end{pmatrix} = BA$. Hence $B_{12} = 0$ and $A_1B_{11} = B_{11}A_1$. Therefore $N(A)$ is an invariant subspace of $B$. With respect to $\overline{R(A)} = N(B_{11}) \oplus \overline{R(B_{11})}$, $A_1$ and $B_{11}$ have operator matrices $A_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}' & A_{22} \end{pmatrix}$ and $B_{11} = \text{diag}(0, B_{11}')$, respectively. From $A_1B_{11} = B_{11}A_1$ we have $A_{12}B_{11}' = 0$. Hence $A_{12} = 0$ and $A \circ B = \text{diag}(0, A_{12}'B_{11}', A_{22}, 0)$. Therefore $N(B_{11}')$ is an invariant subspace of $A$ and $A \circ B \in P(H)$, since $A|_{\overline{R(B_{11}')}} \circ B|_{\overline{R(B_{11}')}} \in P(\overline{R(B_{11}')})$. 

If $A, B \in \varepsilon(H)$, then we can easily get that the part of the necessity condition of Theorem 3.2 is Theorem 1.3 in [6]. The virtue of our proof is that we can know the structure of $A, B$ and $AB$ clearly. One can use this method to prove Corollary 2.4, Theorem 2.5 and Theorem 2.6 in [6].

The following theorem is an answer to the question raised by S. Gudder and G. Nagy in [6], which is also a generalization of Theorem 2.6 (c) in [6]. This question was also independently answered by A. Gheondea and S. Gudder in [7]. Though our proof is essentially the same as A. Gheondea and S. Gudder’s, maybe our presentation of it is better and simpler. We still retain the proof here.

**Theorem 3.3.** If $A, B \in \varepsilon(H)$, then we have $A \circ B \geq B$ if and only if $AB = BA = B$.

**Proof.** It is clear that $AB = BA = B$ implies $A \circ B \geq B$. Conversely, if

$$A \circ B = \begin{pmatrix} A_2 \circ B_{11} & A_2 \circ B_{12} \\ 0 & 0 \end{pmatrix} \geq \begin{pmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{pmatrix},$$

then $B_{12} = B_{22} = 0$. Hence $B = \text{diag}(B_{11}, 0)$ and $A_2 \circ B_{11} \geq B_{11}$. This implies that $A_2 \circ B_{11} \geq B_{11}$, for $n \in \mathbb{N}$. Suppose that $A_1 = \int_0^1 \lambda dE(x)$ is the spectral calculus of $A_1$. Let $H_1 = E(1)$ and $H_2 = E([0, 1))$. Then for each $x \in H_2$, we have $A_1 \circ x \to 0$ as $n \to \infty$. This implies $B_{11}x = 0$, for $x \in H_2$. Hence $A = \text{diag}(I, A_2', 0)$ and $B = \text{diag}(B_{11}', 0, 0)$, with respect to $H = H_1 \oplus H_2 \oplus N(A)$, where $A_2' = A|_{H_2}$.
and $B_{11}' = B|_{H_1}$. Therefore,

$$AB = \begin{pmatrix} B_{11}' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = BA = B.$$ 

\[\square\]

Acknowledgement

We would like to thank Professor Huai-Xin Cao for his useful help. We also thank the referee for sending us the paper [7] of A. Gheondea and S. Gudder and giving many helpful suggestions which improved the presentation of the paper.

References


College of Mathematics and Information Science, Shaanxi Normal University, Xi’an 710062, P. R. China
E-mail address: yangjia0426@sina.com

College of Mathematics and Information Science, Shaanxi Normal University, Xi’an 710062, P. R. China
E-mail address: hkdu@snnu.edu.cn

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use