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PSEUDOCOMPACT SPACES X AND df-SPACES $C_c(X)$

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Abstract. Let X be a completely regular Hausdorff space, and let $C_c(X)$ be the space $C(X)$ of continuous real-valued functions on X endowed with the compact-open topology. We find various equivalent conditions for $C_c(X)$ to be a df-space, resolving an old question of Jarchow and consolidating work by Jarchow, Mazon, McCoy and Todd. Included are analytic characterizations of pseudocompactness and an example that shows that, for $C_c(X)$, Grothendieck's DF-spaces do not coincide with Jarchow's df-spaces. Any such example necessarily answers a thirty-year-old question on weak barrelledness properties for $C_c(X)$, our original motivation.

1. Introduction

We well know that the strong dual E'_β of any bornological locally convex space (lcs) E with a fundamental sequence of bounded sets (fsbs) is a Fréchet space (a complete metrizable lcs). Jarchow states [3, p. 270] that "nothing seems to be known of when precisely $C_c(X)$ is a df-space", while proving [3, Theorem 12.4.1] that, for E an arbitrary lcs, E'_β is Fréchet if and only if E becomes a df-space when given its Mackey topology. For $E = C_c(X)$, Mazon [7] adds the equivalent condition that each regular Borel measure on X has compact support. More recently, McCoy and Todd [8] characterized X for which the uniform dual of $C_c(X)$ is a Banach space. Unwittingly, the four authors unearthed four equivalent conditions that answer Jarchow's ancient query. Our main theorem will establish these four answers and six more given in terms of X, $C_d(X)$, $C_c(X)$ and the weak, strong and uniform duals of $C_c(X)$, including analogues to Warner's celebrated DF-space result (cf. [11, 10.1.22]).

Consequently, we are able to exhibit a $C_c(X)$ space that is a df-space and not a DF-space. Any such $C_c(X)$, as we shall see, must be ℓ^∞-barrelled and not $\aleph_0$-barrelled, answering Buchwalter and Schmets' implied question [1, p. 349, Remark 1], even older than Jarchow's. The example is key to a companion paper [5] that completely resolves the weak barrelledness picture for $C_c(X)$ spaces.

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The Buchwalter-Schmets question comes first not only chronologically, but also in the following sense: If its answer had been the opposite, then for $C_c(X)$, the DF-spaces would coincide with the *df*-spaces, and the Jarchow question would be moot, having been answered already by Warner. Other pertinent examples are given. We observe *en route* two pretty characterizations of pseudocompactness.

First, we formally state the main result, followed by definitions and proof.

Main Theorem. *The following statements about $E = C_c(X)$ are equivalent.*

(1) *E is a df-space.*
(2) *The strong dual E'_β is a Fréchet space.*
(3) *The strong dual E'_β is a Banach space and equals E'_N.*
(4) *The uniform dual E'_N is a Banach space.*
(5) *E'_β is docile and locally complete.*
(6) *The weak dual E'_σ is docile and locally complete.*
(7) *X is pseudocompact and E'_σ is locally complete.*
(8) *Every countable union of support sets in X is relatively compact.*
(9) *For each sequence $\{\mu_n\}_n$ in E' there exists a sequence $\{\varepsilon_n\}_n \subset (0,1]$ such that $\{\varepsilon_n\mu_n : n \in \mathbb{N}\}$ is equicontinuous.*
(10) *$C_d(X)$ has the countable neighborhood property.*
(11) *Each regular Borel measure on X has compact support.*

Recall that an lcs E is a *df-space* if it contains an fsbs and is c_0-quasibarrelled; i.e., every null sequence in E'_β is equicontinuous, and the class of *df*-spaces properly contains Grothendieck's class of DF-spaces. Note also that every sequentially complete *df*-space is strictly webbed [3, 12.4.6]. E is *locally complete* [11, 5.1.6] if every bounded set is contained in a Banach disc. Since a Mackey space E is c_0-quasibarrelled if and only if E'_β is locally complete [11, 8.2.23(c)], Jarchow's Theorem may be rephrased as below.

Theorem 1.1 (Jarchow)**.** *The strong dual E'_β of an lcs E is a Fréchet space if and only if E has an fsbs and E'_β is locally complete.*

Proof. E'_β is metrizable if and only if E has an fsbs, and a metrizable E'_β is complete if and only if it is locally complete [11, 5.1.9]. □

In general, E'_β is locally complete if E is c_0-quasibarrelled [11, 8.2.23(b)]. Clearly, if $(E, \mathcal{T})$ is a *df*-space, then so is E under any locally convex topology γ between $\mathcal{T}$ and the Mackey topology $\tau(E, E')$. Moreover, the existence of a *df*-space topology γ between $\mathcal{T}$ and $\tau(E, E')$, while ensuring that E'_β is a Fréchet space by Jarchow's Theorem, does not generally ensure that $(E, \mathcal{T})$ is a *df*-space. [Try (E, γ) a Banach space, $\mathcal{T} = \sigma(E, E')$.] But in case $(E, \mathcal{T}) = C_c(X)$ and (E, γ) is a *df*-space, then $(E, \mathcal{T})$ is indeed a *df*-space by the equivalence of (1) and (2) in our Main Theorem.

Corollary 1.2. *If $(E, \mathcal{T}) = C_c(X)$, then either all the topologies between $\mathcal{T}$ and $\tau(E, E')$ make E a df-space, or none do.*

The Corollary fails when "*df*" is replaced by "DF". The counterexample (in the last section) is related to the long-standing question, recently answered by a Saxon-Tweddle example [14], of whether there exists a Mackey $\aleph_0$-barrelled lcs that is not barrelled. We still do not know whether there exists a Mackey $\aleph_0$-barrelled $C_c(X)$ space that is not barrelled.

An lcs E is *docile* [4] if every infinite-dimensional subspace contains an infinite-dimensional bounded set. Clearly, every metrizable lcs is docile, and continuous linear images of docile spaces are docile. Note that (6) and (7) suggest a connection between pseudocompactness of X and docility of the weak dual of $C_c(X)$; in fact, we shall show these conditions to be equivalent to each other and also to the condition that $C_c(X)$ does not contain an isomorphic copy of $\mathbb{R}^{\mathbb{N}}$, the Fréchet space of all real sequences.

2. Additional notation and remarks

Given $A \subset X$ and $\varepsilon > 0$, we put $[A, \varepsilon] = \{f \in C(X) : |f(x)| \leq \varepsilon \text{ for all } x \in A\}$. The sets of the form $[K, \varepsilon]$ with K a compact subset of X and $\varepsilon > 0$ constitute a base of neighborhoods of 0 in $C_c(X)$. For $E = C_c(X)$ we let E'_N denote the dual E' endowed with the topology given by the norm $\|\lambda\| = \sup\{|\lambda(f)| : f \in [X, 1]\}$. The subspace of bounded functions in $C(X)$ is denoted by $C^b(X)$, and $C^b_u(X)$ is this space given the familiar Banach space topology of uniform convergence. Because $C^b_u(X)$ dominates a dense subspace of $C_c(X)$, the uniform dual E'_N of $C_c(X)$ is viewed as a subspace of the strong dual of the Banach space $C^b_u(X)$ in [8], where McCoy and Todd proved the equivalence of (4) and (8), denoting E'_N by $\Lambda(X)$. The notation E'_N is consonant with the use of E'_β and E'_σ as the respective strong and weak duals of E.

An lcs E has the *countable neighborhood property* if, for every sequence $\{U_n\}_n$ of neighborhoods of the origin, there exists a sequence $\{a_n\}_n$ of positive scalars such that $\bigcap_n a_n U_n$ is a neighborhood of zero. Recall that Warner [15] (cf. [11, 10.1.22]) showed that $C_c(X)$ is a DF-space if and only if $C_c(X)$ has the countable neighborhood property, if and only if every countable union of compact sets in X is relatively compact. In going from DF-spaces to df-spaces, (8) and (10) change compact sets to support sets. More particularly, (10) changes $C_c(X)$ to $C_d(X)$, where $C_d(X)$ denotes the space $C(X)$ endowed with the topology of uniform convergence on support sets in X. Since support sets are compact, the topology of $C_d(X)$ is between that of $C_c(X)$ and the topology of pointwise convergence. By definition, a support set K in X is the support of some continuous linear form λ on $C_c(X)$, denoted by $\operatorname{supp}\lambda$, and is the intersection of all closed sets L in X having the property that $\lambda(f) = 0$ whenever f is a member of $C(X)$ that vanishes on L. Also, support sets satisfy the countable chain condition (ccc) (cf. [3], [8]).

Following Fremlin, Garling and Haydon [2], a *Borel measure* μ is a σ-additive real-valued (finite) function on all Borel sets in X. It is *regular* if its negative and positive parts satisfy $\mu(A) = \sup\{\mu(K) : K \text{ is a compact subset of } A\}$. For any regular Borel measure μ there exists a smallest closed subset L of X such that μ vanishes on every Borel set that misses L, and L is said to be the support of μ, again denoted $\operatorname{supp}\mu$, but now possibly noncompact. However, $\operatorname{supp}\lambda = \operatorname{supp}\mu$ holds when either of μ, a nonnegative regular Borel measure, and $\lambda \in C_c(X)'$ is given and the other is appropriately chosen. The unmodified term "support set" will always refer to that for a continuous linear form on $C_c(X)$.

3. Pseudocompactness, docility, local completeness and bornivores

Topologists have long known that X is pseudocompact if and only if there is no locally finite sequence $\{U_n\}_n$ of disjoint nonempty open sets in X, defining $\{U_n\}_n$ to be *locally finite* if each $x \in X$ has a neighborhood that meets only finitely many

of the U_n. The straightforward topological proof combines with the notion of weak dual docility to produce a common analytic characterization of both the notion and pseudocompactness.

Theorem 3.1. *The following assertions are equivalent:*

(1) *X is pseudocompact; i.e., $[X,1]$ is absorbing in $C(X)$;*
(2) *$C_c(X)$ does not contain an isomorphic copy of $\mathbb{R}^{\mathbb{N}}$;*
(3) *the weak dual $C_c(X)'_\sigma$ is docile.*

Proof. (1) $\Rightarrow$ (3): If X is pseudocompact, $C_c(X)$ is dominated by the Banach space $C_u(X)$ whose strong dual, being normed, is docile and contains a subspace that dominates $C_c(X)'_\sigma$. Hence $C_c(X)'_\sigma$ is docile.

(3) $\Rightarrow$ (2): Suppose F is a subspace of $C_c(X)$ isomorphic to $\mathbb{R}^{\mathbb{N}}$. It is well known that the weak dual of $\mathbb{R}^{\mathbb{N}}$, and hence F'_σ, is not docile. Extending members of F' to all of $C_c(X)$ by the Hahn-Banach theorem generates a nondocile subpace of $C_c(X)'_\sigma$, and therefore $C_c(X)'_\sigma$ is not docile.

(2) $\Rightarrow$ (1): Suppose X is not pseudocompact; equivalently, there is a sequence $\{U_n\}_n$ of disjoint nonempty open sets in X that is locally finite. Selecting $x_n \in U_n$, we choose $f_n \in C(X)$ with $f_n(x_n) = 1$ and $f_n(X \setminus U_n) = \{0\}$. Because $\{U_n\}_n$ is locally finite, the series $\sum_n a_n f_n$ converges in $C_c(X)$ for each scalar sequence $\{a_n\}_n \in \mathbb{R}^{\mathbb{N}}$. Define the linear map $T : \mathbb{R}^{\mathbb{N}} \to C_c(X)$ by $T(\{a_n\}_n) = \sum_n a_n f_n$. Since each evaluation functional $\delta_{x_m} \in C_c(X)'$ and satisfies $\delta_{x_m}(\sum_n a_n f_n) = a_m$, we note that T is one-to-one and open onto its image. Moreover, each partial sum map T_N taking $\{a_n\}_n$ into $\sum_{n \le N} a_n f_n$ is continuous, and T is the pointwise limit of the sequence $\{T_N\}_N$. Since the Fréchet space $\mathbb{R}^{\mathbb{N}}$ is barrelled, $\{T_N\}_N$ is equicontinuous and T is continuous [16, 9-3-7]. Thus $C_c(X)$ contains $\mathbb{R}^{\mathbb{N}}$, contradicting (2). □

Let us translate parts (a), (d) and (e) of [1, Theorem 4.1] into the more modern terminology of [11]. Recall that a subset A of X is *bounding* if every $f \in C(X)$ is bounded on A [11, 10.1.16], and an lcs E is *ℓ^∞-barrelled* if every $\sigma(E', E)$-bounded sequence is equicontinuous.

Theorem 3.2 (Buchwalter and Schmets)**.** *The following assertions are equivalent:*

(a) *$C_c(X)$ is ℓ^∞-barrelled;*
(d) *the weak dual $C_c(X)'_\sigma$ is locally complete;*
(e) *a countable union of support sets in X is relatively compact, provided it is bounding.*

Our formal proofs ignore a nonetheless deep and fundamental result of Warner [15, Theorem 11] that gives several characterizations for $C_c(X)$ to have an fsbs. One such is that $[X,1]$ be bornivorous (absorb bounded sets) in $C_c(X)$. This is stronger than the previous condition (1) that $[X,1]$ absorb points in $C_c(X)$, and immediately implies that $\{[X,n] : n \in \mathbb{N}\}$ is an fsbs.

Observation. *The following assertions are equivalent:*

(1) *$[X,1]$ is bornivorous in $C_c(X)$;*
(2) *X is pseudocompact and $C_c(X)$ is locally complete.*

Proof. The Banach disk $[X,1]$ is also a barrel in $C_c(X)$ if either (1) or (2) holds, and then it is bornivorous if and only if $C_c(X)$ is locally complete [11, 5.1.6(iv), 5.1.10]. □

Corollary 3.3. *The following assertions are equivalent:*

(1) $C_c(X)$ *is a* df*-space;*
(2) $C_c(X)$ *has an fsbs and both* $C_c(X)$ *and* $C_c(X)'_\sigma$ *are locally complete;*
(3) $C_c(X)$ *has an fsbs and* $C_c(X)'_\sigma$ *is locally complete;*
(4) $C_c(X)$ *has an fsbs and is* ℓ^∞*-barrelled.*

Proof. (1) $\Rightarrow$ (2): If $C_c(X)$ is a df-space, it has an fsbs by definition, and by (3) of the Main Theorem the strong dual $C_c(X)'_\beta$ is normed with unit ball the polar B of $[X,1]$. Thus $B^\circ = [X,1]^{\circ\circ} = [X,1]$ is bornivorous in $C_c(X)$, which is then locally complete by the Observation. By a form of the Banach-Mackey Theorem (cf. [13, Theorems 2.4, 2.5] and [16, 10-4-5]), every weakly bounded subset of $C_c(X)'$ is strongly bounded, and thus the c_0-quasibarrelled $C_c(X)$ is c_0-barrelled by definition [11]. Therefore, $C_c(X)'_\sigma$ is locally complete [11, 8.2.23(b)].

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (4): The Buchwalter-Schmets Theorem.

(4) $\Rightarrow$ (1): In general, ℓ^∞-barrelled $\Rightarrow$ c_0-barrelled $\Rightarrow$ c_0-quasibarrelled [11]. □

Remark 3.1. Note that (4) is the analogue to the well-known result that $C_c(X)$ is a DF-space if and only if it has an fsbs and is $\aleph_0$-barrelled. Also, from part (7) of the Main Theorem and Corollary 3.3, it follows that for pseudocompact X the space $C_c(X)$ is locally complete provided $C_c(X)'_\sigma$ is locally complete. The converse fails: Put $X = \beta\mathbb{N} \setminus \{p\}$, with $p \in \beta\mathbb{N} \setminus \mathbb{N}$. Then X is pseudocompact and locally compact, and $C_c(X)$ is complete [11, 10.1.24], thus locally complete. But $\mathbb{N}$, a countable union of singleton support sets, has noncompact closure X; the equivalence of (7) and (8) means that $C_c(X)'_\sigma$ is not locally complete.

4. Proof of the Main Theorem

As suggested by the proof of Corollary 3.3, the strong and uniform duals of $C_c(X)$ coincide if and only if $[X,1]$ is bornivorous. We further observe

Lemma 4.1. *If every countable subset of* X *is relatively compact, i.e.,* X *is* ω*-bounded, then* $[X,1]$ *is bornivorous in* $C_c(X)$*.*

Proof. If a subset A of $C_c(X)$ is not absorbed by $[X,1]$, then there are sequences $\{f_n\}_n \subset A$ and $\{x_n\}_n \subset X$ such that $\{f_n(x_n)\}_n$ is unbounded. Since the closure K of $\{x_n\}_n$ is compact, $[K,1]$ is a neighborhood of 0 that does not absorb A, and A is not bounded. □

Other useful sufficient conditions include

Lemma 4.2. *With* $E = C_c(X)$*, if one of* E'_N*,* E'_β*,* E'_σ *is both docile and locally complete, then* $[X,1]$ *is bornivorous.*

Proof. Suppose $[X,1]$ is not bornivorous. Then there exist sequences $\{f_n\}_n \subset C_c(X)$ and $\{x_n\}_n \subset X$ such that $\{f_n\}_n$ is bounded in $C_c(X)$ with each $f_n(x_n) > n$. Boundedness of $\{f_n\}_n$ implies that $\{x_n : n \in \mathbb{N}\}$ is infinite. Indeed, we may assume that $x_m \neq x_n$ for $m \neq n$, so that the evaluation functionals δ_{x_n} form a Hamel basis for their span D in E'. Each nonzero $\gamma \in D$ can be written uniquely as the sum $\gamma = \sum_n b_n \delta_{x_n}$, where all but finitely many of the scalars b_n are zero; thus $\operatorname{supp}\gamma = \{x_n : b_n \neq 0\}$ is finite and we may define $\min\gamma = \min\{|b_n| : x_n \in \operatorname{supp}\gamma\}$. Docility yields a linearly independent sequence $\{\gamma_n\}_n \subset D$ bounded in a given one

of the duals where, by local completeness, the series $\sum_n a_n\gamma_n$ converges for each absolutely summable scalar sequence $\{a_n\}_n$ [13, Theorem 2.1]. In all three cases it is clear that the series converges pointwisely to its sum at each point of $[X,1]$. We may inductively choose an absolutely summable sequence $\{a_n\}_n$ of nonzero scalars converging to 0 so fast that, for each n,

$$\sum_{k>n} |a_k| \cdot \|\gamma_k\| < |a_n| \min \gamma_n.$$

The sum μ of the series $\sum_n a_n\gamma_n$ is in E'. So there is a compact set $K \subset X$ such that μ is bounded on the 0-neighborhood $[K,1]$. Linear independence of $\{\gamma_n\}_n$ means that $\bigcup_n \operatorname{supp}\gamma_n$ is an infinite subset of $\{x_n\}_n$. Hence the union is not a subset of K, whereon $\{f_n\}_n$ is uniformly bounded, and there is a smallest positive integer p such that $\operatorname{supp}\gamma_p \nsubseteq K$. Select $y \in \operatorname{supp}\gamma_p \setminus K$. Since $A = K \cup \{x \neq y : x \in \operatorname{supp}\gamma_p\}$ is closed and misses y and X is completely regular, there exists $g \in [X,1]$ with $g(y) = 1$ and $g(A) = \{0\}$. The fact that $cg \in [K,1]$ for all scalars c means that μ, being linear and bounded on $[K,1]$, must vanish at g. But this is contradicted by the fact that

$$\begin{aligned} |\mu g| &= \left|\sum_{n\geq 1} a_n\gamma_n g\right| = \left|\sum_{n\geq p} a_n\gamma_n g\right| \\ &\geq |a_p\gamma_p g| - \sum_{k>p} |a_k\gamma_k g| \geq |a_p| \min\gamma_p - \sum_{k>p} |a_k| \cdot \|\gamma_k\| > 0. \end{aligned}$$

□

We are now in a position to give a complete proof of our main result. We remind the reader (see introduction) that Mazon [7] was the first to prove that (2) ⇔ (11), and McCoy and Todd [8] that (4) ⇔ (8).

Proof of the Main Theorem. (1) ⇒ (2): The strong dual of any c_0-quasibarrelled space is locally complete [11, 8.2.23(b)]; so Jarchow's Theorem applies.

(2)–(6) *are equivalent*: If any one of (2)–(6) holds, then by Lemma 4.2 the set $[X,1]$ is bornivorous. Thus $E'_\beta = E'_N$ and the Banach-Steinhaus theorem ensures that strongly and weakly bounded subsets of E' coincide; it readily follows that each of (2)–(6) must hold if one of them does.

(6) ⇔ (7): Theorem 3.1.

(7) ⇒ (8): X is pseudocompact means that every subset of X is bounding, and so the Buchwalter-Schmets Theorem applies.

(8) ⇒ (1): Since every singleton subset of X is a support set, Lemma 4.1 implies that $\{[X,n] : n \in \mathbb{N}\}$ is an fsbs, and the Buchwalter-Schmets Theorem that E is ℓ^∞-barrelled and thus c_0-quasibarrelled.

So far, we have (1)–(8) are equivalent.

(7) ⇒ (9): Given $\{\mu_n\}_n \subset E'$, let $\varepsilon_n = (\|\mu_n\| + 1)^{-1}$, so that $\{\varepsilon_n\mu_n\}_n$, being uniformly bounded on the barrel $[X,1]$, is $\sigma(E',E)$-bounded; in fact, it is equicontinuous by [(a) ⇐ (d)] of the Buchwalter-Schmets Theorem.

(9) ⇒ (10): Let $\{U_n\}_n$ be a sequence of 0-neighborhoods in $C_d(X)$. For each n let K_n be a support set in X and let δ_n be a positive scalar such that $[K_n,\delta_n] \subset U_n$. Let μ_n be a positive continuous linear form on $C_c(X)$ whose support set is K_n. From (9) there exist positive numbers ε_n such that $\{\varepsilon_n\mu_n\}_n$ is equicontinuous. Then $\sum_n 2^{-n}\varepsilon_n\mu_n$ is $\sigma(E',E)$-convergent to some $\mu \in E'$ with support K. Now if

$f \in C_c(X)$ with $f(K) = \{0\}$, then both the positive part f^+ of f and its negative part f^- vanish on K, so that $0 = \mu(f^+) = \mu(f^-) = \mu(f)$. Thus for each $n \in \mathbb{N}$ we have $0 = \mu_n(f^+) = \mu_n(f^-) = \mu_n(f)$, which means that $K_n \subset K$. Therefore $[K, 1] \subset [K_n, 1] \subset \delta_n^{-1} U_n$, so that $\bigcap_n \delta_n^{-1} U_n$ is a 0-neighborhood in $C_d(X)$, and (10) holds.

(10) $\Rightarrow$ (8): Let $\{K_n\}_n$ be a sequence of support sets. By (10) there is a sequence $\{a_n\}_n$ of positive numbers such that $U = \bigcap_n a_n[K_n, 1]$ is a 0-neighborhood in $C_d(X)$. Then there must be a support set K in X and an $\varepsilon > 0$ such that $[K, \varepsilon] \subset U$. If $x \in X \setminus K$, then, since X is completely regular, there exists some $f \in [X, 1]$ such that $f(x) = 1$ and $f(K) = \{0\}$. But this means that $(a_n + 1) f \in [K, \varepsilon] \subset U \subset [K_n, a_n]$ and hence $x \notin K_n$; i.e., K contains each K_n. Since K is compact, $\bigcup_n K_n$ is relatively compact.

Thus (1)–(10) are equivalent, and the proof is complete if we show that (8) $\Leftrightarrow$ (11).

(8) $\Rightarrow$ (11): For a regular Borel measure μ on X there exist sequences $\{K_n^+\}_n$ and $\{K_n^-\}_n$ of compact subsets of $\operatorname{supp} \mu^\pm$ such that $\mu^\pm(X) = \sup\{\mu^\pm(K_n^\pm) : n \in \mathbb{N}\}$, respectively, and (8) implies that the respective closure $K^\pm$ of $\bigcup_n K_n^\pm$ is compact. Hence $\mu^\pm$ vanishes on the Borel sets that miss $K^\pm$, and $\mu = \mu^+ - \mu^-$ vanishes on the Borel sets that miss $K^+ \cup K^-$. Therefore, $\operatorname{supp} \mu$ is a closed subset of the compact set $K^+ \cup K^-$, and is compact.

(11) $\Rightarrow$ (8): Let $\{K_n\}_n$ be a sequence of nonempty support sets in X and choose each μ_n a nonnegative regular Borel measure on X with $\mu_n(X) = 1$ and $\operatorname{supp} \mu_n = K_n$. The function μ defined on each Borel set A of X by the equation $\mu(A) = \sum_n 2^{-n} \mu_n(A)$ is, routinely, a nonnegative regular Borel measure on X. The support K of μ obviously contains each K_n, and is compact by (11), so that $\bigcup_n K_n$ is relatively compact. $\square$

5. Examples

Five examples complete the paper.

(1) By Theorem 3.1, *every space $C_c(X)$ whose weak dual is docile* is dominated by the Banach space $C_u(X)$, hence *is docile itself.* The converse fails, as the Fréchet space $C_c(\mathbb{R})$ shows: $\mathbb{R}$ is certainly not pseudocompact, and Theorem 3.1 applies. Moreover,

$$\{f \in C(\mathbb{R}) : \text{the restriction of } f \text{ to } [n, n+1] \text{ is linear for each integer } n\}$$

is a copy of $\mathbb{R}^\mathbb{N}$ predicted by Theorem 3.1.

(2) Although not a df-space, $C_c(\mathbb{R})$ is barrelled, hence ℓ^∞-barrelled, and both $C_c(\mathbb{R})$ and $C_c(\mathbb{R})'_\sigma$ are locally complete. Thus in none of the conditions (2)–(4) of Corollary 3.3 can we drop the explicit requirement of an fsbs, nor the requirement in (6) and (7) of the pseudocompactness of X, equivalently, the docility of $C_c(X)'_\sigma$. There are many similar examples: each realcompact, noncompact space X ensures that $C_c(X)$ is barrelled and that X is not pseudocompact.

(3) Many barrelled $C_c(X)$ spaces are not Baire-like; one concrete example takes $X = \mathbb{Q}$, the rationals [11, 10.1.28]. By [12, Theorem 2.1], any such $C_c(X)$ contains a copy of φ, an $\aleph_0$-dimensional space with the strongest locally convex topology. Since φ is clearly not docile, neither is its superspace $C_c(X)$, providing an abundance of nondocile spaces $C_c(X)$. Interestingly, *each space $C_c(X)$ that contains the nondocile φ also contains the docile* $\mathbb{R}^\mathbb{N}$ via Theorem 3.1. In particular, *every*

barrelled non-Baire-like space $C_c(X)$ *contains both* φ *and* $\mathbb{R}^{\mathbb{N}}$, each the strong dual of the other, and *no infinite-dimensional* $C_c(X)$ *has the strongest locally convex topology* (cf. [6, II.4.7(c)]). However, $C_c(\mathbb{R})$ contains $\mathbb{R}^{\mathbb{N}}$ and not φ.

(4) The completely regular Hausdorff space ω_1 of countable ordinals with the interval topology has been well studied. Morris and Wulbert [10] pointed out that $(E, \mathcal{T}) = C_c(\omega_1)$ is a non-Mackey $\aleph_0$-barrelled space. Warner's theorem [11, 10.1.22] shows it is a DF-space, since every countable union of compact sets in ω_1 is relatively compact. Indeed, a fundamental system of compact sets in ω_1 consists of the initial segments $S_\alpha = \{\beta \in \omega_1 : \beta \leq \alpha\}$. Lemma 4.1, e.g., shows that $[\omega_1, 1]$ is bornivorous in $C_c(\omega_1)$.

We will construct a topology γ between $\mathcal{T}$ and the Mackey topology $\tau(E, E')$ such that (E, γ) is not a DF-space, denying the DF-space analogue to Corollary 1.2. First, define

$$X_0 = \{\alpha \in \omega_1 : \alpha \text{ has no immediate predecessor}\},$$

and for $n = 1, 2, \ldots$ set

$$X_n = \{\alpha + n : \alpha \in X_0\}.$$

Note that $X_0, X_1, X_2, \ldots$ partition ω_1 into countably many uncountable sets, and $\{[S_\alpha, \varepsilon] : \alpha \in X_0 \text{ and } \varepsilon > 0\}$ is a base of $\mathcal{T}$-neighborhoods of 0.

Let

$$G = \{f \in C(\omega_1) : f(X_0) = \{0\}\}.$$

For each $\alpha \in \omega_1$, put $\alpha^\times = \min\{\beta \in X_0 : \beta \geq \alpha\}$, and define a subspace H of $C(\omega_1)$ by writing

$$H = \left\{h \in C(\omega_1) : h(\alpha) = h\left(\alpha^\times\right) \text{ for all } \alpha \in \omega_1\right\}.$$

Clearly, $G \cap H = \{0\}$. Given $f \in C(\omega_1)$, define $h : \omega_1 \to \mathbb{R}$ such that $h(\alpha) = f(\alpha^\times)$ for each $\alpha \in \omega_1$. Observe that h is continuous. Also, $h \in H$ since $\alpha^{\times\times} = \alpha^\times$, and $f - h \in G$, proving that $C(\omega_1)$ is the algebraic direct sum of G and H. Moreover, if $\alpha \in X_0$, $\varepsilon > 0$, and $f \in [S_\alpha, \varepsilon/2]$, then h defined as above takes only certain values on S_α achieved by f, so that $h \in [S_\alpha, \varepsilon/2]$, and we have $f - h \in [S_\alpha, \varepsilon] \cap G$; i.e.,

$$[S_\alpha, \varepsilon/2] \subset ([S_\alpha, \varepsilon] \cap G) + H,$$

proving that $C_c(\omega_1)$ is the topological direct sum of G and H with their induced topologies $\mathcal{T}|_G$ and $\mathcal{T}|_H$.

A routine argument shows that the values $f(\alpha)$ of each $f \in G$ eventually vanish, and if $\mu \in G^*$ is bounded on $[\omega_1, 1] \cap G$, then there is some $\alpha \in \omega_1$ such that μ vanishes on the subspace $G_\alpha = \{f \in G : f(S_\alpha) = \{0\}\}$. Thus μ is bounded on $([\omega_1, 1] \cap G) + G_\alpha = [S_\alpha, 1] \cap G$. Consequently, $C_u(\omega_1)$ and $C_c(\omega_1)$ induce topologies on G yielding the same dual G'. We endow G with the topology ξ having as a base of 0-neighborhoods sets of the form $[S_\alpha \cup A_n, \varepsilon] \cap G$, where $\alpha \in \omega_1$, $A_n = \bigcup_{k=0}^n X_k$ for some nonnegative integer n, and $\varepsilon > 0$. Obviously, ξ is between $\mathcal{T}|_G$ and the uniform (Banach) topology on G. So the topological direct sum

$$(E, \gamma) := (G, \xi) \oplus (H, \mathcal{T}|_H)$$

yields γ finer than $\mathcal{T}$ and coarser than the Mackey topology. Also, each $U_n :=$

$([A_n, 2] \cap G) + H$ is a closed absolutely convex γ-neighborhood of 0, and $U := \bigcap_{n \geq 0} U_n = ([\omega_1, 2] \cap G) + H \supset [\omega_1, 1]$ is bornivorous in (E, γ). But U contains no set of the form $[S_\alpha \cup A_n, \varepsilon] \cap G$; indeed, the latter and not the former set contains the point f having value 3 at $\alpha + n + 1$ and 0 elsewhere. Therefore, U is not a γ-neighborhood of the origin, and (E, γ) is not $\aleph_0$-quasibarrelled, hence cannot be a DF-space.

(5) As noted earlier, $C_c(X)$ is a DF-space if and only if each countable union of compact sets in X is relatively compact. We provide a concrete X for which $C_c(X)$ is a df-space but not a DF-space. In particular, this makes $C_c(X)$ an ℓ^∞-barrelled space (Corollary 3.3) which, by definition, is not $\aleph_0$-quasibarrelled, hence not $\aleph_0$-barrelled, answering the Buchwalter-Schmets question.

Recall that $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ is a closed subset of $\beta\mathbb{N}$, and van Mill [9] provided a non-P-point $x_0 \in \mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ with the following property: Given a sequence of closed subsets $K_n \subset \mathbb{N}^*$ not containing x_0 but satisfying the ccc, the closure of $\bigcup_n K_n$ also does not contain x_0. Accordingly, a countable union of support sets of $X := \mathbb{N}^* \setminus \{x_0\}$ has the same closure in $\beta\mathbb{N}$ as in X, and thus is compact. Therefore, (8) implies that $C_c(X)$ is a df-space.

On the other hand, since x_0 is not a P-point of $\mathbb{N}^*$, there exists a sequence $\{U_n\}_n$ of open neighborhoods of x_0 in $\mathbb{N}^*$ such that $\bigcap_n U_n$ is not a neighborhood of x_0. Hence x_0 is in the closure of $X \setminus (\bigcap_n U_n) = \bigcup_n (X \setminus U_n)$, so that $K_n = X \setminus U_n = \mathbb{N}^* \setminus U_n$ gives a sequence of compact sets whose union has noncompact closure in X; i.e., $C_c(X)$ is not a DF-space.

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