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THE POINCARÉ POLYNOMIAL OF AN MP ARRANGEMENT

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ABSTRACT. Let $\mathcal{A} = \{A_i\}_{i\in I}$ be an mp arrangement in a complex algebraic variety X with corresponding complement $Q(\mathcal{A}) = X\setminus\bigcup_{i\in I} A_i$ and intersection poset $L(\mathcal{A})$. Examples of such arrangements are hyperplane arrangements and toral arrangements, i.e., collections of codimension 1 subtori, in an algebraic torus. Suppose a finite group Γ acts on X as a group of automorphisms and stabilizes the arrangement $\{A_i\}_{i\in I}$ setwise. We give a formula for the graded character of Γ on the cohomology of $Q(\mathcal{A})$ in terms of the graded character of Γ on the cohomology of certain subvarieties in $L(\mathcal{A})$.

1. INTRODUCTION

Let Y be a smooth complex algebraic variety and Γ a finite group of automorphisms of Y. For each $g \in \Gamma$ one may consider the (compactly supported) Poincaré polynomial

$$P_c^\Gamma(Y,t)(g) = \sum_{i\geqslant 0} \operatorname{trace}(g, H_c^i(Y,\mathbb{C}))t^i.$$

Here, $H_c^i(Y,\mathbb{C})$ is the compactly supported deRham (or singular) cohomology of Y with complex coefficients. We are interested in the case when Y is the complement, in a smooth complex variety, of a union of closed subvarieties that satisfy a cohomological condition called minimal purity (mp). Examples of such collections are hyperplane arrangements in complex affine space and toral arrangements, i.e., collections of codimension 1 subtori, in an algebraic torus. We prove the following.

Theorem. *Let $\mathcal{A} = \{A_i\}_{i\in I}$ be an mp arrangement in a complex mp variety X with intersection poset $L(\mathcal{A})$, and let Γ be a finite group of automorphisms of X that stabilizes $\mathcal{A}$ setwise. Let $Q(\mathcal{A}) = X\setminus\bigcup_{i\in I} A_i$ and $g \in \Gamma$. Then the Poincaré polynomial is given by*

$$P_c^\Gamma(Q(\mathcal{A}),t)(g) = \sum_{Z\in L^g(\mathcal{A})} \mu_g(X,Z)(-t)^{c_X(Z)} P_c^\Gamma(Z,t)(g)$$

where $L^g(\mathcal{A})$ is the poset of elements of $L(\mathcal{A})$ fixed by g, μ_g its Möbius function and $c_X(Z)$ is the codimension of Z in X.

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Dimca and Lehrer have proved this when $g \in \Gamma$ fixes each element of $L(\mathcal{A})$, i.e., $L^g(\mathcal{A}) = L(\mathcal{A})$ ([3], Corollary). As in [3], the proof of the theorem uses *weight polynomials* $W_c^\Gamma(X,t) \in R(\Gamma)[t]$, where $R(\Gamma)$ is the complex representation ring of Γ. The reason for this is that the weight polynomial satisfies an additive property as follows. If $Z \subset X$ is a closed Γ-invariant subset, then

$$W_c^\Gamma(X,t) = W_c^\Gamma(Z,t) + W_c^\Gamma(U,t) \tag{1.1}$$

in $R(\Gamma)[t]$ where $U = X \backslash Z$. In general, the Poincaré polynomial does not satisfy such an additive property. However, when the variety X is mp the weight polynomial determines the Poincaré polynomial and we are able to pass from one description to the other. The basic idea then is to decompose the mp variety X in an appropriate manner and to use the additive property of the weight polynomial to determine $W_c^\Gamma(Q(\mathcal{A}),t)$ and consequently $P_c^\Gamma(Q(\mathcal{A}),t)$.

2. Preliminaries

Any complex algebraic variety X of dimension n has a *weight filtration*

$$0 = W_{-1}H_c^j(X,\mathbb{C}) \subset W_0H_c^j(X,\mathbb{C}) \subset \cdots \subset W_{2n}H_c^j(X,\mathbb{C}) = H_c^j(X,\mathbb{C})$$

on each cohomology group $H_c^j(X,\mathbb{C})$ as constructed by Deligne ([1], [2]). A group Γ acting on X respects the filtration, and the graded components

$$Gr_m^W H_c^j(X,\mathbb{C}) = W_mH_c^j(X,\mathbb{C})/W_{m-1}H_c^j(X,\mathbb{C})$$

are then Γ-modules.

Definition 2.1 ([3], 3.1)**.**
(i) An irreducible variety X is *minimally pure(mp)* if each nonzero cohomology group $H_c^j(X,\mathbb{C})$ is a pure Hodge structure of weight $2j-2\dim X$, i.e., $Gr_m^W H_c^j(X,\mathbb{C}) = 0$ unless $m = 2j - 2\dim X$.
(ii) A variety X is *minimally pure* if it is equi-dimensional and for any collection $\{X_1, X_2, \ldots, X_r\}$ of irreducible components of X, the irreducible variety $X_1 \setminus (X_2 \cup \cdots \cup X_r)$ is mp.

The definition of an mp arrangement can now be stated.

Definition 2.2 ([3], 3.6)**.** Let X be a complex mp variety and $\mathcal{A} = \{A_i\}_{i\in I}$ a finite collection of closed subsets of X. Let $L(\mathcal{A})$ be the poset of all intersections of elements of $\mathcal{A}$ ordered by reverse inclusion (including X as the empty intersection). We say that $\mathcal{A}$ is an *mp arrangement* if the following conditions hold.

(a) For each $i \in I$, either A_i is a union of irreducible components of X, or else $\operatorname{codim}_{X_j} A_i \cap X_j = 1$ for all irreducible components X_j of X with $A_i \cap X_j \neq \emptyset$.
(b) For each $Z \in L(\mathcal{A})$ we have
  (i) Z is an mp variety.
  (ii) $\dim Z \cap X_j = \dim Z$ for any irreducible component X_j of X with $Z \cap X_j \neq \emptyset$.
  (iii) The family of subsets $\mathcal{A}_Z = \{A_i \cap Z \mid i \in I, A_i \cap Z \neq Z\}$ of Z satisfies the conditions (a), $(b)(i)$ and $(b)(ii)$ above with X replaced by Z.

Given an mp arrangement $\mathcal{A} = \{A_i\}_{i\in I}$ in a variety X, the complement $Q(\mathcal{A})$ is the variety

$$Q(\mathcal{A}) = X \backslash \bigcup_{i\in I} A_i.$$

Remark 2.3.

(i) The complement $Q(\mathcal{A})$ is an mp variety. This was proved in ([3], 3.7).

(ii) The arrangement $\mathcal{A}_Z$ is called the *restriction of* $\mathcal{A}$ from X to Z and is again an mp arrangement in Z.

(iii) The Möbius function of the poset $L(\mathcal{A})$ will be denoted by μ. If g is an automorphism of X that stabilizes $L(\mathcal{A})$ setwise, then $L(\mathcal{A})^g$ will denote the poset of elements of $L(\mathcal{A})$ fixed by g and μ_g its Möbius function. The properties of the Möbius function are enumerated in [6]. We sometimes write L for $L(\mathcal{A})$ and L^g for $L(\mathcal{A})^g$.

Lastly, we recall the definition of weight polynomials. Here we identify the character of a representation of the group Γ with the vector space affording the representation.

Definition 2.4 ([3], 1.5)**.** Let X be a complex algebraic variety and Γ a group acting on X.

(i) The *weight m equivariant Euler characteristic* of X is the element of $R(\Gamma)$ given by
$$E_c^{\Gamma,m}(X) = \sum_j (-1)^j Gr_m^W H_c^j(X,\mathbb{C}).$$

(ii) The *weight (equivariant Euler) polynomial* of X is the element of $R(\Gamma)[t]$ given by
$$W_c^\Gamma(X,t) = \sum_m E_c^{\Gamma,m}(X)t^m.$$

The additive property (1.1) follows from the existence of the long exact sequence ([4], Lemma)

$$\cdots \to H_c^j(U,\mathbb{C}) \to H_c^j(X,\mathbb{C}) \to H_c^j(Z,\mathbb{C}) \to H_c^{j+1}(U,\mathbb{C}) \to \dots \tag{2.1}$$

that respects the Γ-module structure and the mixed Hodge structure on the cohomology.

Remark 2.5. The Poincaré polynomial can also be considered as an element of $R(\Gamma)[t]$. When X is an mp variety of dimension n, the weight polynomial determines the Poincaré polynomial and we have the following relation in $R(\Gamma)[t]$:
$$W_c^\Gamma(X,t) = t^{-2n}P_c^\Gamma(X,-t^2).$$

3. Proof of the Theorem

Let $\mathcal{A}$ be an mp arrangement in a complex mp variety X of dimension n. For $Z \in L(\mathcal{A})$, let $Z^* = Z \setminus \bigcup W$, where $W \in L(\mathcal{A})$ with $W \subset Z, W \neq Z$. There is a decreasing filtration
$$U_n \subseteq U_{n-1} \subseteq \cdots \subseteq U_0 = X$$
with $U_j = \bigcup Z^*$, where the union is over those $Z \in L(\mathcal{A})$ satisfying $c_X(Z) \geqslant j$. In fact, this is a filtration of closed subsets of X since each $Z \in L(\mathcal{A})$ is closed and $U_j = \bigcup Z$, where the union is over those $Z \in L(\mathcal{A})$ with $c_X(Z) \geqslant j$.

For $j = 0, \dots, n-1$, let $V_j = U_j \setminus U_{j+1}$ and set $V_n = U_n$. It may happen that some of the V_j are empty. In this case we set $W_c^\Gamma(V_j,t) = 0$ in $R(\Gamma)[t]$, i.e., $W_c^\Gamma(V_j,t)(g) = 0$ for all $g \in \Gamma$. Also note that the nonempty V_j are mp varieties, being the disjoint union of mp varieties of the same dimension. This follows from the

definition of the mp property. With this notation we now state the main technical lemma used in the proof of the theorem.

Lemma 3.1. *Let $\mathcal{A}$ be an mp arrangement in a complex variety X of dimension n, and let Γ be a finite group of automorphisms of X that stabilizes $\mathcal{A}$ setwise. The varieties V_j are stabilized by Γ and the following relations hold in $R(\Gamma)[t]$:*

(i) $W_c^\Gamma(V_j,t) = W_c^\Gamma(U_j,t) - W_c^\Gamma(U_{j+1},t)$ *for* $j = 0,1,\dots,n-1$*;*
(ii) $W_c^\Gamma(V_n,t) = W_c^\Gamma(U_n,t)$.
(iii) *Let* $g \in \Gamma$. *For nonempty* V_j,

$$W_c^\Gamma(V_j,t)(g) = \sum_{\substack{Z\in L^g,\\ c_X(Z)=j}} W_c^\Gamma(Z^*,t)(g).$$

Proof. The group Γ stabilizes each U_j and so stabilizes V_j. If V_j is nonempty, (i) follows directly from the additive property (1.1) of the weight polynomial . If V_j is empty, observe that (i), with our convention that $W_c^\Gamma(V_j,t) = 0$, is still correct since in this case $U_j = U_{j+1}$. The second relation (ii) is trivial.

To prove (iii) suppose $Z \in L(\mathcal{A})$ with $c_X(Z) = j$. Then Z^* is the complement in Z of the restricted arrangement $\mathcal{A}_Z$ and so is mp. Hence Z^* is an equi-dimensional variety (possibly irreducible). Let W be an irreducible component of Z^*. Since $\dim W = \dim V_j$, W must also be an irreducible component of V_j and so is closed in V_j. Hence Z^* is closed in V_j. Now V_j is a disjoint union of the Z^*. So, in fact, each Z^* is also open in V_j. By the Mayer-Vietoris sequence for cohomology with compact supports we have

$$H_c^k(V_j,\mathbb{C}) \cong \bigoplus H_c^k(Z^*,\mathbb{C}) \tag{3.1}$$

where the sum is over those $Z \in L(\mathcal{A})$ with $c_X(Z) = j$. An element $g \in \Gamma$ stabilizes V_j, permutes the Z^* and respects the isomorphism (3.1). We therefore have

$$P_c^\Gamma(V_j,t)(g) = \sum_{\substack{Z\in L^g,\\ c_X(Z)=j}} P_c^\Gamma(Z^*,t)(g).$$

The varieties appearing in this expression are mp, and by Remark 2.5, this equation is equivalent to

$$W_c^\Gamma(V_j,t)(g) = \sum_{\substack{Z\in L^g,\\ c_X(Z)=j}} W_c^\Gamma(Z^*,t)(g).$$

□

We are now able to complete the proof of the theorem.

Proof of the Theorem. In the notation of Lemma 3.1, we have

$$\begin{aligned}\sum_{j=0}^{n} W_c^\Gamma(V_j,t) &= \sum_{j=0}^{n-1}\left(W_c^\Gamma(U_j,t) - W_c^\Gamma(U_{j+1},t)\right) + W_c^\Gamma(U_n,t) = W_c^\Gamma(U_0,t)\\ &= W_c^\Gamma(X,t).\end{aligned}$$

Hence,

$$W_c^\Gamma(X,t) = \sum_{j=0}^{n} W_c^\Gamma(V_j,t). \tag{3.2}$$

Lemma 3.1(iii) states

$$W_c^\Gamma(V_j,t)(g) = \sum_{\substack{Z\in L^g,\\ c_X(Z)=j}} W_c^\Gamma(Z^*,t)(g).$$

Substituting this expression into (3.2) we obtain

$$W_c^\Gamma(X,t)(g) = \sum_{Z\in L^g} W_c^\Gamma(Z^*,t)(g).$$

Define functions $f,h : L^g \to \mathbb{C}[t]$ by $f(Z) = W_c^\Gamma(Z,t)(g)$ and $h(Z) = W_c^\Gamma(Z^*,t)(g)$. Then for each $U \in L^g$,

$$f(U) = \sum_{L^g \ni Z \subseteq U} h(Z).$$

It follows by Möbius inversion that

$$h(U) = \sum_{L^g \ni Z \subseteq U} \mu_g(U,Z) f(Z)$$

and, consequently,

$$W_c^\Gamma(X^*,t)(g) = \sum_{Z\in L^g} \mu_g(X,Z) W_c^\Gamma(Z,t)(g).$$

Note, by definition, X^* is $Q(\mathcal{A})$. Each of the varieties in the above expression is mp, and recalling Remark 2.5, the theorem follows. □

Remark 3.2. Suppose $\mathcal{A}$ is a hyperplane arrangement in a complex vector space V with complement $M(\mathcal{A})$, and suppose $\Gamma \subset GL(V)$ is a finite group that stabilizes $\mathcal{A}$ setwise. We use the theorem to derive the Orlik-Solomon formula for the graded character of Γ on the deRham cohomology of $M(\mathcal{A})$ ([5, Cor. 6.16]). It is known that hyperplane arrangements are mp ([3, Ex. 3.3]). The compactly supported Poincaré polynomial of affine space of dimension n is t^{2n}. The theorem then yields

$$P_c^\Gamma(M(\mathcal{A}),t)(g) = \sum_{Z\in L^g} \mu_g(V,Z)(-t)^{c(Z)} t^{2\dim Z}.$$

Using Poincaré duality, this is equivalent to

$$P^\Gamma(M(\mathcal{A}),t)(g) = \sum_{Z\in L^g} \mu_g(V,Z)(-t)^{c(Z)}.$$

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