A NOTE ON THE SUPPORT OF A SOBOLEV FUNCTION
ON A $k$-CELL

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Abstract. It is shown that a $k$-cell (the homeomorphic image of a closed ball in $\mathbb{R}^k$) in $\mathbb{R}^n$, $1 \leq k < n$, cannot support a function in $W^{1,p}(\mathbb{R}^n)$ if $p > \left[\frac{k+1}{2}\right]$, the greatest integer in $(k + 1)/2$.

1. Introduction

In this paper we investigate the question of determining whether the homeomorphic image of a $k$-dimensional closed ball in $\mathbb{R}^n$, $1 \leq k < n$, a $k$-cell, can support a Sobolev function $f \in W^{1,p}(\mathbb{R}^n)$. Since a $k$-cell is nowhere dense in $\mathbb{R}^n$, it is natural to first inquire whether a compact, nowhere dense set can support a Sobolev function. Of course, this question is only of interest when the set has positive Lebesgue measure. For the case $p > n$, the answer is obvious, since any function of $W^{1,p}(\mathbb{R}^n)$ has a continuous representative in $\mathbb{R}^n$, and a nonzero continuous function cannot have its support on a nowhere dense compact set. However, for the case $1 < p \leq n$, Polking [Pol72, Theorem 4] showed that there is a nonzero element of $W^{1,p}(\mathbb{R}^n)$ that does have nowhere dense compact support. A characterization of nowhere dense sets that can support $W^{1,p}(\mathbb{R}^n)$ functions in terms of capacity is given in [AH96, Theorem 11.3.2]. The existence of homeomorphisms that carry sets of Lebesgue measure zero into sets of positive measure is well known. Besicovitch [Bes50] constructed a homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^3$ that carries null sets onto sets of positive measure. In [Ron87], a homeomorphism in $W^{1,q}(\mathbb{R}^n;\mathbb{R}^n)$, with $q < n$, was constructed carrying null sets into sets of positive Lebesgue measure. The question we investigate in this paper is whether a $k$-cell in $\mathbb{R}^n$, $0 < k < n$, can support a Sobolev function $u \in W^{1,p}(\mathbb{R}^n)$. The complete answer to this question remains an open problem. Bagby and Gauthier [BG98] proved that for $n > k > 0$ and $p > \max(1, k-1)$, only the zero function in $W^{1,p}(\mathbb{R}^n)$ has its support contained in a $k$-cell. Our contribution to this question is to offer an improvement of this result for $n \geq 3$. In Theorem 5 of this paper it is shown that the Bagby-Gauthier result remains true by requiring $p > \left[\frac{k+1}{2}\right]$ where $\left[\frac{k+1}{2}\right]$ denotes the greatest integer in $(k + 1)/2$. The main ingredient of the proof is that under these restrictions on $p$, if $u \in W^{1,p}(\mathbb{R}^{k+1})$ is not identically zero, then $u$ has a representative that is defined, continuous and strictly positive (or negative) on a pair of linked spheres of dimension $\left[\frac{k+1}{2}\right]$ and $(k + 1)/2$; see Definition 1.
2. Preliminaries

The Lebesgue measure of a set $E \subset \mathbb{R}^n$ is denoted by $|E|$, its $s$-dimensional Hausdorff measure by $H^s(E)$, and its $p$-capacity by $\gamma_p(E)$. We refer the reader to [MZ97, Section 2.1] for the definitions of $p$-capacity, its comparison to Hausdorff measure, and its relationship to functions in the Sobolev class $W^{1,p}$. In particular, we recall that

$$\gamma_p(E) = 0 \quad \text{if and only if} \quad H^{n-p+\varepsilon}(E) = 0 \quad \text{for all} \quad \varepsilon > 0 \quad \text{and} \quad 1 \leq p \leq n.$$  

The restriction of a function $u$ to a set $E$ is denoted by $u \res E$. With $\Omega \subset \mathbb{R}^n$ an open set and $n \geq 1$, the Sobolev space $W^{1,p}(\Omega)$, $p \geq 1$, consists of those functions $u \in L^p(\Omega)$ for which the first-order distributional partial derivatives of $u$ also belong to $L^p(\Omega)$. The norm on $W^{1,p}(\Omega)$ is given by

$$\|u\|_{1,p;\Omega} = \left( \sum_{k=0}^n \int_\Omega |D^k u|^p \, dx \right)^{1/p}.$$  

An alternate definition of the Sobolev space is furnished by the fact that $C^\infty(\Omega) \cap \{ u : \|u\|_{1,p;\Omega} < \infty \}$ is dense in $W^{1,p}(\Omega)$. A sequence of functions that converges except on a set of $\gamma_p$ zero is said to converge $p$-a.e. A function $u$ is called $p$-quasicontinuous if for each $\varepsilon > 0$, there exists an open set $U \subset \mathbb{R}^n$ with $\gamma_p(U) \leq \varepsilon$ such that $u \res \mathbb{R}^n \setminus U$ is continuous. Any function $u \in W^{1,p}(\mathbb{R}^n)$ has a representative that is $p$-quasicontinuous. Indeed, the pointwise limit of a suitable subsequence of smooth functions $\{u_k\}$ that converge strongly to $u$ in $W^{1,p}$ defines a $p$-quasicontinuous representative; cf. [MZ97, Lemma 2.19]. Throughout, we will employ the notation $\mathbf{u}$ (boldface $u$) to denote a $p$-quasicontinuous representative of $u \in W^{1,p}(\mathbb{R}^n)$ and $B_a^p(r)$ to denote the open ball in $\mathbb{R}^n$ with center $x$ and radius $r$. Recall that an arbitrary $u \in L^p(\mathbb{R}^n)$ has an $L^p$-Lebesgue point almost everywhere; that is,

$$\lim_{r \to 0} \frac{1}{|B_a^p(r)|} \int_{B_a^p(r)} |u(x) - u(a)|^p \, dx = 0$$

for almost all $a \in \mathbb{R}^n$. When $u \in W^{1,p}(\mathbb{R}^n)$, this limit is zero for all $a$ in the complement of a $\gamma_p$ null set. If $a$ is a Lebesgue point for $u$ and if $\{u_k\}$ is taken as the standard mollifiers of $u$, then $u_k(a) \to u(a)$. We will use the notation $\mathring{u}(a,r)$ to denote the integral average of $u$ over the the ball $B_a^p(r)$, and $\mathring{u}(a) := \lim_{r \to 0} \mathring{u}(a,r)$ when the limit exists. Likewise, we let $\nabla \mathring{u}(a)$ denote the value of $\nabla u$ in terms of the limit of its integral averages at $a$.

Throughout, we will assume that $1 \leq p \leq n$ since our problem becomes trivial if $p > n$. We will make extensive use of the “coarea formula”, stated below.

**Theorem 1** ([Fed59, Theorem 3.1]). If $X$ and $Y$ are separable Riemannian manifolds of class 1 of respective dimensions $m$ and $k$, $m \geq k$, and $f : X \to Y$ is a Lipschitzian map, then

$$\int_X g(x)Jf(x) \, dH^m(x) = \int_Y \left( \int_{f^{-1}(y)} g(x) \, dH^{m-k}(x) \right) dH^k(y)$$

whenever $g : X \to \mathbb{R}^1$ is $H^m$ integrable. Here, $Jf(x)$ denotes the square root of the sum of the squares of the determinants of the $k \times k$ minors of the differential of $f$ at $x$. 

We will not need the full strength of Federer’s coarea formula, but merely the case when $X$ and $Y$ are subsets of Euclidean space.

3. Linked spheres in $\mathbb{R}^n$

**Definition 1.** With $S^k$ denoting the standard $k$-sphere in $\mathbb{R}^{k+1}$, let $\Sigma^k_1 := h_1(S^k)$ and $\Sigma_{n-1-k}^2 := h_2(S^{n-1-k})$ be the images of disjoint topological embeddings, $h_1, h_2$, of $S^k$ and $S^{n-1-k}$ into $\mathbb{R}^n$. The linking number of $\Sigma^k_1$ and $\Sigma^k_2$ is defined as the topological degree of the mapping

$$S^k \times S^{n-1-k} \xrightarrow{f} S^{n-1}$$

defined by $f(x, y) = \frac{h_1(x) - h_2(y)}{h_1(x) - h_2(y)}$; see [Hir76] or [Rol76].

**Remark 1.** Recall that the topological degree is defined to be that integer, deg($f$), so that the induced homomorphism of homology groups, $f_*: H_{n-1}(S^k \times S^{n-1-k}) \to H_{n-1}(S^{n-1})$, is given by multiplication by deg($f$). Note that both homology groups are isomorphic to $\mathbb{Z}$. Recall also that if $f$ is smooth, then deg($f$) = $\sum_{x \in f^{-1}(y)} Jf(x)$ where $y$ is a regular value of $f$, and where $Jf(x)$ denotes the Jacobian of $f$ at $x$.

**Theorem 2.** Let $\bar{B}^{n-1}$ be a closed ball in $\mathbb{R}^{n-1}$ and suppose that $h: \bar{B}^{n-1} \to \mathbb{R}^n$ is an embedding, with disjoint $\Sigma^k_1, \Sigma_{n-1-k}^2 \subseteq h(\bar{B}^{n-1})$. Then the linking number of $\Sigma^k_1$ and $\Sigma^k_2$ is 0.

**Proof.** The mapping $f$ in (3.1) can be factored as $f = f_2 \circ f_1$ where

$$f_1: S^k \times S^{n-1-k} \to (h(\bar{B}^{n-1}) \setminus \Sigma_{n-1-k}^2) \times \Sigma_{n-1-k}^2$$

and

$$f_2: (h(\bar{B}^{n-1}) \setminus \Sigma_{n-1-k}^2) \times \Sigma_{n-1-k}^2 \to S^{n-1}.$$

Let $H_i(K)$ denote the $i$th homology group of $K$. Recalling the Künneth theorem, [Mas91] Section XI.4, Theorem 4.1, and the fact that $H_i(S^k)$ and $H_i(S^k \setminus S^j)$ are torsion free, we have that

$$H_q \left((h(\bar{B}^{n-1}) \setminus \Sigma_{n-1-k}^2) \times \Sigma_{n-1-k}^2\right)$$

$$= \sum_{j=0}^q H_j \left(h(\bar{B}^{n-1}) \setminus \Sigma_{n-1-k}^2\right) \otimes H_{q-j} \left(\Sigma_{n-1-k}^2\right).$$

Since $h(\bar{B}^{n-1} \setminus \Sigma_{n-1-k}^2)$ is homeomorphic to $\bar{B}^{n-1} \setminus h^{-1}(\Sigma_{n-1-k}^2)$, the complement of an embedded $(n-1-k)$-sphere, we obtain the following homology groups: for $k > 1$,

$$H_q \left(h(\bar{B}^{n-1}) \setminus \Sigma_{n-1-k}^2\right) = \begin{cases} \mathbb{Z} & \text{when } q = 0, k-1, \text{ and } n-2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_q \left(\Sigma_{n-1-k}^2\right) = \begin{cases} \mathbb{Z} & \text{when } q = 0, \text{ and } n-1-k, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore,

$$H_q \left((h(\bar{B}^{n-1}) \setminus \Sigma_{n-1-k}^2) \times \Sigma_{n-1-k}^2\right) = 0$$

except when

$$q \in \{0, k-1, n-2, n-2-k, 2n-3-k\}.$$
Consequently, $H_{n-1} \left( (h(B^{n-1}) \setminus \Sigma_2^{n-1-k}) \times \Sigma_2^{n-1-k} \right) = 0$ unless $2n - 3 - k = n - 1$; that is, if $k = n - 2$. However, without loss of generality, it can be assumed that $k < n/2$, and therefore $H_{n-1} \left( (h(B^{n-1}) \setminus \Sigma_2^{n-1-k}) \times \Sigma_2^{n-1-k} \right) = 0$ except when $n = 3$. When $n = 3$, the Jordan Curve Theorem can be applied to the curves $\Sigma_1^k$ and $\Sigma_2^{n-1-k}$ in $h(B^3)$ to conclude that one of the curves is null homotopic in the complement of the other. Since the degree is a homotopy invariant, in this case the degree will be 0 as well. 

4. QUASICONTINUOUS REPRESENTATIVES ON SPHERES

For $3 \leq m + 2 \leq n$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we will write $x = (x', x'')$ where $x' = (x_1, x_2, \ldots, x_{m+1})$ and $x'' = (x_{m+2}, \ldots, x_n)$. Let $Q: \mathbb{R}^n \to \mathbb{R}^{n-m-1}$ be defined as $Q(x) = x''$. Then $Q^{-1}(x'')$ is an $(m + 1)$-dimensional “horizontal” affine space. Throughout, we will use the notation $S_m^m(r)$ to denote the $m$-sphere centered at $x \in \mathbb{R}^n$ of radius $r$ that is contained in $Q^{-1}(x'')$. Thus,

$$S_m^m(r) = \{ y \in Q^{-1}(x'') : |y - x'| = r \}$$
$$= \{ y \in \mathbb{R}^n : y = (y', x''), |y' - x'| = r \}.$$  

We will also consider spheres in planes orthogonal to $Q^{-1}(x'')$, using the familiar notation for spheres. Thus, for $b \in S_m^m(r)$ we will consider an $(n - m - 1)$-sphere centered at $b$ in the $(n - m)$-plane orthogonal to $Q^{-1}(a'')$ that contains the line through $a$ and $b$; thus, for $b \in S_m^m(r)$ and $0 < \rho < r$ we define

$$S_{\rho}^{n-m-1}(\rho) = \{ y \in \mathbb{R}^n : |y - b| = \rho, y' = \alpha(b' - a') + a', \alpha \in \mathbb{R}^1 \}.$$ 

It can be shown as a direct consequence of the definition that these spheres are linked. Also, see [Gage81] introductory remark.

For any $a \in \mathbb{R}^n$, let $F_a : \mathbb{R}^n \to \mathbb{R}^{n-m}$ be defined as $F_a(x) = F_a(x', x'') = (|x' - a'|, x'') \in \mathbb{R}^{n-m}$. Thus, for $z = (z_1, \ldots, z_{n-m}) \in \mathbb{R}^{n-m}$, we have

$$F_a^{-1}(z) = \{ y \in \mathbb{R}^n : |y' - a'| = z_1, y'' = (z_2, \ldots, z_{n-m}) \}$$
$$= S^{m}_{m}(w', z_2, \ldots, z_{n-m})(z_1).$$

It is not difficult to verify that $JF_a = 1$ and that $F_a$ is Lipschitz with Lipschitz constant 1. Let $I_r \subset \mathbb{R}^{n-m}$ denote the cube in $\mathbb{R}^{n-m}$ of side length $r > 0$ and center $(r, 0, \ldots, 0)$. Then $F_a^{-1}(I_r) := \bigcup_{w \in I_r} F_a^{-1}(w)$ defines a “rectangular torus.” For example, if $n = 3, m = 1, a = 0 \in \mathbb{R}^3$, and $I_r$ is the $r$ by $r$ square in the $(y, z)$-plane centered at $(r, 0)$, then $F_a^{-1}(I_r)$ is the figure obtained by rotating $I_r$ about the $z$-axis.

Theorem 3. Let $u \in W^{1,p}(\mathbb{R}^n)$, let $m$ be an integer with $n \geq m + 2 \geq 3$, $p > m$ and let $u$ denote an arbitrary, but fixed, $p$-quasiconstant representative of $u$ as determined by the pointwise limit of a suitable subsequence of smooth functions $u_k$ that converge strongly to $u$ in $W^{1,p}(\mathbb{R}^n)$. Then:

(i) $u$ is continuous on $F_a^{-1}(w)$ for $H^{n-m}$-a.e. $w \in \mathbb{R}^{n-m}$.

(ii) If $a \in \mathbb{R}^n$ is an $L^p$-Lebesgue point for both $u$ and $|\nabla u|$ and if $\bar{u}(a) > 0$, then there exists $R_0 > 0$ such that for every $0 < r < R_0$ there exists an $H^{n-m}$-measurable set $E_r \subset I_r$ of positive $H^{n-m}$-measure such that $u$ is continuous and positive on $F_a^{-1}(w)$ for $w \in E_r$. 


Proof. (i) Since \( u \in W^{1,p}(\mathbb{R}^n) \), we know that \( u \) is the strong limit of functions \( u_k \in C^\infty(\mathbb{R}^n) \) and therefore, for each \( \varepsilon > 0 \), there exists an open set \( U_\varepsilon \subset \mathbb{R}^n \) and a subsequence such that \( \gamma_p(U_\varepsilon) < \varepsilon \) and that the \( u_k \) converge to \( u \) uniformly on \( \mathbb{R}^n \setminus U_\varepsilon \); cf. [MZ97, Lemma 2.19]. Choosing a sequence \( \varepsilon_j \to 0 \), we see that 

\[
\gamma_p(U) = 0 \quad \text{where} \quad U := \bigcap_{\varepsilon_j} U_\varepsilon.
\]

Since \( F_a \) is Lipschitz, \( \gamma_p(F_a(U_\varepsilon)) \leq C \gamma_p(U_\varepsilon) < C \varepsilon_j \), where \( C = C(p,n) \). [AH96, Theorem 5.2.1]. Let

\[
E := \bigcap_{\varepsilon_j > 0} F_a(U_\varepsilon).
\]

Then \( \gamma_p(E) = 0 \), so that \( H^{n-p+\varepsilon}(E) = 0 \) for all \( \varepsilon > 0 \), by (2.1). Since \( p > m \), there exists \( \varepsilon > 0 \) and \( 0 < \alpha < 1 \) such that \( n - p + \varepsilon = n - m - \alpha \), and therefore \( H^{n-m-\alpha}(E) = 0 \). This, in turn, implies that \( H^{n-m}(E) = 0 \). If \( w \notin E \), then \( w \notin F_a(U_\varepsilon) \) for some \( j \), which implies that \( F_a^{-1}(w) \cap U_\varepsilon = \emptyset \). Thus, \( u \), the uniform pointwise limit of the \( u_k \) on \( \mathbb{R}^n \setminus U_\varepsilon \), is continuous on \( F_a^{-1}(w) \) for \( w \notin E \). That is, \( u \) is continuous on \( F_a^{-1}(w) \) for \( H^{n-m} \)-a.e. \( w \in \mathbb{R}^n \).

(ii) The proof is divided into three parts.

**Step 1.** For \( H^{n-m} \)-a.e. \( w \in I_r \), \( u_w := u \mathbb{1}_{F_a^{-1}(w)} \), we claim that

\[
\sup_{F_a^{-1}(w)} |u| \leq C \left( \int_{F_a^{-1}(w)} r^{p-m} |\nabla (u_w)|^p + r^{-m} |u_w|^p \, dH^m \right)^{1/p},
\]

with \( C \) a constant. For this, observe that the co-area formula yields

\[
\lim_{k \to \infty} \int_{I_r} \int_{F_a^{-1}(w)} |\nabla u_k - \nabla u|^p + |u_k - u|^p \, dH^m \, dH^{n-m}(w)
\]

\[
= \lim_{k \to \infty} \int_{F_a^{-1}(I_r)} (|\nabla u_k - \nabla u|^p + |u_k - u|^p) \, dH^n
\]

\[= 0.
\]

Thus there is a subsequence of the \( u_k \) (still denoted as the full sequence) such that for \( H^{n-m} \)-a.e. \( w \in I_r \),

\[
\lim_{k,l \to \infty} \int_{F_a^{-1}(w)} |\nabla u_k - \nabla u_l|^p + |u_k - u_l|^p \, dH^m = 0.
\]

This subsequence converges strongly to some element of \( W^{1,p}(F_a^{-1}(w)) \), which we denote by

\[
u \mathbb{1}_{F_a^{-1}(w)}.
\]

Since \( u_k \to u \) uniformly on \( F_a^{-1}(w) \) for \( w \notin E \), observe that \( u \mathbb{1}_{F_a^{-1}(w)} \) is a continuous representative of \( u \mathbb{1}_{F_a^{-1}(w)} \). To ease notation, we will write \( u_w \) for \( u \mathbb{1}_{F_a^{-1}(w)} \). For \( g \in C^\infty(\mathbb{R}^n) \), it is well known that

\[
\sup_{S^a_m(1)} |g| \leq C \left( \int_{S^a_m(1)} |\nabla g|^p + |g|^p \, dH^m \right)^{1/p},
\]

with \( C = C(m,p) \), and by a simple scaling argument that

\[
\sup_{S^a_m(r)} |g| \leq C \left( \int_{S^a_m(r)} r^{p-m} |\nabla g|^p + r^{-m} |g|^p \, dH^m \right)^{1/p}.
\]
Since \( u_k \in F_{a_k}^{-1}(w) \) converges uniformly to \( u \in F_{a}^{-1}(w) \) and strongly to \( u_w \) in the sense of (4.4), applying (4.6) with \( g \) replaced by \( u_k \) yields (4.5).

**Step 2.** We will show that there exist a constant \( C_2 > 0 \) and an \( H^{n-m} \)-measurable set \( E_r \subset I_r \) of positive \( H^{n-m} \)-measure such that

\[
\int_{F_{a}^{-1}(w)} |\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \ dH^m \leq C_2 r^n \quad \text{for each } w \in E_r.
\]

From the hypotheses that \( a \) is an \( L^p \)-Lebesgue point for both \( u \) and \( |\nabla u| \) and that \( \bar{u}(a) > 0 \), it follows that there exist positive numbers \( R \) and \( \kappa \) such that for \( r \in (0, R) \) we have

\[
u(a, r) > \kappa > 0
\]

and

\[
\int_{B^n_a(r)} |\nabla u|^p dH^n \leq \left( |\nabla u(a)|^p + 1 \right) H^{n}(B^n_a(r)).
\]

Using Poincaré’s inequality and (4.9), there exists \( C_1 = C_1(n, p) \) such that

\[
\int_{B^n_a(r)} |u - \bar{u}(a, r)|^p dH^n \leq C_1 \int_{B^n_a(r)} |\nabla u|^p r^p dH^n \leq C_1 \alpha_n \left( |\nabla u(a)|^p + 1 \right) r^{n+p}
\]

where \( \alpha_n \) is the volume of the unit ball in \( \mathbb{R}^n \), and consequently,

\[
\int_{B^n_a(r)} \left| \frac{u - \bar{u}(a, r)}{r} \right|^p dH^n \leq C_1 \alpha_n \left( |\nabla u(a)|^p + 1 \right) r^n \quad \text{for } r \in (0, R).
\]

Employing the co-area formula, (4.11) and (4.9), we have for all \( r \in (0, R) \),

\[
\int_{I_r} \int_{F_{a}^{-1}(w)} |\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \ dH^m(t) dH^{n-m}(w)
\]

\[
= \int_{F_{a}^{-1}(I_r)} |JF_a| \left( |\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \right) dH^n
\]

\[
\leq \int_{B^n_a(r + r\sqrt{n}/2)} \left( |\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \right) dH^n
\]

\[
\leq \alpha_n \left( 1 + \frac{\sqrt{n}}{2} \right)^n (C_1 + 1) \left( |\nabla u(a)|^p + 1 \right) r^n.
\]

That is, setting \( C_2 = \alpha_n \left( 1 + \sqrt{n}/2 \right)^n (C_1 + 1) \left( |\nabla u(a)|^p + 1 \right) \), we have

\[
\int_{I_r} \int_{F_{a}^{-1}(w)} |\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \ dH^m dH^{n-m}(w) \leq C_2 r^n.
\]

Let \( G(w) \) denote the inner integral in this expression, so that we have

\[
\int_{I_r} G(w) \ dH^{n-m}(w) \leq C_2 r^n,
\]

which establishes (4.7).
Step 3. Finally, we will establish (ii) of our theorem. Since $E_r \subset I_r$, notice that for $w \in E_r$, $F_{a^{-1}}(w)$ is an $m$-sphere whose radius, $w_1 =: \rho$, has the property that $r/2 \leq \rho \leq 3r/2$. Thus, using (4.3) and (4.4), we obtain

$$
\sup_{F_{a^{-1}}(w)} \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \\
\leq C \int_{F_{a^{-1}}(w)} \rho^{p-m} \left| \nabla \left( \frac{u_w - \bar{u}(a, r)}{r} \right) \right|^p + \rho^{-m} \left| \frac{u_w - \bar{u}(a, r)}{r} \right|^p \ dH^m \\
\leq C \left( \frac{3}{2} \right)^p \rho^{-m} \int_{F_{a^{-1}}(w)} \left( \nabla (u_w) |^p + \left| \frac{u_w - \bar{u}(a, r)}{r} \right|^p \right) \ dH^m \\
\leq C \left( \frac{3}{2} \right)^p 2^{m-p} \rho^{-m} \int_{F_{a^{-1}}(w)} \left( \nabla (u_w) |^p + \left| \frac{u_w - \bar{u}(a, r)}{r} \right|^p \right) \ dH^m \\
\leq C_2 C \left( \frac{3}{2} \right)^p 2^m.
$$

With $K := (C_2 C \left( \frac{3}{2} \right)^p 2^m)^p$, we have $\sup_{F_{a^{-1}}(w)} |u - \bar{u}(a, r)| \leq Kr$ for $w \in E_r$. This, along with (4.8), implies there exists $R_0 > 0$ such that $u > 0$ on $F_{a^{-1}}(w)$ for $w \in E_r, 0 < r < R_0$.

Theorem 4. Let $n \geq 3, n > m$ and $p > m > n - 1 - m \geq 1$. If $u$ is a non-zero element of $W^{1,p}(\mathbb{R}^n)$, then $u$ has a pair of linked spheres of dimensions $m$ and $n - m$ in its support.

Proof. If $u \in W^{1,p}(\mathbb{R}^n)$ is not identically zero, then there exists $a \in \mathbb{R}^n$ such that $a$ is an $L^p$-Lebesgue point for $u$ and $|\nabla u|$. We will assume without loss of generality, that $\bar{u}(a) > 0$. Applying Theorem 3 we obtain $r_0 > 0$ and a Borel set $E_{r_0} \subset I_{r_0}$ of positive $H^{n-m}$-measure such that for $w \in E_{r_0} \subset \mathbb{R}^{n-m}, u$ is continuous and positive everywhere on $F_{a^{-1}}(w) = S_{(a', w_2, \ldots, w_{n-m})}(w_1)$. With a slight abuse of the notation introduced at the beginning of Section 4, we let $w'' := (w_2, \ldots, w_{n-m})$ so that we now have

$$F_{a^{-1}}(w) = S_{(a', w'')(1)}. $$

Let $W_a := \bigcup_{w \in E_{r_0}} F_{a^{-1}}(w)$. Since $E_{r_0}$ is $H^n$-measurable and $JF_a = 1$, we can appeal to the co-area formula to conclude that

$$H^n(W_a) = \int_{E_{r_0}} H^m \left( F_{a^{-1}}(w) \right) \ dH^{n-m}(w) > 0. $$

Note that $u$ is defined and is positive at all points of $W_a$. For suitable $w \in W_a$, we will construct an $(n - m - 1)$-sphere that will link with $S_{(a', w'')}(1)$ and that will lie in a “radial” $(n - m)$-plane emanating from $(a', w'')$ orthogonal to $Q^{-1}(w'')$. For this purpose define

$$P: \mathbb{R}^n \setminus \bar{B}_{r_0}^{n}(a', w'')(r_0/2) \rightarrow S_{(a', w'')}(1) \quad \text{by} \quad P(x) = \left( a' + \frac{x' - a'}{|x' - a'|}, w'' \right). $$

Observe that $P$ is locally Lipschitz and that $P^{-1}(\theta)$ is independent of $w$ for $\theta \in S_{(a', w')}(1)$. Proceeding as in the proof of Theorem 3 Step 1, with $F_a$ replaced by $P$, an application of the co-area formula yields that $u \bigcup P^{-1}(\theta) \in W^{1,p}(P^{-1}(\theta))$ for $H^m$-a.e. $\theta \in S_{(a', w')}(1)$ and that $u \bigcup P^{-1}(\theta)$ is a $p$-quasicontinuous representative for $u \bigcup P^{-1}(\theta)$; see (4.3) and (4.4). Since $H^n(W_a) > 0$, the co-area formula also
implies that \( H^{n-m}(W_a \cap P^{-1}(\theta)) > 0 \) for \( H^{m}\text{-a.e } \theta \in S^m_{(a',w')} (1) \). Thus, for such \( \theta \), there exists
\begin{equation}
  w \in W_a \cap P^{-1}(\theta)
\end{equation}
such that \( w \) is a Lebesgue point for both \( u \mathcal{L} P^{-1}(\theta) \) and \( \nabla (u \mathcal{L} P^{-1}(\theta)) \). Since \( H^{n-m}(W_a \cap P^{-1}(\theta)) > 0 \) and \( u \mathcal{L} P^{-1}(\theta) > 0 \) on \( W_a \cap P^{-1}(\theta) \), it follows that we can also require \( w \) to have been chosen so that
\begin{equation}
  u \mathcal{L} P^{-1}(\theta)(w) > 0.
\end{equation}
With \( w \) determined by (4.11) and (4.12), it follows that \( u \mathcal{L} P^{-1}(\theta) \) satisfies the hypotheses of Theorem 3 (ii) with the ambient space \( \mathbb{R}^n \) replaced by \( P^{-1}(\theta) \) and with \( F_a \) replaced by \( D : P^{-1}(\theta) \rightarrow \mathbb{R}^1 \), defined by \( D(x) = |x - (a',w')| \). Theorem 3 (ii) provides a number \( 0 < \bar{r} < \overline{w}/2 \) and a set \( A \subset (\bar{r}/2,3\bar{r}/2) \) of positive \( H^1 \)-measure such that \( u \mathcal{L} P^{-1}(\theta) \) is defined and positive on each \( D^{-1}(\rho), \rho \in A \). Thus we have that \( u > 0 \) on the \( (n-m-1) \)-sphere \( D^{-1}(\rho) \) and \( u > 0 \) on the \( m \)-sphere \( S^m_{(a',w')} (r) \). These spheres are linked, since they are similar to the linked spheres (4.1) and (4.2).

\section*{Theorem 5.} Let \( h : \bar{B}^k \rightarrow \mathbb{R}^n \) be an embedding of the closed ball \( \bar{B}^k \subset \mathbb{R}^{k+1} \) where \( 1 \leq k < n \) and \( n \geq 3 \). If \( u \in W^{1,p} (\mathbb{R}^n) \), \( p > \frac{k+1}{2} \) and \( \text{spt } u \subset h(\bar{B}^k) \), then \( u \equiv 0 \).

\textit{Proof.} First, assume \( k \) is even, \( k + 1 < n \), and by contradiction, suppose that \( H^m(\text{spt } u) > 0 \). Writing \( x \in \mathbb{R}^n \) as \( x = (x',x'') \) where \( x' \in \mathbb{R}^{k+1} \), recall that \( Q : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k-1} \) is defined as \( Q(x) := x'' \). Then we have \( H^{k+1}(Q^{-1}(x'') \cap \text{spt } u) > 0 \) for all \( x'' \) in a set \( E \) of positive \( H^{n-k-1} \)-measure and, as in (4.5), \( u \) is a nonzero element of \( W^{1,p}(Q^{-1}(x'')) \) for \( H^{n-k-1} \)-a.e. \( x'' \in E \). Redefine \( E \) to include only such \( x'' \). For \( x'' \in E \), we employ Theorem 4 with \( \mathbb{R}^n \) replaced by the \( (k+1) \)-dimensional affine space \( Q^{-1}(x'') \) and \( m \) replaced by \( k/2 \) to conclude that \( u \in W^{1,p}(Q^{-1}(x'')) \) contains a pair of linked spheres, both of dimension \( k/2 \), in its support. Call these spheres \( S_1 \) and \( S_2 \). With \( h \) as in the statement of our theorem, let \( H := h^{-1} \mathcal{L} (Q^{-1}(x'') \cap h(\bar{B}^k)) \); so \( H \) is a homeomorphism of \( (Q^{-1}(x'') \cap h(\bar{B}^k)) \) into \( \mathbb{R}^k \). Since \( S_1 \) and \( S_2 \) are linked spheres in \( (Q^{-1}(x'') \cap h(\bar{B}^k)) \) and since \( H \) is a homeomorphism, it follows from Definition 1 that \( H(S_1) \) and \( H(S_2) \) are linked in \( \mathbb{R}^k \), which contradicts Theorem 2.

The above proof is easily modified and simpler for the case \( k + 1 = n \). A similar argument holds when \( k \) is odd. \hfill \Box

\section*{References}


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