

A NOTE ON THE SUPPORT OF A SOBOLEV FUNCTION ON A k -CELL

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(Communicated by David Preiss)

ABSTRACT. It is shown that a k -cell (the homeomorphic image of a closed ball in \mathbb{R}^k) in \mathbb{R}^n , $1 \leq k < n$, cannot support a function in $W^{1,p}(\mathbb{R}^n)$ if $p > [\frac{k+1}{2}]$, the greatest integer in $(k+1)/2$.

1. INTRODUCTION

In this paper we investigate the question of determining whether the homeomorphic image of a k -dimensional closed ball in \mathbb{R}^n , $1 \leq k < n$, a k -cell, can support a Sobolev function $f \in W^{1,p}(\mathbb{R}^n)$. Since a k -cell is nowhere dense in \mathbb{R}^n , it is natural to first inquire whether a compact, nowhere dense set can support a Sobolev function. Of course, this question is only of interest when the set has positive Lebesgue measure. For the case $p > n$, the answer is obvious, since any function of $W^{1,p}(\mathbb{R}^n)$ has a continuous representative in \mathbb{R}^n , and a nonzero continuous function cannot have its support on a nowhere dense compact set. However, for the case $1 < p \leq n$, Polking [Pol72, Theorem 4] showed that there is a nonzero element of $W^{1,p}(\mathbb{R}^n)$ that does have nowhere dense compact support. A characterization of nowhere dense sets that can support $W^{1,p}(\mathbb{R}^n)$ functions in terms of capacity is given in [AH96, Theorem 11.3.2]. The existence of homeomorphisms that carry sets of Lebesgue measure zero into sets of positive measure is well known. Besicovitch [Bes50] constructed a homeomorphism from \mathbb{R}^2 to \mathbb{R}^3 that carries null sets onto sets of positive measure. In [Pon87], a homeomorphism in $W^{1,q}(\mathbb{R}^n; \mathbb{R}^n)$, with $q < n$, was constructed carrying null sets into sets of positive Lebesgue measure. The question we investigate in this paper is whether a k -cell in \mathbb{R}^n , $0 < k < n$, can support a Sobolev function $u \in W^{1,p}(\mathbb{R}^n)$. The complete answer to this question remains an open problem. Bagby and Gauthier [BG98] proved that for $n > k > 0$ and $p > \max(1, k-1)$, only the zero function in $W^{1,p}(\mathbb{R}^n)$ has its support contained in a k -cell. Our contribution to this question is to offer an improvement of this result for $n \geq 3$. In Theorem 5 of this paper it is shown that the Bagby-Gauthier result remains true by requiring $p > [\frac{k+1}{2}]$ where $[\frac{k+1}{2}]$ denotes the greatest integer in $\frac{k+1}{2}$. The main ingredient of the proof is that under these restrictions on p , if $u \in W^{1,p}(\mathbb{R}^{k+1})$ is not identically zero, then u has a representative that is defined, continuous and strictly positive (or negative) on a pair of linked spheres of dimension $[\frac{k+1}{2}]$ and $[\frac{k}{2}]$; see Definition 1.

Received by the editors May 3, 2001 and, in revised form, March 11, 2003.
2000 *Mathematics Subject Classification*. Primary 46E35.

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2. PRELIMINARIES

The Lebesgue measure of a set $E \subset \mathbb{R}^n$ is denoted by $|E|$, its s -dimensional Hausdorff measure by $H^s(E)$, and its p -capacity by $\gamma_p(E)$. We refer the reader to [MZ97, Section 2.1] for the definitions of p -capacity, its comparison to Hausdorff measure, and its relationship to functions in the Sobolev class $W^{1,p}$. In particular, we recall that

$$(2.1) \quad \gamma_p(E) = 0 \quad \text{if and only if } H^{n-p+\varepsilon}(E) = 0 \text{ for all } \varepsilon > 0 \text{ and } 1 \leq p \leq n.$$

The restriction of a function u to a set E is denoted by $u \llcorner E$. With $\Omega \subset \mathbb{R}^n$ an open set and $n \geq 1$, the Sobolev space $W^{1,p}(\Omega)$, $p \geq 1$, consists of those functions $u \in L^p(\Omega)$ for which the first-order distributional partial derivatives of u also belong to $L^p(\Omega)$. The norm on $W^{1,p}(\Omega)$ is given by

$$\|u\|_{1,p;\Omega} = \left(\sum_{k=0}^n \int_{\Omega} |D^k u|^p dx \right)^{1/p}.$$

An alternate definition of the Sobolev space is furnished by the fact that $C^\infty(\Omega) \cap \{u : \|u\|_{1,p;\Omega} < \infty\}$ is dense in $W^{1,p}(\Omega)$. A sequence of functions that converges except on a set of γ_p zero is said to converge p -q.e. A function u is called p -quasicontinuous if for each $\varepsilon > 0$, there exists an open set $U \subset \mathbb{R}^n$ with $\gamma_p(U) < \varepsilon$ such that $u \llcorner \mathbb{R}^n \setminus U$ is continuous. Any function $u \in W^{1,p}(\mathbb{R}^n)$ has a representative that is p -quasicontinuous. Indeed, the pointwise limit of a suitable subsequence of smooth functions $\{u_k\}$ that converge strongly to u in $W^{1,p}$ defines a p -quasicontinuous representative; cf. [MZ97, Lemma 2.19]. Throughout, we will employ the notation \mathbf{u} (boldface u) to denote a p -quasicontinuous representative of $u \in W^{1,p}(\mathbb{R}^n)$ and $B_x^n(r)$ to denote the open ball in \mathbb{R}^n with center x and radius r . Recall that an arbitrary $u \in L^p(\mathbb{R}^n)$ has an L^p -Lebesgue point almost everywhere; that is,

$$\lim_{r \rightarrow 0} \frac{1}{|B_a^n(r)|} \int_{B_a^n(r)} |u(x) - u(a)|^p dx = 0$$

for almost all $a \in \mathbb{R}^n$. When $u \in W^{1,p}(\mathbb{R}^n)$, this limit is zero for all a in the complement of a γ_p null set. If a is a Lebesgue point for u and if $\{u_k\}$ is taken as the standard mollifiers of u , then $u_k(a) \rightarrow u(a)$. We will use the notation $\bar{u}(a, r)$ to denote the integral average of u over the ball $B_a^n(r)$, and $\bar{u}(a) := \lim_{r \rightarrow 0} \bar{u}(a, r)$ when the limit exists. Likewise, we let $\overline{\nabla u}(a)$ denote the value of ∇u in terms of the limit of its integral averages at a .

Throughout, we will assume that $1 \leq p \leq n$ since our problem becomes trivial if $p > n$. We will make extensive use of the “coarea formula”, stated below.

Theorem 1 ([Fed59, Theorem 3.1]). *If X and Y are separable Riemannian manifolds of class 1 of respective dimensions m and k , $m \geq k$, and $f: X \rightarrow Y$ is a Lipschitzian map, then*

$$(2.2) \quad \int_X g(x) Jf(x) dH^m(x) = \int_Y \int_{f^{-1}(y)} g(x) dH^{m-k}(x) dH^k(y)$$

whenever $g: X \rightarrow \mathbb{R}^1$ is H^m integrable. Here, $Jf(x)$ denotes the square root of the sum of the squares of the determinants of the $k \times k$ minors of the differential of f at x .

We will not need the full strength of Federer’s coarea formula, but merely the case when X and Y are subsets of Euclidean space.

3. LINKED SPHERES IN \mathbb{R}^n

Definition 1. With S^k denoting the standard k -sphere in \mathbb{R}^{k+1} , let $\Sigma_1^k := h_1(S^k)$ and $\Sigma_2^{n-1-k} := h_2(S^{n-1-k})$ be the images of disjoint topological embeddings, h_1, h_2 , of S^k and S^{n-1-k} into \mathbb{R}^n . The linking number of Σ_1^k and Σ_2^k is defined as the topological degree of the mapping

$$(3.1) \quad S^k \times S^{n-1-k} \xrightarrow{f} S^{n-1}$$

defined by $f(x, y) = \frac{h_1(x) - h_2(y)}{|h_1(x) - h_2(y)|}$; see [Hir76] or [Rol76].

Remark 1. Recall that the topological degree is defined to be that integer, $\text{deg}(f)$, so that the induced homomorphism of homology groups, $f_* : H_{n-1}(S^k \times S^{n-1-k}) \rightarrow H_{n-1}(S^{n-1})$, is given by multiplication by $\text{deg}(f)$. Note that both homology groups are isomorphic to \mathbb{Z} . Recall also that if f is smooth, then $\text{deg}(f) = \sum_{x \in f^{-1}(y)} \mathcal{J}f(x)$ where y is a regular value of f , and where $\mathcal{J}f(x)$ denotes the Jacobian of f at x .

Theorem 2. Let \bar{B}^{n-1} be a closed ball in \mathbb{R}^{n-1} and suppose that $h : \bar{B}^{n-1} \rightarrow \mathbb{R}^n$ is an embedding, with disjoint $\Sigma_1^k, \Sigma_2^{n-1-k} \subset h(\bar{B}^{n-1})$. Then the linking number of Σ_1^k and Σ_2^{n-1-k} is 0.

Proof. The mapping f in (3.1) can be factored as $f = f_2 \circ f_1$ where

$$f_1 : S^k \times S^{n-1-k} \rightarrow (h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}) \times \Sigma_2^{n-1-k}$$

and

$$f_2 : (h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}) \times \Sigma_2^{n-1-k} \rightarrow S^{n-1}.$$

Let $H_i(K)$ denote the i th homology group of K . Recalling the Künneth theorem, [Mas91, Section XI.4, Theorem 4.1], and the fact that $H_i(S^k)$ and $H_i(S^k \setminus S^j)$ are torsion free, we have that

$$\begin{aligned} H_q((h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}) \times \Sigma_2^{n-1-k}) \\ = \sum_{j=0}^q H_j(h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}) \otimes H_{q-j}(\Sigma_2^{n-1-k}). \end{aligned}$$

Since $h(\bar{B}^{n-1} \setminus \Sigma_2^{n-1-k})$ is homeomorphic to $\bar{B}^{n-1} \setminus h^{-1}(\Sigma_2^{n-1-k})$, the complement of an embedded $(n - 1 - k)$ -sphere, we obtain the following homology groups: for $k > 1$,

$$H_q(h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}) = \begin{cases} \mathbb{Z} & \text{when } q = 0, k - 1, \text{ and } n - 2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_q(\Sigma_2^{n-1-k}) = \begin{cases} \mathbb{Z} & \text{when } q = 0, \text{ and } n - 1 - k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$H_q((h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}) \times \Sigma_2^{n-1-k}) = 0$$

except when

$$q \in \{0, k - 1, n - 2, n - 2 - k, 2n - 3 - k\}.$$

Consequently, $H_{n-1}((h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}) \times \Sigma_2^{n-1-k}) = 0$ unless $2n-3-k = n-1$; that is, if $k = n-2$. However, without loss of generality, it can be assumed that $k < n/2$, and therefore $H_{n-1}((h(\bar{B}^{n-1}) \setminus \Sigma_2^{n-1-k}) \times \Sigma_2^{n-1-k}) = 0$ except when $n = 3$. When $n = 3$, the Jordan Curve Theorem can be applied to the curves Σ_1^k and Σ_2^{n-1-k} in $h(\bar{B}^2)$ to conclude that one of the curves is null homotopic in the complement of the other. Since the degree is a homotopy invariant, in this case the degree will be 0 as well. \square

4. QUASICONTINUOUS REPRESENTATIVES ON SPHERES

For $3 \leq m+2 \leq n$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we will write $x = (x', x'')$ where $x' = (x_1, x_2, \dots, x_{m+1})$ and $x'' = (x_{m+2}, \dots, x_n)$. Let $Q: \mathbb{R}^n \rightarrow \mathbb{R}^{n-m-1}$ be defined as $Q(x) = x''$. Then $Q^{-1}(x'')$ is an $(m+1)$ -dimensional “horizontal” affine space. Throughout, we will use the notation $S_x^m(r)$ to denote the m -sphere centered at $x \in \mathbb{R}^n$ of radius r that is contained in $Q^{-1}(x'')$. Thus,

$$(4.1) \quad \begin{aligned} S_x^m(r) &= \{y \in Q^{-1}(x'') : |y - x'| = r\} \\ &= \{y \in \mathbb{R}^n : y = (y', x''), |y' - x'| = r\}. \end{aligned}$$

We will also consider spheres in planes orthogonal to $Q^{-1}(x'')$, using the familiar notation for spheres. Thus, for $b \in S_a^m(r)$ we will consider an $(n-m-1)$ -sphere centered at b in the $(n-m)$ -plane orthogonal to $Q^{-1}(a'')$ that contains the line through a and b ; thus, for $b \in S_a^m(r)$ and $0 < \rho < r$ we define

$$(4.2) \quad S_b^{n-m-1}(\rho) = \{y \in \mathbb{R}^n : |y - b| = \rho, y' = \alpha(b' - a') + a', \alpha \in \mathbb{R}^1\}.$$

It can be shown as a direct consequence of the definition that these spheres are linked. Also, see [Gage81, introductory remark].

For any $a \in \mathbb{R}^n$, let $F_a: \mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$ be defined as $F_a(x) = F_a(x', x'') = (|x' - a'|, x'') \in \mathbb{R}^{n-m}$. Thus, for $z = (z_1, \dots, z_{n-m}) \in \mathbb{R}^{n-m}$, we have

$$\begin{aligned} F_a^{-1}(z) &= \{y \in \mathbb{R}^n : |y' - a'| = z_1, y'' = (z_2, \dots, z_{n-m})\} \\ &= S_{(a', z_2, \dots, z_{n-m})}^m(z_1). \end{aligned}$$

It is not difficult to verify that $JF_a = 1$ and that F_a is Lipschitz with Lipschitz constant 1. Let $I_r \subset \mathbb{R}^{n-m}$ denote the cube in \mathbb{R}^{n-m} of side length $r > 0$ and center $(r, 0, \dots, 0)$. Then $F_a^{-1}(I_r) := \bigcup_{w \in I_r} F_a^{-1}(w)$ defines a “rectangular torus”. For example, if $n = 3, m = 1, a = 0 \in \mathbb{R}^3$, and I_r is the r by r square in the (y, z) -plane with center $(r, 0)$, then $F_a^{-1}(I_r)$ is the figure obtained by rotating I_r about the z -axis.

Theorem 3. *Let $u \in W^{1,p}(\mathbb{R}^n)$, let m be an integer with $n \geq m+2 \geq 3, p > m$ and let \mathbf{u} denote an arbitrary, but fixed, p -quasicontinuous representative of u as determined by the pointwise limit of a suitable subsequence of smooth functions u_k that converge strongly to u in $W^{1,p}(\mathbb{R}^n)$. Then:*

- (i) \mathbf{u} is continuous on $F_a^{-1}(w)$ for H^{n-m} -a.e. $w \in \mathbb{R}^{n-m}$.
- (ii) If $a \in \mathbb{R}^n$ is an L^p -Lebesgue point for both u and $|\nabla u|$ and if

$$\bar{u}(a) > 0,$$

then there exists $R_0 > 0$ such that for every $0 < r < R_0$ there exists an H^{n-m} -measurable set $E_r \subset I_r$ of positive H^{n-m} -measure such that \mathbf{u} is continuous and positive on $F_a^{-1}(w)$ for $w \in E_r$.

Proof. (i) Since $u \in W^{1,p}(\mathbb{R}^n)$, we know that u is the strong limit of functions $u_k \in C^\infty(\mathbb{R}^n)$ and therefore, for each $\varepsilon > 0$, there exists an open set $U_\varepsilon \subset \mathbb{R}^n$ and a subsequence such that $\gamma_p(U_\varepsilon) < \varepsilon$ and that the u_k converge to u uniformly on $\mathbb{R}^n \setminus U_\varepsilon$; cf. [MZ97, Lemma 2.19]. Choosing a sequence $\varepsilon_j \rightarrow 0$, we see that $\gamma_p(U) = 0$ where $U := \bigcap_{\varepsilon_j} U_{\varepsilon_j}$. Since F_a is Lipschitz, $\gamma_p(F_a(U_{\varepsilon_j})) \leq C\gamma_p(U_{\varepsilon_j}) < C\varepsilon_j$, where $C = C(p, n)$, [AH96, Theorem 5.2.1]. Let

$$E := \bigcap_{\varepsilon_j > 0} F_a(U_{\varepsilon_j}).$$

Then $\gamma_p(E) = 0$, so that $H^{n-p+\varepsilon}(E) = 0$ for all $\varepsilon > 0$, by (2.1). Since $p > m$, there exists $\varepsilon > 0$ and $0 < \alpha < 1$ such that $n - p + \varepsilon = n - m - \alpha$, and therefore $H^{n-m-\alpha}(E) = 0$. This, in turn, implies that $H^{n-m}(E) = 0$. If $w \notin E$, then $w \notin F_a(U_{\varepsilon_j})$ for some j , which implies that $F_a^{-1}(w) \cap U_{\varepsilon_j} = \emptyset$. Thus \mathbf{u} , the uniform pointwise limit of the u_k on $\mathbb{R}^n \setminus U_{\varepsilon_j}$, is continuous on $F_a^{-1}(w)$ for $w \notin E$. That is, \mathbf{u} is continuous on $F_a^{-1}(w)$ for H^{n-m} -a.e. $w \in \mathbb{R}^{n-m}$.

(ii) The proof is divided into three parts.

Step 1. For H^{n-m} -a.e. $w \in I_r$, $u_w := u \llcorner F_a^{-1}(w)$, we claim that

$$(4.3) \quad \sup_{F_a^{-1}(w)} |\mathbf{u}| \leq C \left(\int_{F_a^{-1}(w)} r^{p-m} |\nabla(u_w)|^p + r^{-m} |u_w|^p \, dH^m \right)^{1/p},$$

with C a constant. For this, observe that the co-area formula yields

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_{I_r} \int_{F_a^{-1}(w)} |\nabla u_k - \nabla u|^p + |u_k - u|^p \, dH^m \, dH^{n-m}(w) \\ &= \lim_{k \rightarrow \infty} \int_{F_a^{-1}(I_r)} (|\nabla u_k - \nabla u|^p + |u_k - u|^p) \, dH^n \\ &= 0. \end{aligned}$$

Thus there is a subsequence of the u_k (still denoted as the full sequence) such that for H^{n-m} -a.e. $w \in I_r$,

$$(4.4) \quad \lim_{k,l \rightarrow \infty} \int_{F_a^{-1}(w)} |\nabla u_k - \nabla u_l|^p + |u_k - u_l|^p \, dH^m = 0.$$

This subsequence converges strongly to some element of $W^{1,p}(F_a^{-1}(w))$, which we denote by

$$(4.5) \quad u \llcorner F_a^{-1}(w).$$

Since $u_k \rightarrow \mathbf{u}$ uniformly on $F_a^{-1}(w)$ for $w \notin E$, observe that $\mathbf{u} \llcorner F_a^{-1}(w)$ is a continuous representative of $u \llcorner F_a^{-1}(w)$. To ease notation, we will write u_w for $u \llcorner F_a^{-1}(w)$. For $g \in C^\infty(\mathbb{R}^n)$, it is well known that

$$\sup_{S_a^m(1)} |g| \leq C \left(\int_{S_a^m(1)} |\nabla g|^p + |g|^p \, dH^m \right)^{1/p},$$

with $C = C(m, p)$, and by a simple scaling argument that

$$(4.6) \quad \sup_{S_a^m(r)} |g| \leq C \left(\int_{S_a^m(r)} r^{p-m} |\nabla g|^p + r^{-m} |g|^p \, dH^m \right)^{1/p}.$$

Since $u_k \llcorner F_a^{-1}(w)$ converges uniformly to $\mathbf{u} \llcorner F_a^{-1}(w)$ and strongly to u_w in the sense of (4.4), applying (4.6) with g replaced by u_k yields (4.3).

Step 2. We will show that there exist a constant $C_2 > 0$ and an H^{n-m} -measurable set $E_r \subset I_r$ of positive H^{n-m} -measure such that

$$(4.7) \quad \int_{F_a^{-1}(w)} |\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p dH^m \leq C_2 r^m \text{ for each } w \in E_r.$$

From the hypotheses that a is an L^p -Lebesgue point for both u and $|\nabla u|$ and that $\bar{u}(a) > 0$, it follows that there exist positive numbers R and κ such that for $r \in (0, R)$ we have

$$(4.8) \quad \bar{u}(a, r) > \kappa > 0$$

and

$$(4.9) \quad \int_{B_a^n(r)} |\nabla u|^p dH^n \leq (|\overline{\nabla u}(a)|^p + 1) H^n(B_a^n(r)).$$

Using Poincarè's inequality and (4.9), there exists $C_1 = C_1(n, p)$ such that

$$\begin{aligned} \int_{B_a^n(r)} |u - \bar{u}(a, r)|^p dH^n &\leq C_1 \int_{B_a^n(r)} |\nabla u|^p r^p dH^n \\ &\leq C_1 \alpha_n (|\overline{\nabla u}(a)|^p + 1) r^{n+p} \end{aligned}$$

where α_n is the volume of the unit ball in \mathbb{R}^n , and consequently,

$$(4.10) \quad \int_{B_a^n(r)} \left| \frac{u - \bar{u}(a, r)}{r} \right|^p dH^n \leq C_1 \alpha_n (|\overline{\nabla u}(a)|^p + 1) r^n \text{ for } r \in (0, R).$$

Employing the co-area formula, (4.10) and (4.9), we have for all $r \in (0, R)$,

$$\begin{aligned} \int_{I_r} \int_{F_a^{-1}(w)} |\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p dH^m(t) dH^{n-m}(w) \\ = \int_{F_a^{-1}(I_r)} |JF_a| \left(|\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \right) dH^n \\ \leq \int_{B_a^n(r + r\sqrt{n}/2)} \left(|\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p \right) dH^n \\ \leq \alpha_n \left(1 + \frac{\sqrt{n}}{2} \right)^n (C_1 + 1) (|\overline{\nabla u}(a)|^p + 1) r^n. \end{aligned}$$

That is, setting $C_2 = \alpha_n (1 + \sqrt{n}/2)^n (C_1 + 1) (|\overline{\nabla u}(a)|^p + 1)$, we have

$$\int_{I_r} \int_{F_a^{-1}(w)} |\nabla u|^p + \left| \frac{u - \bar{u}(a, r)}{r} \right|^p dH^m dH^{n-m}(w) \leq C_2 r^n.$$

Let $G(w)$ denote the inner integral in this expression, so that we have

$$\int_{I_r} G(w) dH^{n-m}(w) \leq C_2 r^n,$$

which establishes (4.7).

Step 3. Finally, we will establish (ii) of our theorem. Since $E_r \subset I_r$, notice that for $w \in E_r$, $F_a^{-1}(w)$ is an m -sphere whose radius, $w_1 =: \rho$, has the property that $r/2 \leq \rho \leq 3r/2$. Thus, using (4.3) and (4.7), we obtain

$$\begin{aligned} & \sup_{F_a^{-1}(w)} \left| \frac{\mathbf{u} - \bar{u}(a, r)}{r} \right|^p \\ & \leq C \int_{F_a^{-1}(w)} \left(\rho^{p-m} \left| \nabla \left(\frac{u_w - \bar{u}(a, r)}{r} \right) \right|^p + \rho^{-m} \left| \frac{u_w - \bar{u}(a, r)}{r} \right|^p \right) dH^m \\ & \leq C \left(\frac{3}{2} \right)^p \rho^{-m} \int_{F_a^{-1}(w)} \left(|\nabla(u_w)|^p + \left| \frac{u_w - \bar{u}(a, r)}{r} \right|^p \right) dH^m \\ & \leq C \left(\frac{3}{2} \right)^p 2^m r^{-m} \int_{F_a^{-1}(w)} \left(|\nabla(u_w)|^p + \left| \frac{u_w - \bar{u}(a, r)}{r} \right|^p \right) dH^m \\ & \leq C_2 C \left(\frac{3}{2} \right)^p 2^m. \end{aligned}$$

With $K := (C_2 C (\frac{3}{2})^p 2^m)^p$, we have $\sup_{F_a^{-1}(w)} |\mathbf{u} - \bar{u}(a, r)| \leq Kr$ for $w \in E_r$. This, along with (4.8), implies there exists $R_0 > 0$ such that $\mathbf{u} > 0$ on $F_a^{-1}(w)$ for $w \in E_r$, $0 < r < R_0$. \square

Theorem 4. *Let $n \geq 3$, $n > m$ and $p > m \geq n - 1 - m \geq 1$. If u is a non-zero element of $W^{1,p}(\mathbb{R}^n)$, then u has a pair of linked spheres of dimensions m and $n - 1 - m$ in its support.*

Proof. If $u \in W^{1,p}(\mathbb{R}^n)$ is not identically zero, then there exists $a \in \mathbb{R}^n$ such that a is an L^p -Lebesgue point for u and $|\nabla u|$. We will assume without loss of generality, that $\bar{u}(a) > 0$. Applying Theorem 3 we obtain $r_0 > 0$ and a Borel set $E_{r_0} \subset I_{r_0}$ of positive H^{n-m} -measure such that for $w \in E_{r_0} \subset \mathbb{R}^{n-m}$, \mathbf{u} is continuous and positive everywhere on $F_a^{-1}(w) = S_{(a', w_2, \dots, w_{n-m})}^m(w_1)$. With a slight abuse of the notation introduced at the beginning of Section 4, we let $w'' := (w_2, \dots, w_{n-m})$ so that we now have

$$F_a^{-1}(w) = S_{(a', w'')}^m(w_1).$$

Let $W_a := \bigcup_{w \in E_{r_0}} F_a^{-1}(w)$. Since E_{r_0} is H^n -measurable and $JF_a = 1$, we can appeal to the co-area formula to conclude that

$$H^n(W_a) = \int_{E_{r_0}} H^m(F_a^{-1}(w)) dH^{n-m}(w) > 0.$$

Note that \mathbf{u} is defined and is positive at all points of W_a . For suitable $w \in W_a$, we will construct an $(n - m - 1)$ -sphere that will link with $S_{(a', w'')}^m(w_1)$ and that will lie in a “radial” $(n - m)$ -plane emanating from (a', w'') orthogonal to $Q^{-1}(w'')$. For this purpose define

$$P: \mathbb{R}^n \setminus \bar{B}_{(a', w'')}^n(r_0/2) \rightarrow S_{(a', w'')}^m(1) \quad \text{by } P(x) = \left(a' + \frac{x' - a'}{|x' - a'|}, w'' \right).$$

Observe that P is locally Lipschitz and that $P^{-1}(\theta)$ is independent of w for $\theta \in S_{(a', w'')}^m(1)$. Proceeding as in the proof of Theorem 3, Step 1, with F_a replaced by P , an application of the co-area formula yields that $u \llcorner P^{-1}(\theta) \in W^{1,p}(P^{-1}(\theta))$ for H^m -a.e. $\theta \in S_{(a', w'')}^m(1)$ and that $\mathbf{u} \llcorner P^{-1}(\theta)$ is a p -quasicontinuous representative for $u \llcorner P^{-1}(\theta)$; see (4.4) and (4.5). Since $H^n(W_a) > 0$, the co-area formula also

implies that $H^{n-m}(W_a \cap P^{-1}(\theta)) > 0$ for H^m -a.e. $\theta \in S_{(a', w'')}^m(1)$. Thus, for such θ , there exists

$$(4.11) \quad w \in W_a \cap P^{-1}(\theta)$$

such that w is a Lebesgue point for both $u \llcorner P^{-1}(\theta)$ and $\nabla(u \llcorner P^{-1}(\theta))$. Since $H^{n-m}(W_a \cap P^{-1}(\theta)) > 0$ and $\mathbf{u} \llcorner P^{-1}(\theta) > 0$ on $W_a \cap P^{-1}(\theta)$, it follows that we can also require w to have been chosen so that

$$(4.12) \quad \overline{u \llcorner P^{-1}(\theta)}(w) > 0.$$

With w determined by (4.11) and (4.12), it follows that $u \llcorner P^{-1}(\theta)$ satisfies the hypotheses of Theorem 3 (ii) with the ambient space \mathbb{R}^n replaced by $P^{-1}(\theta)$ and with F_a replaced by $D: P^{-1}(\theta) \rightarrow \mathbb{R}^1$, defined by $D(x) = |x - (a', w'')|$. Theorem 3 (ii) provides a number $0 < \bar{r} < w_1/2$ and a set $A \subset (\bar{r}/2, 3\bar{r}/2)$ of positive H^1 -measure such that $\mathbf{u} \llcorner P^{-1}(\theta)$ is defined and positive on each $D^{-1}(\rho)$, $\rho \in A$. Thus we have that $\mathbf{u} > 0$ on the $(n - m - 1)$ -sphere $D^{-1}(\rho)$ and $\mathbf{u} > 0$ on the m -sphere $S_{(a', w'')}^m(r)$. These spheres are linked, since they are similar to the linked spheres (4.1) and (4.2). \square

Theorem 5. *Let $h: \bar{B}^k \rightarrow \mathbb{R}^n$ be an embedding of the closed ball $\bar{B}^k \subset \mathbb{R}^{k+1}$ where $1 \leq k < n$ and $n \geq 3$. If $u \in W^{1,p}(\mathbb{R}^n)$, $p > [\frac{k+1}{2}]$ and $\text{spt } u \subset h(\bar{B}^k)$, then $u \equiv 0$.*

Proof. First, assume k is even, $k + 1 < n$, and by contradiction, suppose that $H^n(\text{spt } u) > 0$. Writing $x \in \mathbb{R}^n$ as $x = (x', x'')$ where $x' \in \mathbb{R}^{k+1}$, recall that $Q: \mathbb{R}^n \rightarrow \mathbb{R}^{n-k-1}$ is defined as $Q(x) := x''$. Then we have $H^{k+1}(Q^{-1}(x'') \cap \text{spt } u) > 0$ for all x'' in a set E of positive H^{n-k-1} -measure and, as in (4.5), u is a nonzero element of $W^{1,p}(Q^{-1}(x''))$ for H^{n-k-1} -a.e. $x'' \in E$. Redefine E to include only such x'' . For $x'' \in E$, we employ Theorem 4 with \mathbb{R}^n replaced by the $(k + 1)$ -dimensional affine space $Q^{-1}(x'')$ and m replaced by $k/2$ to conclude that $u \in W^{1,p}(Q^{-1}(x''))$ contains a pair of linked spheres, both of dimension $k/2$, in its support. Call these spheres S_1 and S_2 . With h as in the statement of our theorem, let $H := h^{-1} \llcorner (Q^{-1}(x'') \cap h(\bar{B}^k))$; so H is a homeomorphism of $(Q^{-1}(x'') \cap h(\bar{B}^k))$ into \mathbb{R}^k . Since S_1 and S_2 are linked spheres in $(Q^{-1}(x'') \cap h(\bar{B}^k))$ and since H is a homeomorphism, it follows from Definition 1 that $H(S_1)$ and $H(S_2)$ are linked in \mathbb{R}^k , which contradicts Theorem 2.

The above proof is easily modified and simpler for the case $k + 1 = n$. A similar argument holds when k is odd. \square

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