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ON CONTRACTIBLE POLYHEDRA THAT ARE NOT SIMPLY CONTRACTIBLE

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ABSTRACT. In answer to a question of Michael, Dydak, Segal and Spież have constructed a contractible polyhedron that is not strictly contractible. In the present note we prove a related result; by using alternative methods we show that there exist contractible polyhedra that are not simply (hence not strictly) contractible.

1. INTRODUCTION

Michael [6] introduced and investigated the concept of strict contractibility. The space X is said to be *strictly contractible* to the point $x_0 \in X$ if there exists a homotopy $H : X \times I \to X$ (here I denotes the segment $[0,1]$) such that:

(a) For every $x \in X$, $H(x,0) = x$ and $H(x,1) = x_0$.
(b) If $H(x,t) = x_0$, then $x = x_0$ or $t = 1$.
(c) For all t, $H(x_0,t) = x_0$.

If only conditions (a) and (b) hold, then a space X is said to be *simply contractible* to the point $x_0 \in X$.

Clearly, every strictly contractible space is also simply contractible. However, the converse does not hold: Consider the following compactum, usually called the *Comb Space* (see, e.g., Example 1.4.8 in [8]):

$$E = (\{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\} \times I) \cup (I \times \{0\}).$$

Then E is not strictly contractible to the point $x_0 = (0,1) \in E$, but it is simply contractible to x_0.

Michael formulated the following interesting question:

Problem 1.1. Does there exist a contractible AR space X that is not strictly contractible to one of its points $x_0 \in X$?

Recently, Dydak, Segal and Spież [2] have answered Problem 1.1 in the affirmative. The purpose of the present note is to prove, by applying the methods developed in [5], that compact contractible polyhedra considered in [2], [5] are not simply (hence not strictly) contractible.

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Theorem 1.2. *There exists a compact polyhedron X and a point $v_0 \in X$ such that*

(i) *X is contractible;*
(ii) *X is not simply contractible to v_0; and*
(iii) *X is not strictly contractible to v_0.*

2. Preliminaries

The *suspension* ΣZ of a space Z is the quotient space of the product $Z \times I$ in which the subspaces $Z \times \{0\}$ and $Z \times \{1\}$ are identified to points v_0 and v_1, respectively, and are called the *vertices* of ΣZ.

Let $p : Z \times I \to \Sigma Z$ be the quotient mapping, $p_1 : Z \times I \to Z$ and $p_2 : \Sigma Z \to I$ be the canonical projections. Obviously, the natural mapping $p^{-1} : \Sigma Z \to Z \times I$ is a multivalued mapping.

Definition 2.1. A mapping $g : \Sigma Z \to \Sigma Z$ is said to be flat if $(p_2 \circ g \circ p)(z_1, \tau) = (p_2 \circ g \circ p)(z_2, \tau)$ for every pair of points $z_1, z_2 \in Z$ and every $\tau \in I$.

Definition 2.2. A homotopy $H : \Sigma Z \times I \to \Sigma Z$ is said to be flat if for every $t \in I$, the mapping $H(\_, t) : \Sigma Z \to \Sigma Z$ is a flat mapping (cf. [5]).

Lemma 2.3. *Let Z be a compact space, $f : \Sigma Z \to \Sigma Z$ a homotopically trivial flat mapping and $H : \Sigma Z \times I \to \Sigma Z$ a homotopy between f and a constant mapping. Suppose that for no fixed $\tau, t \in I$, the set $\{H(p(z,\tau), t) \mid z \in Z\}$ contains both vertices v_0 and v_1. Then there exists a flat homotopy $H' : \Sigma Z \times I \to \Sigma Z$ from f to the constant mapping.*

Proof. Let $a(\tau, t)$ and $b(\tau, t)$ be the minimum and the maximum of the function $p_2(H(p(\_,\ \tau), t)) : Z \to I$ for given numbers τ and t, respectively. Define the mapping $H' : \Sigma Z \times I \to \Sigma Z$ by the following formula:

$$H'(p(z,\tau), t) = p\left(p_1 p^{-1}(H(p(z,\tau), t)),\ \frac{a(\tau,t)}{1 + a(\tau,t) - b(\tau,t)}\right).$$

The set $p^{-1}(H(p(z,\tau), t))$ is not a singleton only in the case when $H(p(z,\tau), t) = v_0$ or $H(p(z,\tau), t) = v_1$. In these cases we have $a(\tau, t) = 0$ and $b(\tau, t) = 1$, respectively. Thus the mapping H' is well defined and obviously has the required properties (cf. [5]). □

3. Proofs

Proposition 3.1. *Let Z be any noncontractible compact metric space such that ΣZ is contractible. Let $H : \Sigma Z \times I \to \Sigma Z$ be any contraction to a point. Then there exist points $z_0, z_1 \in Z$ and numbers $\tau_0, t_0 \in I$ such that $H(p(z_0, \tau_0), t_0) = v_0$ and $H(p(z_1, \tau_0), t_0) = v_1$.*

Proof. Suppose that there did not exist points z_0, z_1 in Z and numbers τ, t such that

$$H(p(z_0, \tau), t) = v_0 \quad \text{and} \quad H(p(z_1, \tau), t) = v_1.$$

By Lemma 2.3 there would then exist a flat homotopy $H : \Sigma Z \times I \to \Sigma Z$ that would connect the identity mapping to the constant one.

There corresponds to H a mapping $h : I^2 \to I$ such that

$$h(\tau, t) = p_2 H(p(z, \tau), t),\ z \in Z.$$

Note that $p_2H(p(z,\tau),t)$ does not depend on z since H is a flat mapping.

Let $P_i : I^2 \to I,\ i \in \{1,2\}$ be the projections $P_1(\tau,t) = \tau$ and $P_2(\tau,t) = t$. Let $l : [0,1] \to I^2$ be a path with $l(0) = (\tau_0, 0)$, where $\tau_0 \in (0,1)$, with $l(1) \in \partial(I^2) \setminus [0,1] \times \{0\}$ (here $\partial(I^2)$ denotes the boundary of the square I^2) and which does not intersect with $h^{-1}(\{0\}) \cup h^{-1}(\{1\})$. Such paths exist since $h^{-1}(\{0\})$ and $h^{-1}(\{1\})$ are closed disjoint sets (cf. [5]).

Consider the cone $C(Z,\tau_0) = \{p(z,\tau) \mid z \in Z,\ \tau \geq \tau_0\} \subset \Sigma Z$ and define a mapping $g : C(Z,\tau_0) \to \Sigma Z \setminus \{v_0, v_1\}$ as follows:

$$g(p(z,\tau)) = H\left(p(z, P_1 l(\frac{\tau-\tau_0}{1-\tau_0})),\ P_2 l(\frac{\tau-\tau_0}{1-\tau_0})\right).$$

Identify the base $p(Z,\tau_0)$ of the cone $C(Z,\tau_0)$ with Z. Then the restriction $g|_Z$ is an inessential mapping of Z to $\Sigma Z \setminus \{v_0,v_1\}$ since every cone is contractible. However, its composition with the natural projection $\Sigma Z \setminus \{v_0, v_1\} \to Z$ is the identity mapping on Z. This contradicts the noncontractibility of the space Z. □

Proof of Theorem 1.2. Let P be any acyclic noncontractible polyhedron. Take, for example, the 2-dimensional polyhedron constructed in the standard way (see, e.g., [4]) from one of the following presentations (cf. [1]):

$$\{a, b \mid\ b^{-2}aba,\ b^{-3k}a^{6k-1}\},\quad k = \pm 1, 2, 3...,$$

or (cf. [3]):

$$\{a_1, ..., a_r \mid\ a_1a_2a_1^{-1}a_2^{-2},\ a_2a_3a_2^{-1}a_3^{-2}, ...,\ a_ra_1a_r^{-1}a_1^{-2}\},\quad r > 3.$$

Then by the Mayer-Vietoris exact sequence and by the Seifert-van Kampen theorem, the suspension ΣP is an acyclic simply connected polyhedron. It follows by the Hurewicz theorem that all homotopy groups $\pi_*(\Sigma P)$ are trivial and hence ΣP is a contractible space.

Let v_0 be a vertex of the suspension ΣP, and let $H : \Sigma P \times I \to \Sigma P$ be any homotopy between the identity mapping and the constant mapping to the point v_0. Since P is a noncontractible compact polyhedron, there exist by Proposition 3.1 points $z_0, z_1 \in \Sigma P$ and numbers $\tau_0, t_0 \in I$ such that $H(p(z_0,\tau_0),t_0) = v_0$ and $H(p(z_1,\tau_0),t_0) = v_1$. Since $v_0 \neq v_1$ it follows that $t_0 \neq 1$. If $\tau_0 = 0$ or 1, then $p(z_0,\tau_0) = p(z_1,\tau_0)$ and $v_0 = v_1$. Hence $\tau_0 \in (0,1)$ and $p(z_0,\tau_0) \neq v_0$. However, $H(p(z_0,\tau_0),t_0) = v_0$. Therefore, $X = \Sigma P$ is not simply contractible to the point v_0. □

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