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COUPLINGS OF UNIFORM SPANNING FORESTS

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Abstract. We prove the existence of an automorphism-invariant coupling for the wired and the free uniform spanning forests on connected graphs with residually amenable automorphism groups.

1. Introduction

All the graphs in this paper will be assumed to be locally finite with at most a countable number of vertices. Given a (connected) graph Γ, there exist two natural random subgraphs on Γ known as the Free Uniform Spanning Forest (FSF) and the Wired Uniform Spanning Forest (WSF) (see [BLPS] or [L1] for introductory material and more). They are defined as follows. Consider any increasing sequence Γ_i of finite connected subgraphs of Γ whose union is all of Γ. Then, as shown in [Pe], the uniform distribution on spanning trees on Γ_i converges and is, by definition, the distribution of FSF. To obtain WSF, identify all of the boundary vertices of Γ_i to obtain a graph $\tilde{\Gamma_i}$. As proven in [Pe], the uniform distribution on spanning trees on $\tilde{\Gamma}_i$ converges and the limiting distribution is the distribution of WSF (by definition).

In [BLPS], it is shown that there exists a monotone coupling μ of WSF and FSF (this follows from [FM]); i.e., μ is a probability measure on $2^{E(\Gamma)} \times 2^{E(\Gamma)}$ (where $E(\Gamma)$ denotes the edge set of Γ) whose projection onto the first factor is the distribution of WSF, whose projection onto the second factor is the distribution of FSF, and such that μ is concentrated on the set of all pairs $(T_1, T_2) \in 2^{E(\Gamma)} \times 2^{E(\Gamma)}$ such that $T_1 \subset T_2$. The proof, however, is purely an existence proof. The authors of [BLPS] posed the following problem: does there exist a natural or explicit monotone coupling of WSF and FSF? In particular, does there exist an automorphism-invariant coupling of WSF and FSF?

We show that if Γ is a (connected, locally finite) graph and $G < Aut(\Gamma)$ is a residually amenable group of automorphisms of Γ, then there exists a coupling of WSF and FSF that is invariant under elements of G. We note that all finitely generated linear groups are residually amenable. See Section 2 for more details on residual amenability.

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We mention that for many graphs Γ, WSF = FSF. This is true if and only if there are no nonconstant harmonic Dirichlet functions on Γ (see [BLPS]). In particular, WSF = FSF whenever Γ is the Cayley graph of a cocompact lattice G in $\mathbb{E}^n$ (for all n) or in $\mathbb{H}^n$ (for $n > 2$). However, WSF $\neq$ FSF when G is a cocompact Fuchsian group.

2. Background on residually amenable groups

We will use the following definition of amenable (see [Z]):

A topological group G is **amenable** if and only if for every compact metrizable space Z such that G acts on Z by homeomorphisms and the map $G \to Homeo(Z)$ is continuous, there exists a Borel probability measure μ on Z that is invariant under the action of G. By invariant, we mean that for all Borel sets $E \subset Z$ and for all $g \in G$, $\mu(gE) = \mu(E)$. The topology on $Homeo(Z)$ is the compact open topology. We note that finite groups and finitely generated abelian groups are amenable (given the discrete topology).

If P is a property of groups, then a topological group G is said to be **residually** P if for every $g \in G - \{1\}$, there exists a surjective continuous homomorphism $\phi : G \to H$ such that H has property P and $\phi(g)$ is not the identity.

We need the following two results from [Z], Prop. 4.1.6. First, if G is amenable and H is a closed subgroup of G, then H is amenable. Second, if K is a closed normal subgroup of G, then G is amenable if and only if both K and G/K are amenable. Now suppose that K_1 and K_2 are closed normal subgroups of G such that both G/K_1 and G/K_2 are amenable. We claim that $G/(K_1 \cap K_2)$ is amenable. To see this, note that $K_1/(K_1 \cap K_2)$ is a closed normal subgroup of $G/(K_1 \cap K_2)$ and $[G/(K_1 \cap K_2)]/[K_1/(K_1 \cap K_2)]$ is isomorphic to G/K_1, which is amenable. Also, $K_1/(K_1 \cap K_2)$ maps injectively into G/K_2 (under the quotient map); so it is amenable too (by the first result). The second result now implies that $G/(K_1 \cap K_2)$ is itself amenable. It follows easily that if G is separable (i.e., G contains a countable dense subset), then G is residually amenable if and only if there exists a decreasing sequence of closed normal subgroups G_i in G such that G/G_i is amenable and $\bigcap_i G_i = \{1\}$. The next lemma is presumably well known, but we did not find it in the literature.

Lemma 2.1. *If Γ is a locally finite graph (with at most a countable number of vertices) and G is a group of automorphisms of Γ, then G is separable (in the open-compact topology).*

Proof. Let V denote the vertex set of Γ. We let $B_r(v)$ denote the radius r neighborhood of a vertex $v \in V$. Since Γ is locally finite, for every integer radius $r > 0$ and $(v, w) \in V \times V$, there are only finitely many bijections from $B_r(v)$ to $B_r(w)$. So there exists a countable set $S \subset G$ such that for any $(v, w) \in V \times V$, if there is a $g \in G$ such that $g(v) = w$, then there exists a $g_r \in S$ that agrees with g on the radius r neighborhood of v. In the open-compact topology, g_r converges to g as $r \to \infty$. So S is dense in G. □

We let $Aut(\Gamma)$ denote the full automorphism group of a graph Γ.

Corollary 2.2. *If $G < Aut(\Gamma)$ is residually amenable, then there exists a decreasing sequence $\{G_i\}$ of closed normal subgroups of G such that G/G_i is amenable and $\bigcap_i G_i = \{1\}$.*

We note that since finite groups are amenable, any residually finite group is residually amenable. A well-known result due to Mal'cev [Ma] (or see theorem 2.7 [We]) states that if G is a finitely generated subgroup of $GL_n(\mathbb{C})$, then G is residually finite. In particular, all discrete groups of isometries of hyperbolic space are residually finite. It is an open problem whether or not all word hyperbolic groups are residually finite (see [KW]). Olshanskii has proved [O] that in a certain sense, almost every finitely generated group is word hyperbolic. So, if all word hyperbolic groups are residually finite, then almost every (finitely generated) group is residually amenable.

There is an example, due to E. A. Scott [S] of a finitely generated infinite simple group that has a free nonabelian subgroup. Such a group cannot be residually amenable. There are groups that are residually amenable but not residually finite. For example, the Baumslag-Solitar group $BS(2,3) = \langle x, y \,|\, xy^2x^{-1} = y^3 \rangle$ is one.

3. Background on determinantal probability measures

Let E be any set (with at most countable cardinality), and let $S < l^2(E)$ be a closed subspace of l^2-summable functions on E. (In general, we write $S < T$ to mean that S is a closed subspace of T.) There exists a natural probability measure $\mathbf{P}^S$ on 2^E defined by

$$\mathbf{P}^S(B \subset \mathfrak{B}) = Det[\langle P_S(\chi^e), \chi^f \rangle_{e,f \in B}]. \tag{1}$$

$\mathbf{P}^S$ is said to be a determinantal probability measure (see [L2] for the history and background of such measures). Here, B is a finite subset of E and $\mathbf{P}^S(B \subset \mathfrak{B})$ is, by definition, the probability with respect to $\mathbf{P}^S$ that B is contained in a random subset of Γ, P_S denotes projection onto S, χ^e denotes the characteristic function of an element $e \in B$. The right-hand side is the determinant of the matrix with columns and rows indexed by elements of B whose (e,f)-entry is equal to the l^2-inner product $\langle P_S(\chi^e), \chi^f \rangle$. Using the inclusion-exclusion principle, it can be shown that this determines a probability measure. A determinantal probability measure is defined as one that arises from this construction (see [L2] for introductory material and more). We let $\mathcal{G}_S$ denote a random subset with distribution $\mathbf{P}^S$.

It is shown in [BLPS] that the distributions of WSF and FSF are determinantal. To be precise, let Γ be a graph. We let $l^2_-(\Gamma)$ be the space of l^2-summable antisymmetric functions f on the (directed) edge set of Γ. By antisymmetric we mean that if e is an edge of Γ and $\check{e}$ is equal to e with the opposite orientation, then $f(e) = -f(\check{e})$. For any edge e of Γ, let $\mathbf{1}_e$ be the characteristic function of e. We let $\chi^e = \mathbf{1}_e - \mathbf{1}_{\check{e}}$ be the unit flow along the (directed) edge e.

For any vertex v of Γ, define the star of v to be $\Sigma_e \, \chi^e$ where the sum is over all directed edges e with initial endpoint equal to v. We let $\bigstar$ be the space in $l^2_-(\Gamma)$ generated by all stars of all vertices.

We say that $(e_1, \ldots, e_k)$ is a cycle of edges in Γ if the destination vertex of e_i is equal to the initial vertex of e_{i+1} for all i mod k. In this case, we say that the function $\Sigma_i \, \chi^e \in l^2_-(\Gamma)$ is a cycle. We let $\diamondsuit$ be the subspace of $l^2_-(\Gamma)$ generated by all cycles in Γ. It is shown in [BLPS] that

$$l^2_-(\Gamma) = \bigstar \oplus \diamondsuit \oplus \nabla HD \tag{2}$$

where ∇HD is the gradient of the space of all harmonic Dirichlet functions on Γ. It is proven in [BLPS] that the distributions of WSF and FSF are the determinantal probability measures associated to $\bigstar$ and $\bigstar \oplus \nabla HD = \diamondsuit^\perp$ respectively. To be precise, if $\vec{B}$ is a set of directed edges such that if $e \in \vec{B}$, then $\check{e} \notin \vec{B}$, then let B denote the same set of edges of $\vec{B}$ but without orientation. Then we define $\mathbf{P}^S$ by

$$\mathbf{P}^S(B \subset \mathfrak{B}) = Det[\langle P_S(\chi^e), \chi^f \rangle_{e,f \in \vec{B}}] \tag{3}$$

where S is a subspace of $l^2_-(\Gamma)$. It follows from elementary properties of deteminants that this is well defined (i.e., it is independent of the choice of $\vec{B}$). It is shown in [BLPS] that $\mathbf{P}^{\bigstar}$ is the distribution of WSF and $\mathbf{P}^{\diamondsuit^\perp}$ is the distribution of FSF. For our purposes, it suffices to assume this as the definition of the distributions of WSF and FSF. One easily deduces that WSF = FSF if and only if $\nabla HD = 0$. This occurs, for example, when Γ is the Cayley graph of an amenable group or of a Kazhdan group. See Bekka and Valette [BV] (Theorem D) for a more complete list of groups whose Cayley graphs satisfy $\nabla HD = 0$.

We will need the following facts from [L2] (Theorem 5.2). If $S < T < l^2_-(\Gamma)$ and S and T are closed, then for any increasing event $\mathcal{A} \subset 2^{E(\Gamma)}$ ($\mathcal{A}$ is increasing whenever $A \in A$ and $A \subset B \subset E(\Gamma)$ implies $B \in \mathcal{A}$), $\mathbf{P}^S(\mathcal{A}) \leq \mathbf{P}^T(\mathcal{A})$. By Strassen's theorem [St], this implies the existence of a monotone coupling between $\mathbf{P}^S$ and $\mathbf{P}^T$.

4. THE THEOREM

In this section we state and prove the main theorem.

Theorem 4.1. *If* Γ *is a connected locally finite graph (with at most a countable number of vertices) and* $G < Aut(\Gamma)$ *is residually amenable (with respect to the compact-open topology), then there exists a* G*-invariant monotone coupling between the WSF and the FSF on* Γ*.*

Since G is residually amenable and separable, there exists a decreasing sequence $\{G_i\}$ of closed normal subgroups of G such that G/G_i is amenable for all i and $\bigcap_i G_i = \{1\}$.

We let Γ/G_i be the quotient graph. Its vertices (edges) are equivalence classes of vertices (edges) in Γ where v is equivalent to w if there is a $g \in G_i$ such that $gv = w$. If $[v]$ and $[w]$ are vertices of Γ/G_i, then there is an edge between them for every equivalence class $[e]$ of edges such that one endpoint of e is in $[v]$ and the other is in $[w]$. Consider $l^2_-(\Gamma/G_i)$. We let $\bigstar_i$ be the subspace generated by all the stars of Γ/G_i and $\diamondsuit_i$ be the subspace generated by all the cycles of Γ/G_i. We let ∇HD_i denote the gradient of the harmonic Dirichlet functions on Γ/G_i. So,

$$l^2_-(\Gamma/G_i) = \bigstar_i \oplus \diamondsuit_i \oplus \nabla HD_i. \tag{4}$$

Let $\pi_i : \Gamma \to \Gamma/G_i$ denote the quotient map. Suppose c is a cycle in $l^2_-(\Gamma/G_i)$. Then $c = \sum_{i=1}^n \chi^{e_i}$ where $e_1, e_2, \ldots, e_n$ is a sequence of directed edges such that the terminal vertex of e_i is equal to the initial vertex of e_{i+1} (mod n). We will say that c is a true cycle if every directed component of $\pi_i^{-1}(\{e_1, \ldots, e_n\})$ forms a (finite) cycle in Γ. We let C_i denote the closed subspace of $\diamondsuit_i$ spanned by all the true cycles and let H_i denote its orthocomplement in $\diamondsuit_i$. Now we have

$$l^2_-(\Gamma/G_i) = \bigstar_i \oplus C_i \oplus H_i \oplus \nabla HD_i. \tag{5}$$

Now let W_i be the random subgraph of Γ/G_i associated to $\bigstar_i$. Let F_i be the random subgraph associated to $\bigstar_i \oplus H_i \oplus \nabla HD_i = C_i^\perp$.

The strategy of the proof is as follows. We show that in a certain sense, a lift of W_i converges to WSF and a lift of F_i converges to FSF. Using the amenability of G/G_i, we can average any coupling of W_i and F_i over G/G_i to obtain a G/G_i-invariant coupling. By a compactness argument, a limit point of the sequence of lifted couplings exists and we will show that any limit point is a G-invariant monotone coupling of WSF and FSF.

We need to introduce some topological spaces. First, we let $Z = 2^{E(\Gamma)}$ be the space of subgraphs of Γ. It is compact in the product topology. The group G acts on Z in the natural way, and this action is bi-continuous. We let M_Z denote the set of Borel probability measures on Z, and let M_Z^i denote the subset of M_Z of measures that are G-invariant. By invariance we mean if $\mu \in M_Z^i$, $E \subset Z$ and $g \in G$, then $\mu(gE) = \mu(E)$. We say that a sequence $\{\mu_i\}$ in M_Z converges to μ in the weak* topology if

$$\int_Z f\, d\mu_i \to \int_Z f\, d\mu \tag{6}$$

for every continuous function $f : Z \to \mathbb{C}$. It follows from standard functional analysis that both M_Z and M_Z^i are compact under the weak* topology. Note that for every finite subset $B \subset E(\Gamma)$, the function $f_B : Z \to \mathbb{C}$ defined by $f_B(z) = 1$ if $B \subset z$ and $f_B(z) = 0$ otherwise is continuous. It is not hard to show that if

$$\int_Z f_B\, d\mu_i \to \int_Z f_B\, d\mu \tag{7}$$

for every finite set $B \subset E(\Gamma)$, then μ_i converges to μ in the weak* topology. In this case, if $\mathcal{G}_i$ is a random subgraph with distribution μ_i and $\mathcal{G}$ is a random subgraph with distribution μ, then we will say that $\mathcal{G}_i$ converges to $\mathcal{G}$ (weak*). Similarly, we define $M^i_{Z\times Z}$ to be the space of G-invariant Borel probablity measures on $Z \times Z$. It is compact under the weak* topology.

If $\{S_i\}$ is a sequence of closed subspaces in $l^2_-(\Gamma)$, we will say that $P_{S_i} \to P_S$ in the strong operator topology (SOT) if for every $f \in l^2_-(\Gamma)$, $||P_{S_i}(f) - P_S(f)|| \to 0$. We sometimes express this by writing $S_i \to S$ (SOT). It can be shown that the set of subspaces of $l^2_-(\Gamma)$ is compact under the SOT. Note that if $P_{S_i} \to P_S$ (SOT), then $\mathbf{P}^{S_i} \to \mathbf{P}^S$ in the weak* topology. It is well known that if $S_i < l^2_-(\Gamma)$ are closed subspaces and $S_i \nearrow S$ (meaning that $S_i \subset S_{i+1}$ and $\bigcup S_i$ is dense in S), then $P_{S_i}(T) \to P_S(T)$ (SOT) for any closed subspace T (in $l^2_-(\Gamma)$).

If $S < l^2_-(\Gamma/G_i)$, let $\tilde{\mathbf{P}}^S$ be the distribution of the random subgraph $\pi_i^{-1}(\mathcal{G}_S)$. For ease of notation, we also denote $\pi_i^{-1}(W_i)$ by $\tilde{W_i}$ and $\pi_i^{-1}(F_i)$ by $\tilde{F_i}$. We will show that $\tilde{W_i} \to$ WSF and $\tilde{F_i} \to$ FSF. In order to do this, we introduce sequences of random subgraphs of Γ that equal $\tilde{W_i}$ or $\tilde{F_i}$ on fundamental domains of G_i and have the advantage that their distributions are determinantal. For this we let D_i be a connected subgraph of Γ such that the covering map π_i restricted to the edge set of D_i is bijective. Also, assume that $D_i \subset D_{i+1}$ and that $\bigcup_i D_i = \Gamma$. This is possible since $\bigcap_i G_i = \{1\}$.

If $\mathcal{G}_S$ is a random subgraph of Γ/G_i, let $\widehat{\mathcal{G}_S}$ be the random subgraph of Γ that is contained in D_i and is equal to $\widetilde{\mathcal{G}_S}$ on D_i. Note that π_i restricted to D_i induces an isomorphism $\pi_{i*} : l^2_-(D_i) \to l^2_-(\Gamma/G_i)$. If $S < l^2_-(\Gamma/G_i)$ is a (closed) subspace, let $\hat{S}$ denote $\pi_{i*}^{-1}(S)$. It is easy to check that $\hat{\mathcal{G}}_S$ is the random subgraph associated to $\hat{S}$ (i.e., $\hat{\mathcal{G}}_S = \mathcal{G}_{\hat{S}}$).

Lemma 4.2. *Suppose for each i that $\mathcal{G}_i$ is a random subgraph on Γ/G_i. Then $\tilde{\mathcal{G}}_i \to \mathcal{G}$ (weak*) if and only if $\hat{\mathcal{G}}_i \to \mathcal{G}$ (weak*).*

Proof. Let B be a finite set of edges of Γ. Suppose that $\tilde{\mathcal{G}}_i \to \mathcal{G}$ (weak*). Then, by definition of weak* convergence,

$$\int_Z f_B \, d\tilde{\mu}_i \to \int_Z f_B \, d\mu \tag{8}$$

where $\tilde{\mu}_i$ is the distribution of $\tilde{\mathcal{G}}_i$ and μ is the distribution of $\mathcal{G}$. There exists an N such that $i > N$ implies that $B \subset D_i$. Since $\hat{\mathcal{G}}_i$ is equal to $\tilde{G}_i$ on D_i, it follows that

$$\int_Z f_B \, d\hat{\mu}_i \to \int_Z f_B \, d\mu \tag{9}$$

where $\hat{\mu}_i$ is the distribution of $\hat{\mathcal{G}}_i$. Since B is arbitrary, $\hat{\mathcal{G}}_i \to \mathcal{G}$ (weak*). The proof in the other direction is similar. $\square$

Theorem 4.3. *$\tilde{W}_i \to$ WSF and $\tilde{F}_i \to$ FSF (weak*).*

Proof. By the above lemma, it suffices to prove that $\hat{W}_j \to$ WSF and $\hat{F}_j$ to FSF. To this end, it suffices to prove that $\hat{\bigstar}_j \to \bigstar$ and $\widehat{C_j^{\perp}} \to \Diamond^{\perp}$. Let B be a finite set of (directed) edges of Γ, and let S be the subspace generated by χ^e for $e \in B$.

We show first that $P_S(\bigstar) < P_S(\hat{\bigstar}_j)$ for all large enough j. Suppose that v is a vertex incident to an edge $e \in B$. Let f_v be the star of v. It suffices to show that $P_S(f_v) \in P_S(\hat{\bigstar}_j)$ for all large enough j. So let j be large enough so that all edges incident to v are in D_j. Let g_v be the star of $\pi_j(v)$ in Γ/G_j. By definition $g_v \in \bigstar_j$. It is clear that $f_v = \pi_{j*}^{-1}(g_v)$. So $P_S(f_v) = P_S(\pi_{j*}^{-1}(g_v))$, which implies that $P_S(f_v) \in P_S(\hat{\bigstar}_j)$. Since v is arbitrary, the claim is proven.

To show that $P_S(\hat{\bigstar}_j) < P_S(\bigstar)$ for j large enough we follow the same argument as above in reverse. So, $P_S(\hat{\bigstar}_j) = P_S(\bigstar)$ for all j large enough. Similarly, it can be shown that $P_S(\hat{C}_j) = P_S(\Diamond)$ for all j large enough.

We let S_i be the subspace generated by the functions χ^e for $e \in D_i$. Then $S_i \nearrow l^2_-(\Gamma)$; so $P_{S_i}(\bigstar) \to \bigstar$ (SOT). Since $P_{S_i}(\bigstar) = P_{S_i}(\hat{\bigstar}_j)$ for j large enough, we have that for any subsequential limit T for $\{\hat{\bigstar}_j\}$, $P_{S_i}(T) \to \bigstar$ (SOT). But $P_{S_i}(T) \to T$ (SOT); so $T = \bigstar$, i.e., $\hat{\bigstar}_j \to \bigstar$ (SOT). This implies that $\hat{W}_i \to$ WSF (weak*).

Similarly, it can be shown that $\hat{C}_i \to \Diamond$ (SOT). This implies that $(\hat{C}_i)^{\perp} \to \Diamond^{\perp}$ (SOT). Since $S_i \nearrow l^2_-(\Gamma)$, it follows that $P_{S_i}((\hat{C}_i)^{\perp}) \to \Diamond^{\perp}$. But note that $P_{S_i}((\hat{C}_i)^{\perp}) = \widehat{C_i^{\perp}}$. Hence we have shown that $\widehat{C_i^{\perp}} \to \Diamond^{\perp}$. This implies that $\hat{F}_i \to$ FSF (weak*) since $\widehat{C_i^{\perp}}$ is the distribution of $\hat{F}_i$. Now we are done. $\square$

Proof of Theorem 4.1. By Strassen's lemma and [L2], Theorem 6.1, for all i, there exists a monotone coupling of W_i and F_i. The space $\mathcal{C}_i$ of all such monotone couplings is compact under the weak* topology. G/G_i acts on this space by $gG_i\mu(F) \to \mu(gG_iF)$ for $\mu \in \mathcal{C}_i$ and $F \subset 2^{E(\Gamma/G_i)} \times 2^{E(\Gamma/G_i)}$. Since G/G_i is amenable, there exists a G/G_i-invariant measure σ_i on $\mathcal{C}_i$. Let μ_i be defined by

$$\mu_i(F) = \int_{\mathcal{C}_i} \nu(F) \, d\sigma_i(\nu) \tag{10}$$

for any Borel set $F \subset 2^{E(\Gamma/G_i)}$. Since σ_i is G/G_i-invariant, μ_i is a G/G_i-invariant monotone coupling of W_i and F_i.

Since $\pi_i : \Gamma \to \Gamma/G_i$ is equivariant with respect to the G-action, and μ_i is invariant under G/G_i, we may pull μ_i back via π_i to a G-invariant monotone coupling $\tilde{\mu}_i$ between $\tilde{W}_i$ and $\tilde{F}_i$.

The space of G-invariant monotone couplings is naturally a closed subspace of $M^i_{Z\times Z}$. This space is compact. So there exists a weak* limit point μ of $\tilde{\mu}_i$. Since $\mu_i \to \mu$, the first and second marginals of μ_i must converge to the first and second marginals of μ respectively. Thus μ is a coupling between WSF and FSF. $\square$

5. Speculations

(1) Suppose that Γ is a planar graph embedded in the hyperbolic plane so that $Aut(\Gamma)$ acts by hyperbolic isometries. We may then take $G = Aut(\Gamma)$ and choose G_i so that G/G_i is finite. It can be shown that the distribution of F_i is the uniform distribution on subgraphs $\mathcal{G}$ of Γ/G_i such that $\mathcal{G}$ contains a spanning tree and $\mathbb{H}^2/G_i - \mathcal{G}$ is a topological disk. If there exists a "natural" or explicit coupling between W_i and F_i, then one may hope to obtain a natural coupling between WSF and FSF as a limit of lifts of couplings of W_i and F_i. It seems that it should be possible to find such a coupling (if it exists) if only because Γ/G_i is a finite graph.

(2) Suppose that S is a G-invariant subspace of $l^2_-(\Gamma)$. Do there exist G/G_i-invariant subspaces S_i in $l^2_-(\Gamma/G_i)$ such that $\hat{S}_i \to S$?

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References

[BLPS] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, *Uniform Spanning Forests*, Ann. Probab. 29 (2001), no. 1, 1–65. MR **2003a:**60015

[BV] M. E. B. Bekka and A. Valette, *Group cohomology, harmonic functions and the first L^2-Betti number*, Potential Anal. 6 (1997), 313–326. MR **98e:**20056

[FM] T. Feder and M. Mihail, *Balanced Matroids*, Proc. 24th Annual ACM Sympos. Theory Computing (Victoria, BC, Canada), pp. 26-38, ACM Press, New York, 1992.

[L1] R. Lyons, *A bird's-eye view of uniform spanning trees and forests*, in Microsurveys in Discrete Probability, D. Aldous and J. Propp (eds.), Amer. Math. Soc., Providence, RI, 1998, pp. 135–162. MR **99e:**60029

[L2] R. Lyons, *Determinantal Probability Measures*, preprint.

[KW] Ilya Kapovich and Daniel T. Wise, *The equivalence of some residual properties of word-hyperbolic groups*, J. Algebra 223 (2000), no. 2, 562–583. MR **2001f:**20086

[Ma] A. I. Mal'cev, *On isomorphic matrix representations of infinite groups* (Russian). Mat. Sbornik 8, 405-422 (1940); Amer. Math. Soc. Transl. (2) 45, 1-18 (1965). MR **2:**216d

[O] A. Yu. Olshanskii, *Almost every group is hyperbolic*, Internat. J. Algebra Comput. 2 (1992), no. 1, 1–17. MR **93j:**20068

[Pe] Robert Pemantle, *Choosing a spanning tree for the integer lattice uniformly*, Ann. Probab. 19 (1991), 1559-1574. MR **92g:**60014

[Pi] Jean-Paul Pier, *Amenable locally compact groups*, Pure and Applied Mathematics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1984. x+418 pp. MR **86a:**43001

[S] E. A. Scott, *A tour around finitely presented infinite simple groups*, Algorithms and classification in combinatorial group theory (Berkeley, 1989), pp. 83–119, Springer-Verlag, New York, 1992. MR **94j:**20030

[St] V. Strassen, *The existence of probability measures with given marginals*, Ann. Math. Statist. 36 (1965), 423–439. MR **31:**1693

[We] B. A. F. Wehrfritz, *Infinite Linear Groups, An account of the group-theoretic properties of infinite groups of matrices*, Ergebnisse der Matematik und ihrer Grenzgebiete, Band 76, Springer-Verlag, New York, Heidelberg, Berlin, 1973. MR **49:**436

[Z] Robert J. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Mathematics, 81. Birkhäuser Verlag, Basel, 1984. x+209 pp. MR **86j:**22014

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