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GENERALIZED FUNCTION ALGEBRAS AS SEQUENCE SPACE ALGEBRAS

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Abstract. A topological description of various generalized function algebras over corresponding basic locally convex algebras is given. The framework consists of algebras of sequences with appropriate ultra(pseudo)metrics defined by sequences of exponential weights. Such an algebra with embedded Dirac's delta distribution induces the discrete topology on the basic space.

1. Introduction

After Schwartz' "impossibility result" [18] for algebras of generalized functions with a prescribed list of (natural) assumptions, several new approaches have appeared with the aim of applications in nonlinear problems. We refer to the recent monograph [9] for the historical background as well as for the list of relevant references, mainly for algebras of generalized functions today called Colombeau type algebras (see [1], [2], [3], [8], [12]). Colombeau and all other successors introduce algebras of generalized functions through purely algebraic methods. By now, these algebras have become an important tool in the theory of PDEs, stochastic analysis, differential geometry and general relativity. We show that such algebras fit in the general theory of well-known sequence spaces forming appropriate algebras. These classes of algebras of sequences are simply determined by a locally convex algebra E, and a sequence of weights (or sequence of sequences), which serve to construct ultra(pseudo)metrics.

From the beginning, the topological questions concerning such algebras were important. We refer to the papers [2] where the classical topology and a uniform structure were introduced in order to consider generalized functions as smooth functions in appropriate quotient spaces. Then the sharp topology [13] was introduced in connection with the well-posedness of the Carleman system with measures as initial data. Later it was independently reintroduced and analyzed in [17], where the name "sharp topology" appeared. It remained an open question whether the introduced topologies were "good enough", because they induced always the discrete topology on the underlying space.

We show that the topology of a Colombeau type algebra containing Dirac's delta distribution δ as an embedded Colombeau generalized function must induce

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the discrete topology on the basic space E. In the authors' opinion, this result is somehow in analogy to Schwartz' "impossibility result" concerning the product of distributions, as explained in Remark 4.2.

We mention that distribution, ultradistribution and hyperfunction type spaces can be embedded in corresponding algebras of sequences with exponential weights (cf. [4]). More general concepts of generalized functions not anticipating embeddings as well as regularity properties of generalized functions can be found in [16] and [11].

Let us point out that our sequential approach presents some analogy with nonstandard analysis, which can be viewed as refinement of an ultrapower construction. Therefore, the sequences of weights we consider are related to some infinitesimal which appears in the non-Archimedean field of Robinson asymptotic numbers and in spaces of asymptotic functions. We refer the reader to [15, 10] for an introduction to nonstandard analysis and the related question of non-Archimedean fields, and to [14, 19] for a nonstandard approach to nonlinear theories of generalized functions and applications.

To be short, we give most examples only for spaces of functions defined on $\mathbb{R}^s$, although the generalization to an open subset of $\mathbb{R}^s$ is straightforward.

2. General construction

Consider a positive sequence $r = (r_n)_n \in (\mathbb{R}_+)^{\mathbb{N}}$ decreasing to zero. (We use $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{N} = \{0, 1, 2, ...\}$.) If p is a seminorm on a vector space E, we define for $f = (f_n)_n \in E^{\mathbb{N}}$,

$$\| f \|_{p,r} = \limsup_{n\to\infty} \left(p(f_n)\right)^{r_n}$$

with values in $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. Denote $\tilde{E}^{\mathbb{N}} = \{f \in E^{\mathbb{N}} \mid \| f \|_{p,r} < \infty\}$.

Let $(E_\nu^\mu, p_\nu^\mu)_{\mu,\nu\in\mathbb{N}}$ be a family of semi-normed algebras over $\mathbb{R}$ or $\mathbb{C}$ such that

$$\forall \mu, \nu \in \mathbb{N} : \; E_\nu^{\mu+1} \hookrightarrow E_\nu^\mu \text{ and } E_{\nu+1}^\mu \hookrightarrow E_\nu^\mu \text{ (resp. } E_\nu^\mu \hookrightarrow E_{\nu+1}^\mu \text{)} ,$$

where $\hookrightarrow$ means continuously embedded. (For the ν index we consider inclusions in the two directions.) Then let $\overleftarrow{E} = \operatorname*{proj\,lim}_{\mu\to\infty} \operatorname*{proj\,lim}_{\nu\to\infty} E_\nu^\mu = \operatorname*{proj\,lim}_{\nu\to\infty} E_\nu^\nu$, (resp. $\overrightarrow{E} = \operatorname*{proj\,lim}_{\mu\to\infty} \operatorname*{ind\,lim}_{\nu\to\infty} E_\nu^\mu$). Such projective and inductive limits are usually considered with norms instead of seminorms, and with the additional assumption that in the projective case sequences are reduced, while in the inductive case for every $\mu \in \mathbb{N}$ the inductive limit is regular, i.e., a set $A \subset \operatorname*{ind\,lim}_{\nu\to\infty} E_\nu^\mu$ is bounded iff it is contained in some E_ν^μ and bounded there.

Define (with $p \equiv (p_\nu^\mu)_{\nu,\mu}$)

$$\overleftarrow{\mathcal{F}}_{p,r} = \left\{ f \in \overleftarrow{E}^{\mathbb{N}} \;\middle|\; \forall \mu, \nu \in \mathbb{N} : \| f \|_{p_\nu^\mu, r} < \infty \right\} ,$$

$$\overleftarrow{\mathcal{K}}_{p,r} = \left\{ f \in \overleftarrow{E}^{\mathbb{N}} \;\middle|\; \forall \mu, \nu \in \mathbb{N} : \| f \|_{p_\nu^\mu, r} = 0 \right\}$$

$$\text{(resp. } \overrightarrow{\mathcal{F}}_{p,r} = \bigcap_{\mu\in\mathbb{N}} \overrightarrow{\mathcal{F}}^\mu_{p,r} , \; \overrightarrow{\mathcal{F}}^\mu_{p,r} = \bigcup_{\nu\in\mathbb{N}} \left\{ f \in (E_\nu^\mu)^{\mathbb{N}} \;\middle|\; \| f \|_{p_\nu^\mu, r} < \infty \right\} ,$$

$$\overrightarrow{\mathcal{K}}_{p,r} = \bigcap_{\mu\in\mathbb{N}} \overrightarrow{\mathcal{K}}^\mu_{p,r} , \; \overrightarrow{\mathcal{K}}^\mu_{p,r} = \bigcup_{\nu\in\mathbb{N}} \left\{ f \in (E_\nu^\mu)^{\mathbb{N}} \;\middle|\; \| f \|_{p_\nu^\mu, r} = 0 \right\} \text{)} .$$

Proposition–Definition 2.1.

(i) *Writing* $\overleftrightarrow{\cdot}$ *for both,* $\overleftarrow{\cdot}$ *or* $\overrightarrow{\cdot}$*, we have that* $\overleftrightarrow{\mathcal{F}}_{p,r}$ *is an algebra and* $\overleftrightarrow{\mathcal{K}}_{p,r}$ *is an ideal of* $\overleftrightarrow{\mathcal{F}}_{p,r}$*; thus,* $\overleftrightarrow{\mathcal{G}}_{p,r} = \overleftrightarrow{\mathcal{F}}_{p,r}/\overleftrightarrow{\mathcal{K}}_{p,r}$ *is an algebra.*

(ii) *For every* $\mu, \nu \in \mathbb{N}$, $d_{p^\mu_\nu} : (E^\mu_\nu)^{\mathbb{N}} \times (E^\mu_\nu)^{\mathbb{N}} \to \overline{\mathbb{R}}_+$ *defined by* $d_{p^\mu_\nu}(f,g) = \| f - g \|_{p^\mu_\nu, r}$ *is an ultrapseudometric on* $(E^\mu_\nu)^{\mathbb{N}}$*. Moreover,* $(d_{p^\mu_\nu})_{\mu,\nu}$ *induces a topological algebra*[1] *structure on* $\overleftarrow{\mathcal{F}}_{p,r}$ *such that the intersection of the neighborhoods of zero equals* $\overleftarrow{\mathcal{K}}_{p,r}$*.*

(iii) *From (ii),* $\overleftarrow{\mathcal{G}}_{p,r} = \overleftarrow{\mathcal{F}}_{p,r}/\overleftarrow{\mathcal{K}}_{p,r}$ *becomes a topological algebra (over generalized numbers* $\mathbb{C}_r = \mathcal{G}_{|\cdot|,r}$*) whose topology can be defined by the family of ultrametrics* $(\tilde{d}_{p^\mu_\nu})_{\mu,\nu}$ *where* $\tilde{d}_{p^\mu_\nu}([f],[g]) = d_{p^\mu_\nu}(f,g)$*,* $[f]$ *standing for the class of* f*.*

(iv) *If* τ_μ *denote the inductive limit topology on* $\mathcal{F}^\mu_{p,r} = \bigcup_{\nu\in\mathbb{N}}((\tilde{E^\mu_\nu})^{\mathbb{N}}, d_{\mu,\nu})$*,* $\mu \in \mathbb{N}$*, then* $\overrightarrow{\mathcal{F}}_{p,r}$ *is a topological algebra for the projective limit topology of the family* $(\mathcal{F}^\mu_{p,r}, \tau_\mu)_\mu$*.*

Proof. We use the following properties of $\| \cdot \|$:

$$\forall x, y \in E^{\mathbb{N}} : \| x + y \| \le \max(\| x \|, \| y \|) \ , \tag{1}$$

$$\forall x, y \in E^{\mathbb{N}} : \| x \cdot y \| \le \| x \| \cdot \| y \| \ , \tag{2}$$

$$\forall \lambda \in \mathbb{C}^*, x \in E^{\mathbb{N}} : \| \lambda\, x \| = \| x \|. \tag{3}$$

They are consequences of basic properties of seminorms and of $p(x_n + y_n)^{r_n} \le 2^{r_n} \max(p(x_n), p(y_n))^{r_n}$ for (1). Using the above three inequalities, (i)–(iv) follow straightforwardly from the respective definitions. □

Example 2.1 (Colombeau-generalized numbers and ultracomplex numbers)**.** Take $E^\mu_\nu = \mathbb{R}$ or $\mathbb{C}$, and $p^\mu_\nu = |\cdot|$ (absolute value) for all $\mu, \nu \in \mathbb{N}$. Then, for $r_n = \frac{1}{\log n}$, we get the ring of Colombeau's numbers $\overline{\mathbb{R}}$ or $\overline{\mathbb{C}}$.

With the sequence $r_n = n^{-1/m}$ for some fixed $m > 0$, we obtain rings of ultracomplex numbers $\overline{\mathbb{C}}^{p!^m}$ (cf. [4]).

Example 2.2. Consider a Sobolev space $E = W^{s,\infty}(\Omega)$ for some $s \in \mathbb{N}$. The corresponding Colombeau-type algebra is defined by $\mathcal{G}_{W^{s,\infty}} = \mathcal{F}/\mathcal{K}$, where

$$\mathcal{F} = \left\{ u \in (W^{s,\infty})^{\mathbb{N}} \mid \limsup \|u_n\|_{s,\infty}^{\frac{1}{\log n}} < \infty \right\} \ ,$$

$$\mathcal{K} = \left\{ u \in (W^{s,\infty})^{\mathbb{N}} \mid \limsup \|u_n\|_{s,\infty}^{\frac{1}{\log n}} = 0 \right\}.$$

Example 2.3 (simplified Colombeau algebra)**.** Take $E^\mu_\nu = \mathcal{C}^\infty(\mathbb{R}^s)$,

$$p^\mu_\nu(f) = \sup_{|\alpha|\le\nu,\, |x|\le\mu} |f^{(\alpha)}(x)|,$$

and $r_n = \frac{1}{\log n}$. Then, $\overleftarrow{\mathcal{G}}_{p,r} = \overleftarrow{\mathcal{F}}_{p,r}/\overleftarrow{\mathcal{K}}_{p,r}$ is the simplified Colombeau algebra.

The full algebra of Colombeau-generalized functions can be described in an analogous but more complicated setting, as will be explained in a forthcoming paper (see [5], Example 5).

[1] over $(\mathbb{C}^{\mathbb{N}}, \| \cdot \|_{|\cdot|})$, not over $\mathbb{C}$: scalar multiplication is not continuous because of (3).

3. Completeness

Without assuming completeness of $\overleftrightarrow{E}$, we have

Proposition 3.1.
(i) $\overleftarrow{\mathcal{F}}_{p,r}$ *is complete.*

(ii) *Assume that for every $\mu \in \mathbb{N}$, a subset of $\overrightarrow{\mathcal{F}}^{\mu}_{p,r}$ is bounded iff it is a bounded subset of $(E^{\mu}_{\nu})^{\mathbb{N}}$ for some $\nu \in \mathbb{N}$. Then $\overrightarrow{\mathcal{F}}_{p,r}$ is sequentially complete.*

Proof. If $(f^m)_{m\in\mathbb{N}} \in \overleftarrow{\mathcal{F}}_{p,r}$ is a Cauchy sequence, then there exists a strictly increasing sequence $(m_\mu)_{\mu\in\mathbb{N}}$ of integers such that

$$\forall \mu \in \mathbb{N}\ \forall k, \ell \geq m_\mu : \limsup_{n\to\infty} p^{\mu}_{\mu}\left(f^k_n - f^\ell_n\right)^{r_n} < \frac{1}{2^\mu}\ .$$

Thus, there exists a strictly increasing sequence $(n_\mu)_{\mu\in\mathbb{N}}$ of integers such that

$$\forall \mu \in \mathbb{N}\ \forall k, \ell \in [m_\mu, m_{\mu+1}]\ \forall n \geq n_\mu : p^{\mu}_{\mu}\left(f^k_n - f^\ell_n\right)^{r_n} < \frac{1}{2^\mu}\ .$$

(Restricting k,ℓ to $[m_\mu, m_{\mu+1}]$ allows us to take n_μ independent of k,ℓ.) Let $\boldsymbol{\mu}(n) = \sup\{\,\mu \mid n_\mu \leq n\,\}$, and consider the diagonalized sequence

$$\bar{f} = \left(f^{m_{\boldsymbol{\mu}(n)}}_n\right)_n \text{ , i.e., } \bar{f}_n = \begin{cases} f^{m_0}_n & \text{if } n \in [n_0, n_1) \\ \dots & \\ f^{m_\mu}_n & \text{if } n \in [n_\mu, n_{\mu+1}) \\ \dots & \end{cases}.$$

Now let us show that $f^m \to \bar{f}$ in $\overleftarrow{\mathcal{F}}_{p,r}$ as $m \to \infty$. Indeed, for ε and $p^{\mu_0}_{\nu}$ given, choose $\mu > \mu_0, \nu$ such that $\frac{1}{2^\mu} < \frac{1}{2}\varepsilon$. Since p^{μ}_{ν} is increasing in both indices, we have for $m > m_\mu$ (say $m \in [m_{\mu+s}, m_{\mu+s+1}]$) :

$$\begin{aligned} p^{\mu_0}_{\nu}(f^m_n - \bar{f}_n)^{r_n} &\leq p^{\mu}_{\mu}(f^m_n - f^{m_{\boldsymbol{\mu}(n)}}_n)^{r_n} \\ &\leq p^{\mu}_{\mu}(f^m_n - f^{m_{\mu+s+1}}_n)^{r_n} + \sum_{\mu'=\mu+s+1}^{\boldsymbol{\mu}(n)-1} p^{\mu'}_{\mu'}(f^{m_{\mu'}}_n - f^{m_{\mu'+1}}_n)^{r_n} \end{aligned}$$

and for $n > n_{\mu+s}$, we have of course $n \geq n_{\boldsymbol{\mu}(n)}$. Thus finally

$$p^{\mu_0}_{\nu}(f^m_n - \bar{f}_n)^{r_n} < \sum_{\mu'=\mu+s}^{\boldsymbol{\mu}(n)} \frac{1}{2^{\mu'}} < \frac{2}{2^\mu} < \varepsilon$$

and therefore $f^m \to \bar{f}$ in $\overleftarrow{\mathcal{F}}$.

For $(f^m)_m$ a Cauchy net in $\overrightarrow{\mathcal{F}}_{p,r}$, the proof requires some additional considerations. We know that for every μ there is $\nu(\mu)$ such that

$$p^{\mu}_{\nu(\mu)}\left(f^m_n - f^p_n\right)^{r_n} < \varepsilon_\mu,$$

where $(\varepsilon_\mu)_\mu$ decreases to zero. For every μ we can choose $\nu(\mu)$ so that $p^{\mu}_{\nu(\mu)} \leq p^{\mu+1}_{\nu(\mu+1)}$. Now by the same arguments as above, we prove the completeness in the case of $\overrightarrow{\mathcal{F}}_{p,r}$. $\square$

4. General remarks on embeddings of duals

Under mild assumptions on $\overleftrightarrow{E}$, we show that our algebras of (classes of) sequences contain embedded elements of strong dual spaces $\overleftrightarrow{E}'$. First we consider the embedding of the delta distribution. We show that general assumptions on test spaces or on a delta sequence lead to the non-boundedness of a delta sequence in $\overleftrightarrow{E}$.

We consider $F = \mathcal{C}^0(\mathbb{R}^s)$, the space of continuous functions with the projective topology given by sup norms on the balls $B(0,n)$, $n \in \mathbb{N}^*$, or $F = \mathcal{K}(\mathbb{R}^s) = \operatorname{ind\,lim}_{n\to\infty}(\mathcal{K}_n, \|\cdot\|_\infty)$, where

$$\mathcal{K}_n = \{\, \phi \in \mathcal{C}(\mathbb{R}^s) \mid \operatorname{supp}\psi \subset B(0,n) \,\} \ .$$

(Recall that $\mathcal{K}'(\mathbb{R}^s)$ is the space of Radon measures.)

We assume that $\overleftrightarrow{E}$ is dense in F and $\overleftrightarrow{E} \hookrightarrow F$. This implies that $\delta \in F' \subset \overleftrightarrow{E}'$.

Proposition 4.1.

(i) *For $F = \mathcal{C}^0(\mathbb{R}^s)$, a sequence $(\delta_n)_n$ with elements in $\overleftrightarrow{E} \cap (\mathcal{C}^0(\mathbb{R}^s))'$ such that*

$$\exists M > 0, \forall n \in \mathbb{N} : \sup_{|x|>M} |\delta_n(x)| < M \ ,$$

converging weakly to δ in $\overleftrightarrow{E}'$, cannot be bounded in $\overleftrightarrow{E}$. With $F = \mathcal{K}(\mathbb{R}^s)$, if for $(\delta_n)_n \in (\overleftrightarrow{E})^{\mathbb{N}}$ there exists a compact set K so that $\operatorname{supp}\delta_n \subset K$, $n \in \mathbb{N}^*$, then this sequence cannot be bounded in $\overleftrightarrow{E}$.

(ii) *Assume:*

1. Any $\phi \in \overleftrightarrow{E}$ defines an element of F' by $\psi \mapsto \int_{\mathbb{R}^s} \phi(x)\,\psi(x)\,\mathrm{d}x$.

2. If $(\phi_n)_n$ is a bounded sequence in $\overleftrightarrow{E}$, then $\sup\limits_{n\in\mathbb{N}, x\in\mathbb{R}^s} |\phi_n(x)| < \infty$.

Then, if $\overleftrightarrow{E}$ is sequentially weakly dense in $\overleftrightarrow{E}'$ and $(\delta_n)_n$ is a sequence in $\overleftrightarrow{E}$ converging weakly to δ in $\overleftrightarrow{E}'$, then $(\delta_n)_n$ cannot be bounded in $\overleftrightarrow{E}$.

Proof. (i) We will prove the assertion only for $F = \mathcal{C}^0(\mathbb{R}^s)$. Let us show that $(\delta_n)_n$ is not bounded in $\overleftrightarrow{E}$. First consider $\overleftarrow{E}$. Boundedness of $(\delta_n)_n$ in $\overleftarrow{E}$ implies: $\forall\mu \in \mathbb{N}$, $\forall\nu \in \mathbb{N}$, $\exists C_1 > 0$, $\forall n \in \mathbb{N} : p^\mu_\nu(\delta_n) < C_1$. Continuity of $\overleftarrow{E} \hookrightarrow \mathcal{C}^0(\mathbb{R}^s)$ gives

$$\forall k \in \mathbb{N}, \exists\mu \in \mathbb{N}, \exists\nu \in \mathbb{N}, \exists C_2 > 0, \forall\psi \in \overleftarrow{E} : \sup_{|x|<k} |\psi(x)| \le C_2\, p^\mu_\nu(\psi) \ .$$

It follows that $\exists C > 0, \forall n \in \mathbb{N} : \sup_{x\in\mathbb{R}^s} |\delta_n(x)| < C$, which is impossible. To show this, take $\psi \in \mathcal{C}^0(\mathbb{R}^s)$ so that it is positive, $\psi(0) = C+1$, and $\int\psi < 1$. The assumption $\delta_n \in (\mathcal{C}^0(\mathbb{R}^s))'$ implies that it acts on $\mathcal{C}^0(\mathbb{R}^s)$ by $\psi \mapsto \int \delta_n(x)\,\psi(x)\,\mathrm{d}x$. This gives $C+1 = \psi(0) \leftarrow \left|\int \delta_n\psi\,\mathrm{d}x\right| \le C$.

For $\overrightarrow{E}$, simply exchange $\forall\nu \leftrightarrow \exists\nu$ in the above.

(ii) Assumption 2 and the boundedness of $(\delta_n)_n$ in $\overleftrightarrow{E}$ would imply that $\exists C > 0, \forall n \in \mathbb{N} : \sup_{x\in\mathbb{R}^s} |\delta_n(x)| < C$. Now, by assumption 1, we conclude the proof as in part (i). ☐

Remark 4.1. One can take for $\overleftrightarrow{E}$ one of Schwartz' test function spaces or a Beurling or Roumieau test function space of ultradifferentiable functions. Since the delta distribution lives on all functions that are continuous at zero, one can consider also

F and $\overleftrightarrow{E}$ to consist of holomorphic functions with appropriate topologies. This was the reason for considering $\mathcal{C}^0$, although there are many classes of test spaces which would imply the necessary accommodation of conditions of the previous assertion.

Thus, the appropriate choice of a sequence r decreasing to 0 appears to be important to have at least δ embedded into the corresponding algebra. It can be chosen such that for all $\mu \in \mathbb{N}$ and all $\nu \in \mathbb{N}$ (resp. some $\nu \in \mathbb{N}$ in the $\overrightarrow{E}$ case), $\limsup_{n\to\infty} p_\nu^\mu(\delta_n)^{r_n} = A_\nu^\mu$ and $\exists \mu_0, \nu_0 : A_{\nu_0}^{\mu_0} \neq 0$.

So the embedding of duals into corresponding algebras is realized on the basis of two demands:

(1) $\overleftrightarrow{E}$ is weakly sequentially dense in $\overleftrightarrow{E}'$.
(2) There exists a sequence $(r_n)_n$ decreasing to zero, such that for all $f \in \overleftrightarrow{E}'$ and corresponding sequence $(f_n)_n$ in $\overleftrightarrow{E}$, $f_n \to f$ weakly in $\overleftrightarrow{E}'$, we have for all μ and all ν (resp. some ν), $\limsup\limits_{n\to\infty} p_\nu^\mu(f_n)^{r_n} < \infty$.

Remark 4.2. In the definition of sequence spaces $\overleftrightarrow{\mathcal{F}}_{p,r}$, we assumed $r_n \searrow 0$ as $n \to \infty$. In principle, one could consider more general sequences of weights. For example, if $r_n \in (\alpha, \beta)$, $0 < \alpha < \beta$, then $\overleftrightarrow{E}$ can be embedded, in the set-theoretical sense, via the canonical map $f \mapsto (f)_n$ ($f_n = f$). Moreover, this embedding is continuous (thus, topological embedding). It is clear for the projective case $\overleftarrow{E}$. For the inductive case $\overrightarrow{E}$, the assumption that for every μ the inductive limit is regular implies the continuity. Indeed, if there is a function f and a sequence $(f_n)_n$ in E_ν^μ such that $\limsup p_\nu^\mu(f_n - f) < 1$, then

$$\| f_n - f \|_{p_\nu^\mu, r} = \limsup_{n\to\infty} \left(p_\nu^\mu(f_n - f)\right)^{r_n} \leq \left(\limsup_{n\to\infty} p_\nu^\mu(f_n - f)\right)^{\liminf\limits_{n\to\infty} r_n}$$

and $p_\nu^\mu(f_n - f) \to 0$ implies $\| f_n - f \|_{p_\nu^\mu, r} \to 0$.

If $r_n \to \infty$, $\overleftrightarrow{E}$ is no more included in $\overleftrightarrow{\mathcal{F}}_{p,r}$.

In the case we consider ($r_n \to 0$), the induced topology on $\overleftrightarrow{E}$ is obviously a discrete topology. But this is necessarily so, since we want to have "divergent" sequences in $\overleftrightarrow{\mathcal{F}}_{p,r}$. Thus, in our construction, in order to have an appropriate topological algebra containing "δ", it is unavoidable that $(r_n)_n$ tends to zero and so our generalized topological algebra induces a discrete topology on the original algebra $\overleftrightarrow{E}$.

In some sense, this is the price to pay, in our construction, to overcome the conclusion of Schwartz' impossibility statement for multiplication of distributions [18]. Of course, our arguments are completely different and are not related to Schwartz' arguments.

5. Sequences of scales

The analysis of previous sections can be extended to the case where we consider a sequence $(r^m)_m$ of decreasing null sequences $(r_n^m)_n$, satisfying one of the following additional conditions:

$$\forall m, n \in \mathbb{N} : r_n^{m+1} \geq r_n^m \quad \text{or} \quad \forall m, n \in \mathbb{N} : r_n^{m+1} \leq r_n^m \ .$$

Then, in the first (resp. second) case, let

$$\overleftrightarrow{\mathcal{F}}_{p,r} = \bigcap_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{F}}_{p,r^m}\ ,\ \overleftrightarrow{\mathcal{K}}_{p,r} = \bigcup_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{K}}_{p,r^m}\ ,$$

$$(\text{resp. } \overleftrightarrow{\mathcal{F}}_{p,r} = \bigcup_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{F}}_{p,r^m}\ ,\ \overleftrightarrow{\mathcal{K}}_{p,r} = \bigcap_{m\in\mathbb{N}} \overleftrightarrow{\mathcal{K}}_{p,r^m}\ ), \text{ where } p = (p^\mu_\nu)_{\nu,\mu}\ .$$

Proposition 5.1. *With the above notation,* $\overleftrightarrow{\mathcal{G}}_{p,r} = \overleftrightarrow{\mathcal{F}}_{p,r}/\overleftrightarrow{\mathcal{K}}_{p,r}$ *is an algebra.*

Proof. In the first case, $r^{m+1} \geq r^m \implies \|\!|\, f \,\|\!|_{r^{m+1}} \geq \|\!|\, f \,\|\!|_{r^m}$ if $p(f_n) \geq 1$, hence $\mathcal{F}_{m+1} \subset \mathcal{F}_m$, and conversely, $p(k_n) \leq 1 \rightsquigarrow \|\!|\, k \,\|\!|_{r^{m+1}} \leq \|\!|\, k \,\|\!|_{r^m} \rightsquigarrow \mathcal{K}_{m+1} \supset \mathcal{K}_m$. Thus, $\mathcal{F}$ is obviously a subalgebra. To see that $\mathcal{K}$ is an ideal, take $(k,f) \in \mathcal{K}\times\mathcal{F}$. Then $\exists m : k \in \mathcal{K}_m$, but also $f \in \mathcal{F}_m$, in which $\mathcal{K}_m$ is an ideal. Thus $k\cdot f \in \mathcal{K}_m \subset \mathcal{K}$.

If r^m is decreasing, $\mathcal{F}_{m+1} \supset \mathcal{F}_m$ and $\mathcal{K}_{m+1} \subset \mathcal{K}_m$. Because of this inclusion property, $\mathcal{F}$ is a subalgebra. To prove that $\mathcal{K}$ is an ideal, take $(k,f) \in \mathcal{K}\times\mathcal{F}$, i.e., $\forall m : k \in \mathcal{K}_m$, and $\exists m' : f \in \mathcal{F}_{m'}$. We need that $\forall m : k\cdot f \in \mathcal{K}_m$. Indeed, if $m \leq m'$, then $\mathcal{K}_{m'} \subset \mathcal{K}_m$; thus $k\cdot f \in \mathcal{K}_{m'}\cdot\mathcal{F}_{m'} \subset \mathcal{K}_{m'} \subset \mathcal{K}_m$, and if $m' \leq m$, then $\mathcal{F}_{m'} \subset \mathcal{F}_m$; thus $k\cdot f \in \mathcal{K}_m\cdot\mathcal{F}_{m'} \subset \mathcal{K}_m\cdot\mathcal{F}_m \subset \mathcal{K}_m$. □

Example 5.1. $r^m_n = 1/|\log a_m(n)|$, where $(a_m : \mathbb{N} \to \mathbb{R}_+)_{m\in\mathbb{Z}}$ is an asymptotic scale, i.e., $\forall m \in \mathbb{Z} : a_{m+1} = o(a_m)$, $a_{-m} = 1/a_m$, $\exists M \in \mathbb{Z} : a_M = o(a_m^2)$. This gives back the asymptotic algebras of [6].

Example 5.2. Colombeau-type ultradistribution and periodic hyperfunction algebras will be considered in [4].

Example 5.3. $r^m = \chi_{[0,m]}$, i.e., $r^m_n = 1$ if $n \leq m$ and 0 else, gives the Egorov-type algebras [7], where the "subalgebra" contains everything, and the ideal contains only stationary null sequences (with the convention $0^0 = 0$).

In the case of sequences of scales our second demand of the previous section should read: "There exists a sequence $((r^m_n)_n)_m$ of sequences decreasing to 0, such that for all $f \in \overleftrightarrow{E}'$ and the corresponding sequence $(f_n)_n$ in $\overleftrightarrow{E}$, $f_n \to f$ weakly in $\overleftrightarrow{E}'$, there exists an m_0 such that for all μ and all ν (resp. some ν), $\limsup_{n\to\infty} p^\mu_\nu(f_n)^{r^{m_0}_n} < \infty$."

Topological properties for such algebras are a little more complex, but the ideas of constructing families of ultra(pseudo)metrics are now clear. An important feature of our general concept is to show how various classes of ultradistribution and hyperfunction type spaces can be embedded in a natural way into sequence space algebras as considered in Section 2, [4].

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