

## NOTES ON THE LARGEST IRREDUCIBLE CHARACTER DEGREE OF A FINITE GROUP

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ABSTRACT. Let  $G$  be a finite group and  $b(G)$  the largest irreducible character degree of  $G$ . In this note, we show the following results: if  $b(G) < p^2$ , then  $|G : O_p(G)|_p \leq p$ ; if  $b(G) < p^m$  and, in addition,  $G$  is  $p$ -solvable with abelian Sylow  $p$ -subgroup, then  $|G : O_p(G)|_p < p^m$ .

### 1. INTRODUCTION

Let  $b(G)$  be the largest irreducible character degree of the finite group  $G$ . If  $b(G) < p^2$  for a prime  $p$ , then [1, Theorem A] states that  $|G : O_p(G)|_p < p^2$  in case  $G$  is  $p$ -solvable. Our first result shows that the condition  $p$ -solvable is not necessary.

**Theorem 1.** *Let  $b(G)$  be the largest irreducible character degree of a finite group  $G$ . If  $b(G) < p^2$  for a prime  $p$ , then  $|G : O_p(G)|_p \leq p$ .*

Recall that in paper [1], the key step is to show that  $b(G) \geq p^2$  for a finite  $p$ -solvable group  $G$  in which  $|G : O_p(G)|_p = p^2$ . Another generalization of the main theorem in [1] is the following result.

**Theorem 2.** *Let  $b(G)$  be the largest irreducible character degree of a finite  $p$ -solvable group  $G$ . If  $G/O_p(G)$  possesses an abelian Sylow  $p$ -subgroup of order  $p^m$ , then  $p^m \leq b(G)$ .*

### 2. PROOFS OF THEOREMS

Let  $H$  and  $G/N$  be a subgroup and a quotient group of the group  $G$ , respectively. It is easy to see that both  $b(H)$  and  $b(G/N)$  are less than or equal to  $b(G)$ . So, we can use induction on subgroups and quotient groups in our argument. Applying the classification theorem of finite simple groups, we now prove Theorem 2.

*Proof of Theorem 2.* By induction, we may assume that  $O_p(G) = 1$  and  $G = O_{p'}(G)$ . If  $M$  is a normal  $p'$ -subgroup of  $G$ , we may assume that  $O_p(G/M)$  is nontrivial because otherwise we conclude by induction that  $|G/M|_p \leq b(G/M)$  and so that  $|G|_p \leq b(G)$ ; we are done. It also implies that  $\Phi(G)$ , the Frattini subgroup of  $G$  is trivial.

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Case 1. Suppose that all minimal normal subgroups of  $G$  are abelian.

Let  $F$  be the Fitting subgroup of  $G$ . Then  $F$  is a nontrivial  $p'$ -subgroup; also, we see that  $F$  is abelian because  $\Phi(G) = 1$ . We claim that  $C_G(F) = F$ . If this is not true, we may choose a chief factor  $A/F$  of  $G$  such that  $A \leq C_G(F)$ . Note that if  $A$  is solvable, then the Fitting subgroup of  $A$  is equal to  $F$  and hence  $C_A(F) \leq F$ , a contradiction. It follows that  $A/F$  is nonsolvable. Since  $\Phi(A) \leq \Phi(G)$  is trivial,  $A$  is split over  $F$ . Hence we may write  $A = YF$ , where  $Y \cap F = 1$  and  $A/F \cong Y$ . Since  $Y$  centralizes  $F$ , we conclude that  $A = Y \times F$ . It is easy to see that  $Y \triangleleft G$ , and so is a nonsolvable minimal normal subgroup of  $G$ , which contradicts our assumption.

Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Since  $C_G(F) = F$ , we see that the abelian group  $P$  acts faithfully on the abelian  $p'$ -group  $F$ . It is well known that there is  $\chi \in \text{Irr}(PF)$  such that  $\chi(1) = |P|$ . So  $b(G) \geq b(PF) \geq \chi(1) = |P|$ , and we are done.

Case 2. Suppose that there is a minimal normal subgroup, say  $M$ , that is nonsolvable.

Clearly,  $M$  is a  $p'$ -group and  $p$  is an odd prime. Let  $A = C_G(M)$ . Set  $|A|_p = p^t$ ,  $|G/A|_p = p^s$  where  $p^t + p^s = |G|_p$ . It is clear that  $O_p(A) = 1$ . So by induction, there is  $\psi \in \text{Irr}(A)$  such that  $\psi(1) \geq p^t$ .

Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Note that  $O_p(PAM)$  centralizes  $M$ . It follows that  $O_p(PAM) = O_p(A)$  is trivial. Now if  $PAM < G$ , then induction yields  $b(PAM) \geq |P|$  and so  $b(G) \geq |P|$ , and we are done. So we may assume that  $G = PAM$ . Let  $M = M_1 \times M_2 \times \cdots \times M_k$  be the direct product of isomorphic simple groups  $M_i$ . Note that both  $N_G(M_1)$  and  $C_G(M_1) \times M_1$  contain  $A \times M$ . Since  $P$  is abelian, it forces both  $N_G(M_1)$  and  $C_G(M_1) \times M_1$  to be normal subgroups of  $G$ . It is well known that  $G/N_G(M_1)$  acts faithfully and transitively on  $\{M_1, \dots, M_k\}$ . Now since  $G$  transitively acts on  $\{M_1, \dots, M_k\}$ , we see that  $k = |G/N_G(M_1)|$ ,  $N_G(M_1) = \cdots = N_G(M_k)$ ,  $C_G(M_1) \times M_1 = \cdots = C_G(M_k) \times M_k$ . Hence,  $C_G(M_i) \times M_i = A \times M$  for any  $1 \leq i \leq k$ .

Suppose that  $N_G(M_1) = C_G(M_1) \times M_1 = A \times M$ . Then  $k = |G/(A \times M)| = p^s$ . Let  $\lambda_i \in \text{Irr}(M_i)$  with degree  $\lambda_i(1) \geq 3$ , and set  $\lambda = \lambda_1 \times \cdots \times \lambda_k$ . Then  $\lambda$  is extendible to  $A \times M$ . By [4, Corollary 6.17], there is an irreducible constituent  $\eta$  of  $\lambda^{A \times M}$  with degree  $\eta(1) = \lambda(1)\psi(1)$ . Thus

$$b(G) \geq b(AM) \geq \lambda(1)\psi(1) \geq 3^{p^s} p^t \geq p^s p^t = |P|,$$

and we are done.

Suppose that  $N_G(M_1) > C_G(M_1) \times M_1 = A \times M$ . Note that since  $N_G(M_1)/(C_G(M_1) \times M_1)$  is a  $p$ -group, we see that  $p \mid |\text{Out}(M_1)|$ . By a well-known theorem (see p. 303 and 304 of [3]), we know that  $M_1$  is a simple group of Lie type over a field  $GF(q)$ , where  $q = r^m$  is a power of prime  $r$ . Also we know that all elements of  $N_G(M_1)/(A \times M)$  are field automorphisms of  $M_1$ , and that  $p^e := |N_G(M_1)/(A \times M)|$  divides  $m$ . Let  $\lambda_i$  be the Steinberg-character of  $M_i$  and set  $\lambda = \lambda_1 \times \cdots \times \lambda_k$ . We have  $\lambda_i(1) = |M_i|_r \geq r^m \geq r^{p^e} > p^e$ . Thus  $\lambda(1) > p^{ek} \geq kp^e = |G/(A \times M)|$ . Arguing as in the above paragraph, we conclude that  $b(G) \geq b(AM) \geq \psi(1)\lambda(1) \geq |P|$ ; we are done.  $\square$

Now we begin to prove Theorem 1.

**Lemma 3.** *If  $G$  is a nonabelian simple group, then Theorem 1 holds.*

*Proof.* If  $G$  is either a simple group of Lie type or one of the sporadic simple groups, then by a well-known theorem of [5],  $G$  has an irreducible character  $\chi$  of  $p$ -defect zero. Hence the hypothesis of the theorem yields that  $|G|_p \leq p$ ; we are done. Suppose that  $G$  is the alternating group  $A_n$  for  $n \geq 5$ . If Theorem 1 is not true, then  $|G|_p \geq p^2$ ,  $p$  is an odd prime and  $n \geq 2p \geq 6$ . Corresponding to the partition  $(n - 3, 1, 1, 1)$ , we have the character  $\chi_{(n-3,1,1,1)} \in \text{Irr}(S_n)$  with degree  $\chi_{(n-3,1,1,1)}(1) = (n - 1)(n - 2)(n - 3)/6$ . Hence  $G$  has an irreducible constituent  $\psi$  of  $(\chi_{(n-3,1,1,1)})_G$ . We see that

$$p^2 > \psi(1) \geq (n - 1)(n - 2)(n - 3)/12.$$

This implies that  $(2p - 1)(2p - 2)(2p - 3) < 12p^2$ , and so  $p = 3$ . Then  $3^2 > (n - 1)(n - 2)(n - 3)/12$  forces that  $n \leq 9$ . However, for  $6 \leq n \leq 9$ ,  $A_n$  always has an irreducible character  $\eta$  with degree  $\eta(1) \geq p^2 = 9$ , a contradiction. By the classification theorem of finite simple groups, the proof is complete.  $\square$

**Lemma 4.** *If Theorem 1 is not true, to work for a contradiction, we may assume  $G$ , satisfying the following conditions:*

- (1)  $O_p(G)=1; G = O^{p'}(G)$ .
- (2)  $|G|_p = p^2$ .
- (3) If  $N$  is a normal  $p'$ -subgroup of  $G$ , then  $O_p(G/N)$  is nontrivial.

*Proof.* (1) By induction, we may assume that our theorem holds for any proper subgroup and proper quotient group. Hence we may assume that  $O_p(G) = 1$  and  $G = O^{p'}(G)$ .

(2) By (1), we may assume that  $O_p(G) = 1$  and  $G = O^{p'}(G)$ . Let  $N$  be a minimal normal subgroup of  $G$ . By Lemma 3, we may assume that  $N < G$ . Let  $A/N = O_p(G/N)$ . If  $A < G$ , then induction yields that  $|A|_p, |G/A|_p \leq p$ . So if our theorem is not true, then we may assume that  $|G|_p = p^2$ . In case  $A = G$  let  $M$  be a maximal normal subgroup of  $G$  such that  $N \subseteq M$ . Then  $|G/M| = p$  and by induction we conclude that  $|M|_p \leq p$ . Thus if our theorem is not true, we may also assume that  $|G|_p = p^2$ .

(3) If  $O_p(G/N)$  is trivial for a normal  $p'$ -subgroup  $N$  of  $G$ , then by induction we have  $|G/N|_p \leq p$  and hence our theorem holds. So we may assume that  $O_p(G/N)$  is nontrivial.  $\square$

*Proof of Theorem 1.* Suppose that the theorem is not true. To work for a contradiction, we may assume that  $G$  satisfies all conditions of Lemma 4. By Theorem 2 and Lemma 3, we may also assume that  $G$  is neither simple nor  $p$ -solvable.

Case 1. Suppose that  $G$  possesses a minimal normal  $p'$ -subgroup  $N$ .

Let  $M$  be the maximal normal  $p'$ -subgroup of  $G$  such that  $N \leq M$ . Since  $G$  is not  $p$ -solvable, by (2) and (3) of Lemma 4, both  $A/M = O_p(G/M)$  and  $B/N = O_p(G/N)$  are of order  $p$ . We claim that  $C_A(M) \leq M, C_B(N) \leq N$  and  $|C_G(M)|_p = p$ . Suppose that  $C_A(M) \not\leq M$  (or  $C_B(N) \not\leq N$ ). Then  $O_p(A)$  (or  $O_p(B)$ ) has order  $p$ , and so  $O_p(G) > 1$ , a contradiction. Thus  $C_A(M) \leq M$  and so  $|C_G(M)|_p \leq p$ . Suppose that  $|C_G(M)|_p = 1$ . Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . We see that  $P$  acts faithfully on  $M$ . Then by Theorem 2, there is  $\psi \in \text{Irr}(PM)$  with degree  $\psi(1) \geq p^2$ , clearly a contradiction. Hence,  $|C_G(M)|_p = p$ , and since  $C_G(M)A \triangleleft G$ , it follows that  $G = AC_G(M)$  because  $G = O^{p'}(G)$  and  $|G : AC_G(M)|$  is a  $p'$ -number. Suppose that  $M \not\leq C_G(M)$ . We claim that  $A \cap C_G(M)$  is trivial. Otherwise, we also have  $G = A_1C_G(M)$  where  $A_1/(A \cap C_G(M)) = O_p(G/(A \cap$

$C_G(M))$  is of order  $p$  (note that  $A \cap C_G(M) = C_A(M) \leq M \cap C_G(M) < M$  and that  $A_1 \cap C_G(M) = A \cap C_G(M)$ ). This implies that  $A = A_1$ , a contradiction. Therefore,  $A \cap C_G(M)$  is trivial and so  $G = A \times C_G(M)$ . Clearly,  $O_p(A)$  and  $O_p(C_G(M))$  are trivial. So we can choose  $\chi_1 \in \text{Irr}(A)$  and  $\chi_2 \in \text{Irr}(C_G(M))$  such that  $\chi_1(1)_p = \chi_2(1)_p = p$ . Then for  $\chi = \chi_1 \times \chi_2 \in \text{Irr}(G)$ , we have  $\chi(1) \geq p^2$ , a contradiction. Hence  $M \leq C_G(M)$ . Let  $L/M$  be a chief factor of  $G$  such that  $L \leq C_G(M)$ . By the maximality of  $M$ , we see that  $p$  divides  $|L/M|$ . Thus  $|G : AL|$  is a  $p'$ -number. It follows that  $G = AL$  and so  $L = C_G(M)$  and  $L/M$  is a nonabelian simple group with  $|L/M|_p = p$ . Also we see that  $L' = L$  because  $G = O^{p'}(G)$ . Thus  $M$  is a subgroup of the Schur multiplier, say  $S$ , of the nonabelian simple group  $L/M$ .

Clearly  $N \leq M$  is solvable, and hence  $N$  is an elementary abelian  $v$ -group of order  $v^t$  for some prime  $v$ . Note that since a Sylow  $p$ -subgroup  $P_1$  of  $B$  acts non-trivially on  $N$ , it forces that  $p|(v^t - 1)$ . Clearly  $p > 2$  because  $L/M$  is nonsolvable with  $|L/M|_p = p$ . So  $S$  has an elementary abelian subgroup of order  $v^t$  such that  $p|(v^t - 1)$  where  $p \geq 3$ . If  $L/M$  is either an alternating group  $A_n$  or one of the sporadic simple groups, consulting the table on p. 302 and 303 of [3], we see that  $S$  is a cyclic group and the largest prime divisor of its order is at most 3, which is impossible. If  $G$  is a simple group of Lie type over a field  $GF(q)$  where  $q = r^m$  is a power of prime  $r$ , then  $S = S_1 \times S_2$ , where  $S_1, S_2$  are the diagonal multiplier and the exceptional multiplier of  $L/M$ , respectively. Note that since  $S_1$  is an  $r'$ -group and  $S_2$  is an  $r$ -group, it follows that either  $N \leq S_1$  or  $N \leq S_2$ .

Suppose that  $N \leq S_2$ . Note that  $|N| - 1$  has an odd prime divisor  $p$ . Consulting the table of multipliers of simple groups of Lie type (see [3, p. 302]), we conclude that  $L/M$  is one of following groups:  $A_2(4)$  with  $S_2 = Z_4 \times Z_4$ ;  $D_4(2)$  with  $S_2 = Z_2 \times Z_2$ ;  ${}^2A_5(2)$  with  $S_2 = Z_2 \times Z_2$ ;  ${}^2B_2(8)$  with  $S_2 = Z_2 \times Z_2$ ;  ${}^2E_6(2)$  with  $S_2 = Z_2 \times Z_2$ . In each of these cases, we see that  $N$  is an elementary abelian 2-group of order 4 and  $p = 3$ . Checking the degree of the Steinberg-character  $\chi$  of  $L/M$  in each of these cases, we see that  $\chi(1) \geq 3^2$ , a contradiction.

Suppose that  $N \leq S_1$ . Note that  $|N| - 1$  has an odd prime divisor  $p$ . Consulting the table on p. 302 of [3], we see that  $L/M$  is either a simple group of type  $A_l(q)$  with  $S_1 = Z_{(l+1, q-1)}$  or a simple group of type  ${}^2A_l(q)$ ,  $l \geq 2$  with  $S = Z_{(l+1, q+1)}$ . In each of these cases, we see that  $p \leq |N| - 1 \leq l$  and that  $L/M$  has the Steinberg-character  $\chi$  with degree  $q^{l(l+1)/2} \geq q^p \geq p^2$ , a contradiction.

Case 2. Suppose that  $G$  has no minimal normal  $p'$ -subgroup.

Let  $N$  be a minimal normal subgroup of  $G$ . By Lemma 3,  $N < G$ . Since  $G = O^{p'}(G)$  and  $|G|_p = p^2$ , we see that  $|N|_p = p$  and so that  $N$  is a nonabelian simple group.

Suppose that  $M = C_G(N) > 1$ . Since  $O_{p'}(G) = 1$  and  $O_p(G) = 1$ , we see that  $|M|_p = p$  and  $M$  is also a nonabelian simple group. Then  $G = O^{p'}(G)$  implies that  $G = M \times N$ . By [4, Theorem 12.29], we may choose  $\chi_1 \in \text{Irr}(M)$ ,  $\chi_2 \in \text{Irr}(N)$  such that  $\chi_1(1), \chi_2(1) \geq p$ . So  $G$  has an irreducible character  $\chi = \chi_1 \times \chi_2$  with degree  $\chi(1) \geq p^2$ , a contradiction.

Suppose that  $C_G(N) = 1$ . Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Let us investigate the subgroup  $H = PN$ . Clearly,  $O_p(H) = 1$  since  $C_G(N) = 1$ . Hence if  $H < G$ , then induction yields  $p^2 = |H|_p \leq b(H) \leq b(G)$ , a contradiction. So we may assume  $G = PN$ . Clearly,  $|G/N| = p$  is an odd prime. Let  $x \in G/N$  be an outer automorphism of  $N$  with order  $p$ . Applying [3, Theorem 4.239, Theorem 4.240] and the classification theorem of finite simple groups, we conclude that  $N$  is a simple

group of Lie type over a field  $GF(q)$ , where  $q = r^m$  is a power of prime  $r$ . Also, we know that  $x$  is either a diagonal automorphism, or a field automorphism, or a graph automorphism.

Suppose that  $x$  is a field automorphism. It is well known that  $p$ , the order of  $x$ , must divide  $m$ . Let  $\chi$  be the Steinberg-character of  $N$ . We see that  $\chi(1) \geq q = r^m \geq 2^p \geq p^2$ , a contradiction.

Suppose that  $x$  is a graph automorphism. Consulting the table of automorphisms of the Chevalley groups in [2, p. xvi], and since  $p > 2$ , we see that  $N$  is the type  $D_4(q)$  with  $p = 3$ . In this case, we see that the Steinberg-character  $\chi$  of the simple group  $N$  has degree  $\chi(1) \geq q^{12} > 3^2$ , a contradiction.

Suppose that  $x$  is a diagonal automorphism. Consulting the table in [2, p. xvi], we see that  $N$  is one of the following types:  $A_l(q), l \geq 2$  with  $p|(l+1, q-1)$ ;  ${}^2A_l(q), l \geq 2$  with  $p|(l+1, q+1)$ ;  $E_6(q)$  with  $p|(3, q-1)$ ,  ${}^2E_6(q)$  with  $p|(3, q+1)$ . Arguing as in the last paragraph of case 1, we conclude that in each of the above cases the Steinberg-character  $\chi$  of  $N$  has degree  $\chi(1) \geq p^2$ , a contradiction.  $\square$

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