GORENSTEIN DERIVED FUNCTORS

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Abstract. Over any associative ring $R$ it is standard to derive $\text{Hom}_R(-,-)$ using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains $\text{Ext}_R^n(-,-)$ in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product $- \otimes_R -$ using Gorenstein flat modules.

1. Introduction

When $R$ is a two-sided Noetherian ring, Auslander and Bridger [2] introduced in 1969 the $G$-dimension, $G\dim R M$, for every finite (that is, finitely generated) $R$-module $M$. They proved the inequality $G\dim R M \leq \text{pd}_R M$, with equality $G\dim R M = \text{pd}_R M$ when $\text{pd}_R M < \infty$, along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the $G$-dimension.

The (finite) modules with $G$-dimension zero are called Gorenstein projectives. Over a general ring $R$, Enochs and Jenda in [6] defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if $R$ is two-sided Noetherian, and $G$ is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following [4, Theorem 4.2.6]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary $R$-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- $R$ is an associative ring. All modules are—if not specified otherwise—left $R$-modules, and the category of all $R$-modules is denoted $\mathcal{M}$. We use $\mathcal{A}$ for the category of abelian groups (that is, $\mathbb{Z}$-modules).
- We use $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$ for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat $R$-modules; please see [6] and [8], or Definition 2.7 below.
- Furthermore, for each $R$-module $M$ we write $\text{Gpd}_R M$, $\text{Gid}_R M$ and $\text{Gfd}_R M$ for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of $M$, respectively.

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Now, given our base ring $R$, the usual right derived functors $\Ext^n_R(\cdot, \cdot)$ of $\Hom_R(\cdot, \cdot)$ are important in homological studies of $R$. The material presented here deals with the Gorenstein right derived functors $\Ext^n_{\GP}(\cdot, \cdot)$ and $\Ext^n_{\GI}(\cdot, \cdot)$ of $\Hom_R(\cdot, \cdot)$.

More precisely, let $N$ be a fixed $R$-module. For an $R$-module $M$ that has a proper left $\GP$-resolution $G = \cdots \to G_1 \to G_0 \to 0$ (please see [2.1] below for the definition of proper resolutions), we define

$$\Ext^n_{\GP}(M, N) := H^n(\Hom_R(G, N)).$$

From [2.3] it will follow that $\Ext^n_{\GP}(\cdot, N)$ is a well-defined contravariant functor, defined on the full subcategory $\text{LeftRes}_M(\GP)$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper left $\GP$-resolution.

For a fixed $R$-module $M'$ there is a similar definition of the functor $\Ext^n_{\GI}(M', \cdot)$, which is defined on the full subcategory $\text{RightRes}_M(\GI)$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper right $\GI$-resolution. Now, the best one could hope for is the existence of isomorphisms,

$$\Ext^n_{\GP}(M, N) \cong \Ext^n_{\GI}(M, N),$$

which are functorial in each variable $M \in \text{LeftRes}_M(\GP)$ and $N \in \text{RightRes}_M(\GI)$. The aim of this paper is to show a slightly weaker result.

When $R$ is $n$-Gorenstein (meaning that $R$ is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda \cite{EnochsJenda2} Theorem 12.1.4 have proved the existence of such functorial isomorphisms $\Ext^n_{\GP}(M, N) \cong \Ext^n_{\GI}(M, N)$ for all $R$-modules $M$ and $N$.

It is important to note that for an $n$-Gorenstein ring $R$, we have $\text{Gpd}_RM < \infty$, $\text{Gid}_RM < \infty$, and also $\text{Gpd}_R M < \infty$ for all $R$-modules $M$; please see \cite{EnochsJenda2} Theorems 11.2.1, 11.5.1, 11.7.6. For any ring $R$, \cite{Gorenstein} Proposition 2.18 (which is restated in this paper as Proposition 5.1) implies that the category $\text{LeftRes}_M(\GP)$ contains all $R$-modules $M$ with $\text{Gpd}_RM < \infty$; that is, every $R$-module with finite G-projective dimension has a proper left $\GP$-resolution. Also, every $R$-module with finite G-injective dimension has a proper right $\GI$-resolution. So $\text{RightRes}_M(\GI)$ contains all $R$-modules $N$ with $\text{Gid}_RN < \infty$.

Theorem 5.6 in this text proves that the functorial isomorphisms $\Ext^n_{\GP}(M, N) \cong \Ext^n_{\GI}(M, N)$ hold over arbitrary rings $R$, provided that $\text{Gpd}_RM < \infty$ and $\text{Gid}_RN < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product $\otimes_R \cdot$, using proper left $\GP$-resolutions and proper left $\GF$-resolutions. This has also been proved by Enochs and Jenda \cite{EnochsJenda2} Theorem 12.2.2] in the case when $R$ is $n$-Gorenstein.

2. Preliminaries

Let $T : \mathcal{C} \to \mathcal{E}$ be any additive functor between abelian categories. One usually derives $T$ using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors $T$ with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let \( \mathcal{X} \subseteq \mathcal{C} \) be a full subcategory. A proper left \( \mathcal{X} \)-resolution of \( M \in \mathcal{C} \) is a complex \( X = \cdots \to X_1 \to X_0 \to 0 \) where \( X_i \in \mathcal{X} \), together with a morphism \( X_0 \to M \), such that \( X^+ := \cdots \to X_1 \to X_0 \to M \to 0 \) is also a complex, and such that the sequence

\[
\text{Hom}_\mathcal{C}(X, X^+) = \cdots \to \text{Hom}_\mathcal{C}(X, X_1) \to \text{Hom}_\mathcal{C}(X, X_0) \to \text{Hom}_\mathcal{C}(X, M) \to 0
\]

is exact for every \( X \in \mathcal{X} \). We sometimes refer to \( X^+ = \cdots \to X_1 \to X_0 \to M \to 0 \) as an augmented proper left \( \mathcal{X} \)-resolution. We do not require that \( X^+ \) itself is exact. Furthermore, we use \( \text{LeftRes}_\mathcal{C}(\mathcal{X}) \) to denote the full subcategory of \( \mathcal{C} \) consisting of those objects that have a proper left \( \mathcal{X} \)-resolution. Note that \( \mathcal{X} \) is a subcategory of \( \text{LeftRes}_\mathcal{C}(\mathcal{X}) \).

Proper right \( \mathcal{X} \)-resolutions are defined dually, and the full subcategory of \( \mathcal{C} \) consisting of those objects that have a proper right \( \mathcal{X} \)-resolution is \( \text{RightRes}_\mathcal{C}(\mathcal{X}) \).

The importance of working with proper resolutions comes from the following:

**Proposition 2.2.** Let \( f: M \to M' \) be a morphism in \( \mathcal{C} \), and consider the diagram

\[
\begin{array}{c}
\cdots \to X_2 \to X_1 \to X_0 \to M \to 0 \\
\downarrow f_2 \downarrow f_1 \downarrow f_0 \downarrow f \\
\cdots \to X'_2 \to X'_1 \to X'_0 \to M' \to 0
\end{array}
\]

where the upper row is a complex with \( X_n \in \mathcal{X} \) for all \( n \geq 0 \), and the lower row is an augmented proper left \( \mathcal{X} \)-resolution of \( M' \). Then the following conclusions hold:

(i) There exist morphisms \( f_n: X_n \to X'_n \) for all \( n \geq 0 \), making the diagram above commutative. The chain map \( \{f_n\}_{n \geq 0} \) is called a lift of \( f \).

(ii) If \( \{f'_n\}_{n \geq 0} \) is another lift of \( f \), then the chain maps \( \{f_n\}_{n \geq 0} \) and \( \{f'_n\}_{n \geq 0} \) are homotopic.

**Proof.** The proof is an exercise; please see [9, Exercise 8.1.2].

**Remark 2.3.** A few comments are in order:

- In our applications, the class \( \mathcal{X} \) contains all projectives. Consequently, all the augmented proper left \( \mathcal{X} \)-resolutions occurring in this paper will be exact. Also, all augmented proper right \( \mathcal{Y} \)-resolutions will be exact, when \( \mathcal{Y} \) is a class of \( R \)-modules containing all injectives.

- Recall (see [15, Definition 1.2.2]) that an \( \mathcal{X} \)-precover of \( M \in \mathcal{C} \) is a morphism \( \varphi: X \to M \), where \( X \in \mathcal{X} \), such that the sequence

\[
\text{Hom}_\mathcal{C}(X', X) \xrightarrow{\text{Hom}_\mathcal{C}(X', \varphi)} \text{Hom}_\mathcal{C}(X', M) \xrightarrow{0}
\]

is exact for every \( X' \in \mathcal{X} \). Hence, in an augmented proper left \( \mathcal{X} \)-resolution \( X^+ \) of \( M \), the morphisms \( X_{i+1} \to \text{Ker}(X_i \to X_{i-1}) \), \( i > 0 \), and \( X_0 \to M \) are \( \mathcal{X} \)-precovers.

- What we have called proper \( \mathcal{X} \)-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply call \( \mathcal{X} \)-resolutions. We have adopted the terminology proper from [3, Section 4].

2.4 (Derived Functors). Consider an additive functor \( T: \mathcal{C} \to \mathcal{E} \) between abelian categories. Let us assume that \( T \) is covariant, say. Then (as usual) we can define the \( n \)th left derived functor

\[
L_n^X T: \text{LeftRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E}
\]
of $T$, with respect to the class $\mathcal{X}$, by setting $L^n_T(M) = H_n(T(X))$, where $X$ is any proper left $\mathcal{X}$-resolution of $M \in \text{LeftRes}_C(\mathcal{X})$. Similarly, the $n^{th}$ right derived functor

$$R^n_T : \text{RightRes}_C(\mathcal{X}) \to \mathcal{E}$$

of $T$ with respect to $\mathcal{X}$ is defined by $R^n_T(N) = H_n(T(Y))$, where $Y$ is any proper right $\mathcal{X}$-resolution of $N \in \text{RightRes}_C(\mathcal{X})$. These constructions are well-defined and functorial in the arguments $M$ and $N$ by Proposition 2.2.

The situation where $T$ is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category $\mathcal{D}$, together with a full subcategory $\mathcal{Y} \subseteq \mathcal{D}$ and an additive functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ in two variables. We will assume that $F$ is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of $F$ is not important, and the definitions and results below can easily be modified to fit the situation where $F$ is covariant in both variables, say.

For fixed $M \in \mathcal{C}$ and $N \in \mathcal{D}$ we can then consider the two right derived functors

$$R^n_X F(-, N) : \text{LeftRes}_C(\mathcal{X}) \to \mathcal{E} \quad \text{and} \quad R^n_Y F(M, -) : \text{RightRes}_D(\mathcal{Y}) \to \mathcal{E}.$$  

If furthermore $M \in \text{LeftRes}_C(\mathcal{X})$ and $N \in \text{RightRes}_D(\mathcal{Y})$, we can ask for a sufficient condition to ensure that

$$R^n_X F(M, N) \cong R^n_Y F(M, N),$$

functorial in $M$ and $N$. Here we wrote $R^n_X F(M, N)$ for the functor $R^n_X F(-, N)$ applied to $M$. Another, and perhaps better, notation could be

$$R^n_X F(-, N)(M).$$

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

**Theorem 2.6.** Consider the functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which is contravariant in the first variable and covariant in the second variable, together with the full subcategories $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$. Assume that we have full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\text{LeftRes}_C(\mathcal{X})$ and $\text{RightRes}_D(\mathcal{Y})$, respectively, satisfying:

(i) $\mathcal{X} \subseteq \bar{\mathcal{X}}$ and $\mathcal{Y} \subseteq \bar{\mathcal{Y}}$.

(ii) Every $M \in \bar{\mathcal{X}}$ has an augmented proper left $\mathcal{X}$-resolution $\cdots \to X_1 \to X_0 \to M \to 0$, such that $0 \to F(M, Y) \to F(X_0, Y) \to F(X_1, Y) \to \cdots$ is exact for all $Y \in \mathcal{Y}$.

(iii) Every $N \in \bar{\mathcal{Y}}$ has an augmented proper right $\mathcal{Y}$-resolution $0 \to N \to Y^0 \to Y^1 \to \cdots$, such that $0 \to F(X, N) \to F(X, Y^0) \to F(X, Y^1) \to \cdots$ is exact for all $X \in \mathcal{X}$.

Then we have functorial isomorphisms

$$R^n_X F(M, N) \cong R^n_Y F(M, N),$$

for all $M \in \bar{\mathcal{X}}$ and $N \in \bar{\mathcal{Y}}$.  


Proof. Please see [3, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [3, Proposition 8.2.14], or the proofs of [13, Theorems 2.7.2 and 2.7.6]. □

In the next paragraphs we apply the results above to special categories \( \mathcal{X} \), \( \mathcal{X} \), \( \mathcal{C} \) and \( \mathcal{C} \), \( \mathcal{D} \), including the categories mentioned in [11]. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

**Definition 2.7.** A complete projective resolution is an exact sequence of projective modules,

\[
P = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots,
\]

such that \( \text{Hom}_R(P, Q) \) is exact for every projective \( R \)-module \( Q \). An \( R \)-module \( M \) is called Gorenstein projective (G-projective for short), if there exists a complete projective resolution \( P \) with \( M \cong \text{Im}(P_0 \to P_{-1}) \). Gorenstein injective (G-injective for short) modules are defined dually.

A complete flat resolution is an exact sequence of flat (left) \( R \)-modules,

\[
F = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots,
\]

such that \( I \otimes R F \) is exact for every injective right \( R \)-module \( I \). An \( R \)-module \( M \) is called Gorenstein flat (G-flat for short), if there exists a complete flat resolution \( F \) with \( M \cong \text{Im}(F_0 \to F_{-1}) \).

3. Gorenstein deriving \( \text{Hom}_R(-, -) \)

We now return to categories of modules. We use \( \widetilde{\mathcal{GP}}, \widetilde{\mathcal{GI}} \) and \( \widetilde{\mathcal{GF}} \) to denote the class of \( R \)-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, \( \mathcal{GP} \) -precovers are always surjective, and \( \widetilde{\mathcal{GP}} \) contains all modules with finite projective dimension.

We now consider the functor \( \text{Hom}_R(-, -) : \mathcal{M} \times \mathcal{M} \to \mathcal{A} \), together with the categories

\[
\mathcal{X} = \mathcal{GP}, \quad \mathcal{X} = \mathcal{GP}, \quad \mathcal{Y} = \mathcal{GI}, \quad \mathcal{Y} = \mathcal{GI}.
\]

In this case we define, in the sense of section 2.4

\[
\text{Ext}^n_{\mathcal{GP}}(-, N) = R^n_{\mathcal{GP}} \text{Hom}_R(-, N) \quad \text{and} \quad \text{Ext}^n_{\mathcal{GI}}(M, -) = R^n_{\mathcal{GI}} \text{Hom}_R(M, -),
\]

for fixed \( R \)-modules \( M \) and \( N \). We wish, of course, to apply Theorem 2.6 to this situation. Note that by [12, Proposition 2.18], we have:

**Proposition 3.1.** If \( M \) is an \( R \)-module with \( \text{Gpd}_R M < \infty \), then there exists a short exact sequence \( 0 \to K \to G \to M \to 0 \), where \( G \to M \) is a \( \mathcal{GP} \)-precover of \( M \) (please see Remark 2.2), and \( \text{pd}_R K = \text{Gpd}_R M - 1 \) (in the case where \( M \) is Gorenstein projective, this should be interpreted as \( K = 0 \)).

Consequently, every \( R \)-module with finite Gorenstein projective dimension has a proper left \( \mathcal{GP} \)-resolution (that is, there is an inclusion \( \widetilde{\mathcal{GP}} \subseteq \text{LeftRes}_M(\mathcal{GP}) \)).

Furthermore, we will need the following from [12, Theorem 2.13]:

**Theorem 3.2.** Let \( M \) be any \( R \)-module with \( \text{Gpd}_R M < \infty \). Then

\[
\text{Gpd}_R M = \sup \{ n \in \mathbb{Z} \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R \text{-module } L \text{ with } \text{pd}_R L < \infty \}.
\]
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an $R$-module $M$ is given by

$$\text{pd}_R M = \{ n \geq 0 \mid \text{Ext}_R^n(M, L) \neq 0 \text{ for some } R\text{-module } L \}.$$ 

It also follows that if $\text{pd}_R M < \infty$, then every projective resolution of $M$ is actually a proper left $\mathcal{GP}$-resolution of $M$.

Lemma 3.4. Assume that $M$ is an $R$-module with finite Gorenstein projective dimension, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $\mathcal{GP}$-resolution of $M$ (which exists by Proposition 3.1). Then $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective modules $H$.

Proof. We split the proper resolution $G^+$ into short exact sequences. Hence it suffices to show exactness of $\text{Hom}_R(S, H)$ for all Gorenstein injective modules $H$ and all short exact sequences

$$S = 0 \to K \to G \to M \to 0,$$

where $G \to M$ is a $\mathcal{GP}$-precover of some module $M$ with $\text{Gpd}_R M < \infty$ (recall that $\mathcal{GP}$-precovers are always surjective). By Proposition 3.1 there is a special short exact sequence,

$$S' = 0 \to K' \xrightarrow{\iota} G' \xrightarrow{\pi} M \to 0,$$

where $\pi: G' \to M$ is a $\mathcal{GP}$-precover and $\text{pd}_R K' < \infty$.

It is easy to see (as in Proposition 2.2) that the complexes $S$ and $S'$ are homotopy equivalent, and thus so are the complexes $\text{Hom}_R(S, H)$ and $\text{Hom}_R(S', H)$ for every (Gorenstein injective) module $H$. Hence it suffices to show the exactness of $\text{Hom}_R(S', H)$ whenever $H$ is Gorenstein injective.

Now let $H$ be any Gorenstein injective module. We need to prove the exactness of

$$\text{Hom}_R(G', H) \xrightarrow{\text{Hom}_R(\iota, H)} \text{Hom}_R(K', H) \to 0.$$

To see this, let $\alpha: K' \to H$ be any homomorphism. We wish to find $\varphi: G' \to H$ such that $\varphi \iota = \alpha$. Now pick an exact sequence

$$0 \to \tilde{H} \to E \xrightarrow{g} H \to 0,$$

where $E$ is injective, and $\tilde{H}$ is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines $H$). Since $\tilde{H}$ is Gorenstein injective and $\text{pd}_R K' < \infty$, we get $\text{Ext}_R^1(K', \tilde{H}) = 0$ by Lemma 1.3, and thus a lifting $\varepsilon: K' \to E$ with $g \varepsilon = \alpha$.

Next, injectivity of $E$ gives $\varepsilon': G' \to E$ with $\varepsilon' \iota = \varepsilon$. Now $\varphi = g \varepsilon': G' \to H$ is the desired map. \qed

With a similar proof we get:
Lemma 3.5. Assume that $N$ is an $R$-module with finite Gorenstein injective dimension, and let $H^+ = 0 \to N \to H^0 \to H^1 \to \cdots$ be an augmented proper right $G$-resolution of $N$ (which exists by the dual of Proposition 3.4). Then $\text{Hom}_R(G, H^+)$ is exact for all Gorenstein projective modules $G$.

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. For all $R$-modules $M$ and $N$ with $\text{Gpd}_RM < \infty$ and $\text{Gid}_RN < \infty$, we have isomorphisms

$$\text{Ext}_{G^P}^n(M, N) \cong \text{Ext}_{G^I}^n(M, N),$$

which are functorial in $M$ and $N$.

3.7 (Definition of GExt). Let $M$ and $N$ be $R$-modules with $\text{Gpd}_RM < \infty$ and $\text{Gid}_RN < \infty$. Then we write

$$\text{GExt}_{G^P}^n(M, N) := \text{Ext}_{G^P}^n(M, N) \cong \text{Ext}_{G^I}^n(M, N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare GExt with the classical Ext. This is done in:

Theorem 3.8. Let $M$ and $N$ be any $R$-modules. Then the following conclusions hold:

(i) There are natural isomorphisms $\text{Ext}_{G^P}^n(M, N) \cong \text{Ext}_R^n(M, N)$ under each of the conditions

- (i) $\text{pd}_RM < \infty$ or (i) $M \in \text{LeftRes}_M(GP)$ and $\text{id}_RN < \infty$.

(ii) There are natural isomorphisms $\text{Ext}_{G^I}^n(M, N) \cong \text{Ext}_R^n(M, N)$ under each of the conditions

- (i) $\text{id}_RN < \infty$ or (i) $N \in \text{RightRes}_N(GI)$ and $\text{pd}_RM < \infty$.

(iii) Assume that $\text{Gpd}_RM < \infty$ and $\text{Gid}_RN < \infty$. If either $\text{pd}_RM < \infty$ or $\text{id}_RN < \infty$, then

$$\text{GExt}_R^n(M, N) \cong \text{Ext}_R^n(M, N)$$

is functorial in $M$ and $N$.

Proof. (i) Assume that $\text{pd}_RM < \infty$, and pick any projective resolution $P$ of $M$. By Remark 3.3, $P$ is also a proper left $GP$-resolution of $M$, and thus

$$\text{Ext}_{G^P}^n(M, N) = H^n(\text{Hom}_R(P, N)) = \text{Ext}_R^n(M, N).$$

In the case where $M \in \text{LeftRes}_M(GP)$ and $\text{id}_RN = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\text{Hom}_R(-, N)$, that is, $\text{Ext}_R^i(G, N) = 0$ (the usual Ext) for every Gorenstein projective module $G$, and every integer $i > 0$.

This is because, if $G$ is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \to G \to Q^0 \to \cdots \to Q^{m-1} \to C \to 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\text{Hom}_R(\cdot, N)$, we get $\text{Ext}_R^i(G, N) \cong \text{Ext}_R^{m+i}(C, N) = 0$, as claimed.

Therefore [11 Chapter III, Proposition 1.2A] implies that $\text{Ext}_R^n(-, N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of $\text{GExt}_R^n(-, -)$.
4. Gorenstein deriving — $\otimes_R$ —

In dealing with the tensor product we need, of course, both left and right $R$-modules. Thus the following addition to Notation 1.1 is needed:

If $\mathcal{C}$ is any of the categories in Notation 1.1 ($\mathcal{M}$, $\mathcal{P}$, etc.), we write $R\mathcal{C}$, respectively, $\mathcal{C}R$, for the category of left, respectively, right, $R$-modules with the property describing the modules in $\mathcal{C}$.

Now we consider the functor $\mathcal{G}_R^P: \mathcal{M}_R \to \mathcal{A}$. For fixed $M \in \mathcal{M}_R$ and $N \in R M$ we define, in the sense of section 2.4:

\[
\text{Tor}^n_{\mathcal{G}_R^P}(-, N) := L^n_{\mathcal{G}_R^P}(\otimes_R N) \quad \text{and} \quad \text{Tor}^n_{\mathcal{G}_R^F}(M, -) := L^n_{\mathcal{G}_R^F}(M \otimes_R -),
\]

The first two $\text{Tor}$s use proper left Gorenstein projective resolutions, and the last two $\text{Tor}$s use proper left Gorenstein flat resolutions. In order to compare these different $\text{Tor}$s, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of $(X, \hat{X}) = (\mathcal{G}_R^P, \hat{\mathcal{G}_R^P})$ or $(\mathcal{G}_R^F, \widehat{\mathcal{G}_R^F})$, namely, the covariant-covariant version of Theorem 2.6 instead of the stated contravariant-covariant version. We will need the classical notion:

**Definition 4.1.** The left finitistic projective dimension $\text{LeftFPD}(R)$ of $R$ is defined as

\[
\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}.
\]

The right finitistic projective dimension $\text{RightFPD}(R)$ of $R$ is defined similarly.

**Remark 4.2.** When $R$ is commutative and Noetherian, the dimensions $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ coincide and are equal to the Krull dimension of $R$, by [10 Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12 Proposition 3.3], [12 Theorem 3.5] and [12 Proposition 3.18], respectively:

**Proposition 4.3.** If $R$ is right coherent with finite $\text{LeftFPD}(R)$, then every Gorenstein projective left $R$-module is also Gorenstein flat. That is, there is an inclusion $\mathcal{G}_R^P \subseteq \mathcal{G}_R^F$.

**Theorem 4.4.** For any left $R$-module $M$, we consider the following three conditions:

(i) The left $R$-module $M$ is G-flat.

(ii) The Pontryagin dual $\text{Hom}_{\mathbb{Q}}(M, \mathbb{Q}/\mathbb{Z})$ (which is a right $R$-module) is G-injective.

(iii) $M$ has an augmented proper right resolution $0 \to M \to F^0 \to F^1 \to \cdots$ consisting of flat left $R$-modules, and $\text{Tor}_i^R(I, M) = 0$ for all injective right $R$-modules $I$, and all $i > 0$.

The implication (i) $\Rightarrow$ (ii) always holds. If $R$ is right coherent, then also (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and hence all three conditions are equivalent.
Proposition 4.5. Assume that $R$ is right coherent. If $M$ is a left $R$-module with $\text{Gfd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is an $R\mathcal{G}F$-precover of $M$, and $\text{fd}_R K = \text{Gfd}_R M - 1$ (in the case where $M$ is Gorenstein flat, this should be interpreted as $K = 0$).

In particular, every left $R$-module with finite Gorenstein flat dimension has a proper left $R\mathcal{G}F$-resolution (that is, there is an inclusion $R\mathcal{G}F \subseteq \text{LeftRes}_R(M)$).

Our first result is:

Lemma 4.6. Let $M$ be a left $R$-module with $\text{Gpd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\mathcal{G}F$-resolution of $M$ (which exists by Proposition [4.4]). Then the following conclusions hold:

(i) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(ii) If $R$ is left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) By Theorem [4.4] above, the Pontryagin dual $H = \text{Hom}_Z(T, Q/\mathbb{Z})$ is a Gorenstein injective left $R$-module. Hence $\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, Q/\mathbb{Z})$ is exact by Proposition [3.3]. Since $Q/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.

(ii) With the given assumptions on $R$, the dual of Proposition [4.3] implies that every Gorenstein projective right $R$-module is also Gorenstein flat.

Lemma 4.7. Assume that $R$ is right coherent with finite $\text{LeftFPD}(R)$. Let $M$ be a left $R$-module with $\text{Gpd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\mathcal{G}F$-resolution of $M$ (which exists by Proposition [4.4] since $R$ is right coherent). Then the following conclusions hold:

(i) $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective left $R$-modules $H$.

(ii) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(iii) If $R$ is also left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) Since $\text{Gfd}_R M < \infty$ and $R$ is right coherent, Proposition [4.5] gives a special short exact sequence $0 \to K' \to G' \to M \to 0$, where $G' \to M$ is an $R\mathcal{G}F$-precover of $M$, and $\text{fd}_R K' < \infty$. Since $R$ has $\text{LeftFPD}(R) < \infty$, Proposition 6] implies that also $\text{pd}_R K' < \infty$. Now the proof of Lemma [4.4] applies.

(ii) If $T$ is a Gorenstein flat right $R$-module, then the left $R$-module $H = \text{Hom}_Z(T, Q/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem [4.4] above. By the result (i), just proved, we have exactness of

$$\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, Q/\mathbb{Z}).$$

Since $Q/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R G^+$, as desired.

(iii) Under the extra assumptions on $R$, the dual of Proposition [4.3] implies that every Gorenstein projective right $R$-module is also Gorenstein flat. Thus (iii) follows from (ii).

Theorem 4.8. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:
(i) If Gfd$_RM < \infty$ and Gfd$_RN < \infty$, then
\[ \text{Tor}^G_{n,R}(M,N) \cong \text{Tor}^G_{n,F}(M,N). \]

(ii) If Gpd$_RM < \infty$ and Gfd$_RN < \infty$, then
\[ \text{Tor}^G_{n,R}(M,N) \cong \text{Tor}^G_{n,F}(M,N) \cong \text{Tor}^G_{n,F}(M,N). \]

(iii) If Gfd$_RM < \infty$ and Gpd$_RN < \infty$, then
\[ \text{Tor}^G_{n,R}(M,N) \cong \text{Tor}^G_{n,F}(M,N) \cong \text{Tor}^G_{n,F}(M,N). \]

(iv) If Gpd$_RM < \infty$ and Gpd$_RN < \infty$, then
\[ \text{Tor}^G_{n,R}(M,N) \cong \text{Tor}^G_{n,F}(M,N) \cong \text{Tor}^G_{n,F}(M,N). \]

All the isomorphisms are functorial in $M$ and $N$.

**Proof.** Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6. \qed

**4.9 (Definition of $g\text{Tor}$ and $\text{GTor}$).** Assume that $R$ is both left and right coherent, and that both LeftFPD($R$) and RightFPD($R$) are finite. Furthermore, let $M$ be a right $R$-module, and let $N$ be a left $R$-module. If Gfd$_RM < \infty$ and Gfd$_RN < \infty$, then we write
\[ g\text{Tor}^R_n(M,N) := \text{Tor}^G_{n,R}(M,N) \cong \text{Tor}^G_{n,F}(M,N) \]
for the isomorphic abelian groups in Theorem 4.8(i). If Gpd$_RM < \infty$ and Gpd$_RN < \infty$, then we write
\[ \text{GTor}^R_n(M,N) := \text{Tor}^G_{n,R}(M,N) \cong \text{Tor}^G_{n,F}(M,N) \]
for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8.

**Theorem 4.10.** Assume that $R$ is both left and right coherent, and that both LeftFPD($R$) and RightFPD($R$) are finite. For every right $R$-module $M$ with finite Gpd$_RM$, and for every left $R$-module $N$ with Gpd$_RN < \infty$, we have isomorphisms:
\[ g\text{Tor}^R_n(M,N) \cong \text{GTor}^R_n(M,N) \]
that are functorial in $M$ and $N$.

Finally we compare $g\text{Tor}$ (and hence $\text{GTor}$) with the usual Tor.

**Theorem 4.11.** Assume that $R$ is both left and right coherent, and that both LeftFPD($R$) and RightFPD($R$) are finite. Furthermore, let $M$ be a right $R$-module with Gfd$_RM < \infty$, and let $N$ be a left $R$-module with Gfd$_RN < \infty$. If either fd$_RM < \infty$ or fd$_RN < \infty$, then there are isomorphisms
\[ g\text{Tor}^R_n(M,N) \cong \text{Tor}^R_n(M,N) \]
that are functorial in $M$ and $N$.

**Proof.** If fd$_RM < \infty$, then we also have pd$_RM < \infty$ by [13 Proposition 6] (since RightFPD($R$) < $\infty$). Let $P$ be any projective resolution of $M$. As noted in Remark 3.8, $P$ is also a proper left $\mathcal{GP}_R$-resolution of $M$. Hence, Theorem 4.8(ii) and the definitions give:
\[ g\text{Tor}^R_n(M,N) = \text{Tor}^G_{n,R}(M,N) = H_n(P \otimes_R N) = \text{Tor}^R_n(M,N), \]
as desired. \qed
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References


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