

GORENSTEIN DERIVED FUNCTORS

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ABSTRACT. Over any associative ring R it is standard to derive $\text{Hom}_R(-, -)$ using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains $\text{Ext}_R^n(-, -)$ in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product $- \otimes_R -$ using Gorenstein flat modules.

1. INTRODUCTION

When R is a two-sided Noetherian ring, Auslander and Bridger [2] introduced in 1969 the G-dimension, $\text{G-dim}_R M$, for every *finite* (that is, finitely generated) R -module M . They proved the inequality $\text{G-dim}_R M \leq \text{pd}_R M$, with equality $\text{G-dim}_R M = \text{pd}_R M$ when $\text{pd}_R M < \infty$, along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called *Gorenstein projectives*. Over a general ring R , Enochs and Jenda in [6] defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if R is two-sided Noetherian, and G is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following [4, Theorem (4.2.6)]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary R -modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- R is an associative ring. All modules are—if not specified otherwise—*left* R -modules, and the category of all R -modules is denoted \mathcal{M} . We use \mathcal{A} for the category of abelian groups (that is, \mathbb{Z} -modules).
- We use \mathcal{GP} , \mathcal{GI} and \mathcal{GF} for the categories of *Gorenstein projective*, *Gorenstein injective* and *Gorenstein flat* R -modules; please see [6] and [8], or Definition 2.7 below.
- Furthermore, for each R -module M we write $\text{Gpd}_R M$, $\text{Gid}_R M$ and $\text{Gfd}_R M$ for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of M , respectively.

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Now, given our base ring R , the usual right derived functors $\text{Ext}_R^n(-, -)$ of $\text{Hom}_R(-, -)$ are important in homological studies of R . The material presented here deals with the Gorenstein right derived functors $\text{Ext}_{\mathcal{GP}}^n(-, -)$ and $\text{Ext}_{\mathcal{GI}}^n(-, -)$ of $\text{Hom}_R(-, -)$.

More precisely, let N be a fixed R -module. For an R -module M that has a *proper left \mathcal{GP} -resolution* $\mathbf{G} = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow 0$ (please see 2.1 below for the definition of proper resolutions), we define

$$\text{Ext}_{\mathcal{GP}}^n(M, N) := \text{H}^n(\text{Hom}_R(\mathbf{G}, N)).$$

From 2.4 it will follow that $\text{Ext}_{\mathcal{GP}}^n(-, N)$ is a well-defined contravariant functor, defined on the full subcategory, $\text{LeftRes}_{\mathcal{M}}(\mathcal{GP})$, of \mathcal{M} , consisting of all R -modules that have a proper left \mathcal{GP} -resolution.

For a fixed R -module M' there is a similar definition of the functor $\text{Ext}_{\mathcal{GI}}^n(M', -)$, which is defined on the full subcategory, $\text{RightRes}_{\mathcal{M}}(\mathcal{GI})$, of \mathcal{M} , consisting of all R -modules that which have a proper right \mathcal{GI} -resolution. Now, the best one could *hope* for is the existence of isomorphisms,

$$\text{Ext}_{\mathcal{GP}}^n(M, N) \cong \text{Ext}_{\mathcal{GI}}^n(M, N),$$

which are functorial in each variable $M \in \text{LeftRes}_{\mathcal{M}}(\mathcal{GP})$ and $N \in \text{RightRes}_{\mathcal{M}}(\mathcal{GI})$. The aim of this paper is to show a slightly weaker result.

When R is n -Gorenstein (meaning that R is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda [9, Theorem 12.1.4] have proved the existence of such functorial isomorphisms $\text{Ext}_{\mathcal{GP}}^n(M, N) \cong \text{Ext}_{\mathcal{GI}}^n(M, N)$ for all R -modules M and N .

It is important to note that for an n -Gorenstein ring R , we have $\text{Gpd}_R M < \infty$, $\text{Gid}_R M < \infty$, and also $\text{Gfd}_R M < \infty$ for all R -modules M ; please see [9, Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring R , [12, Proposition 2.18] (which is restated in this paper as Proposition 3.1) implies that the category $\text{LeftRes}_{\mathcal{M}}(\mathcal{GP})$ contains all R -modules M with $\text{Gpd}_R M < \infty$; that is, every R -module with finite G -projective dimension has a proper left \mathcal{GP} -resolution. Also, every R -module with finite G -injective dimension has a proper right \mathcal{GI} -resolution. So $\text{RightRes}_{\mathcal{M}}(\mathcal{GI})$ contains all R -modules N with $\text{Gid}_R N < \infty$.

Theorem 3.6 in this text proves that the functorial isomorphisms $\text{Ext}_{\mathcal{GP}}^n(M, N) \cong \text{Ext}_{\mathcal{GI}}^n(M, N)$ hold over *arbitrary* rings R , provided that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product $- \otimes_R -$, using proper left \mathcal{GP} -resolutions and proper left \mathcal{GF} -resolutions. This has also been proved by Enochs and Jenda [9, Theorem 12.2.2] in the case when R is n -Gorenstein.

2. PRELIMINARIES

Let $T: \mathcal{C} \rightarrow \mathcal{E}$ be any additive functor between abelian categories. One usually derives T using resolutions consisting of projective or injective objects (if the category \mathcal{C} has enough projectives or injectives). This section is a very brief note on how to derive functors T with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.

2.1 (Proper Resolutions). Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory. A *proper left \mathcal{X} -resolution* of $M \in \mathcal{C}$ is a complex $\mathbf{X} = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0$ where $X_i \in \mathcal{X}$, together with a morphism $X_0 \rightarrow M$, such that $\mathbf{X}^+ := \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$ is also a complex, and such that the sequence

$$\mathrm{Hom}_{\mathcal{C}}(X, \mathbf{X}^+) = \cdots \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, X_1) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, X_0) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, M) \rightarrow 0$$

is exact for every $X \in \mathcal{X}$. We sometimes refer to $\mathbf{X}^+ = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$ as an *augmented proper left \mathcal{X} -resolution*. We do not require that \mathbf{X}^+ itself is exact. Furthermore, we use $\mathrm{LeftRes}_{\mathcal{C}}(\mathcal{X})$ to denote the full subcategory of \mathcal{C} consisting of those objects that have a proper left \mathcal{X} -resolution. Note that \mathcal{X} is a subcategory of $\mathrm{LeftRes}_{\mathcal{C}}(\mathcal{X})$.

Proper right \mathcal{X} -resolutions are defined dually, and the full subcategory of \mathcal{C} consisting of those objects that have a proper right \mathcal{X} -resolution is $\mathrm{RightRes}_{\mathcal{C}}(\mathcal{X})$.

The importance of working with *proper* resolutions comes from the following:

Proposition 2.2. *Let $f: M \rightarrow M'$ be a morphism in \mathcal{C} , and consider the diagram*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ \cdots & \longrightarrow & X'_2 & \longrightarrow & X'_1 & \longrightarrow & X'_0 & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

where the upper row is a complex with $X_n \in \mathcal{X}$ for all $n \geq 0$, and the lower row is an augmented proper left \mathcal{X} -resolution of M' . Then the following conclusions hold:

- (i) There exist morphisms $f_n: X_n \rightarrow X'_n$ for all $n \geq 0$, making the diagram above commutative. The chain map $\{f_n\}_{n \geq 0}$ is called a *lift of f* .
- (ii) If $\{f'_n\}_{n \geq 0}$ is another lift of f , then the chain maps $\{f_n\}_{n \geq 0}$ and $\{f'_n\}_{n \geq 0}$ are homotopic.

Proof. The proof is an exercise; please see [9, Exercise 8.1.2]. □

Remark 2.3. A few comments are in order:

- In our applications, the class \mathcal{X} contains all projectives. Consequently, all the augmented proper left \mathcal{X} -resolutions occurring in this paper will be exact. Also, all augmented proper right \mathcal{Y} -resolutions will be exact, when \mathcal{Y} is a class of R -modules containing all injectives.
- Recall (see [15, Definition 1.2.2]) that an *\mathcal{X} -precover* of $M \in \mathcal{C}$ is a morphism $\varphi: X \rightarrow M$, where $X \in \mathcal{X}$, such that the sequence

$$\mathrm{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X', \varphi)} \mathrm{Hom}_{\mathcal{C}}(X', M) \longrightarrow 0$$

is exact for every $X' \in \mathcal{X}$. Hence, in an augmented proper left \mathcal{X} -resolution \mathbf{X}^+ of M , the morphisms $X_{i+1} \rightarrow \mathrm{Ker}(X_i \rightarrow X_{i-1})$, $i > 0$, and $X_0 \rightarrow M$ are \mathcal{X} -precovers.

- What we have called *proper \mathcal{X} -resolutions*, Enochs and Jenda [9, Definition 8.1.2] simply call *\mathcal{X} -resolutions*. We have adopted the terminology *proper* from [3, Section 4].

2.4 (Derived Functors). Consider an additive functor $T: \mathcal{C} \rightarrow \mathcal{E}$ between abelian categories. Let us assume that T is covariant, say. Then (as usual) we can define the n^{th} left derived functor

$$L_n^{\mathcal{X}}T: \mathrm{LeftRes}_{\mathcal{C}}(\mathcal{X}) \rightarrow \mathcal{E}$$

of T , with respect to the class \mathcal{X} , by setting $L_n^{\mathcal{X}}T(M) = H_n(T(\mathbf{X}))$, where \mathbf{X} is any proper left \mathcal{X} -resolution of $M \in \text{LeftRes}_{\mathcal{C}}(\mathcal{X})$. Similarly, the n^{th} right derived functor

$$R_{\mathcal{X}}^n T: \text{RightRes}_{\mathcal{C}}(\mathcal{X}) \rightarrow \mathcal{E}$$

of T with respect to \mathcal{X} is defined by $R_{\mathcal{X}}^n T(N) = H_n(T(\mathbf{Y}))$, where \mathbf{Y} is any proper right \mathcal{X} -resolution of $N \in \text{RightRes}_{\mathcal{C}}(\mathcal{X})$. These constructions are well-defined and functorial in the arguments M and N by Proposition 2.2.

The situation where T is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category \mathcal{D} , together with a full subcategory $\mathcal{Y} \subseteq \mathcal{D}$ and an additive functor $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ in *two* variables. We will assume that F is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of F is not important, and the definitions and results below can easily be modified to fit the situation where F is covariant in both variables, say.

For fixed $M \in \mathcal{C}$ and $N \in \mathcal{D}$ we can then consider the two right derived functors as in 2.4:

$$R_{\mathcal{X}}^n F(-, N): \text{LeftRes}_{\mathcal{C}}(\mathcal{X}) \rightarrow \mathcal{E} \quad \text{and} \quad R_{\mathcal{Y}}^n F(M, -): \text{RightRes}_{\mathcal{D}}(\mathcal{Y}) \rightarrow \mathcal{E}.$$

If furthermore $M \in \text{LeftRes}_{\mathcal{C}}(\mathcal{X})$ and $N \in \text{RightRes}_{\mathcal{D}}(\mathcal{Y})$, we can ask for a sufficient condition to ensure that

$$R_{\mathcal{X}}^n F(M, N) \cong R_{\mathcal{Y}}^n F(M, N),$$

functorial in M and N . Here we wrote $R_{\mathcal{X}}^n F(M, N)$ for the functor $R_{\mathcal{X}}^n F(-, N)$ applied to M . Another, and perhaps better, notation could be

$$R_{\mathcal{X}}^n F(-, N)[M].$$

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term *left/right balanced functor* (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

Theorem 2.6. *Consider the functor $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ which is contravariant in the first variable and covariant in the second variable, together with the full subcategories $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$. Assume that we have full subcategories $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ of $\text{LeftRes}_{\mathcal{C}}(\mathcal{X})$ and $\text{RightRes}_{\mathcal{D}}(\mathcal{Y})$, respectively, satisfying:*

- (i) $\mathcal{X} \subseteq \tilde{\mathcal{X}}$ and $\mathcal{Y} \subseteq \tilde{\mathcal{Y}}$.
- (ii) *Every $M \in \tilde{\mathcal{X}}$ has an augmented proper left \mathcal{X} -resolution $\cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$, such that $0 \rightarrow F(M, Y) \rightarrow F(X_0, Y) \rightarrow F(X_1, Y) \rightarrow \cdots$ is exact for all $Y \in \mathcal{Y}$.*
- (iii) *Every $N \in \tilde{\mathcal{Y}}$ has an augmented proper right \mathcal{Y} -resolution $0 \rightarrow N \rightarrow Y^0 \rightarrow Y^1 \rightarrow \cdots$, such that $0 \rightarrow F(X, N) \rightarrow F(X, Y^0) \rightarrow F(X, Y^1) \rightarrow \cdots$ is exact for all $X \in \mathcal{X}$.*

Then we have functorial isomorphisms

$$R_{\mathcal{X}}^n F(M, N) \cong R_{\mathcal{Y}}^n F(M, N),$$

for all $M \in \tilde{\mathcal{X}}$ and $N \in \tilde{\mathcal{Y}}$.

Proof. Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [9, Proposition 8.2.14], or the proofs of [14, Theorems 2.7.2 and 2.7.6]. \square

In the next paragraphs we apply the results above to special categories \mathcal{X} , $\tilde{\mathcal{X}}$, \mathcal{C} and \mathcal{Y} , $\tilde{\mathcal{Y}}$, \mathcal{D} , including the categories mentioned in 1.1. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

Definition 2.7. A *complete projective resolution* is an exact sequence of projective modules,

$$\mathbf{P} = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow \cdots,$$

such that $\text{Hom}_R(\mathbf{P}, Q)$ is exact for every projective R -module Q . An R -module M is called *Gorenstein projective* (G -projective for short), if there exists a complete projective resolution \mathbf{P} with $M \cong \text{Im}(P_0 \rightarrow P_{-1})$. *Gorenstein injective* (G -injective for short) modules are defined dually.

A *complete flat resolution* is an exact sequence of flat (left) R -modules,

$$\mathbf{F} = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots,$$

such that $I \otimes_R \mathbf{F}$ is exact for every injective right R -module I . An R -module M is called *Gorenstein flat* (G -flat for short), if there exists a complete flat resolution \mathbf{F} with $M \cong \text{Im}(F_0 \rightarrow F_{-1})$.

3. GORENSTEIN DERIVING $\text{Hom}_R(-, -)$

We now return to categories of *modules*. We use $\tilde{\mathcal{G}}\mathcal{P}$, $\tilde{\mathcal{G}}\mathcal{I}$ and $\tilde{\mathcal{G}}\mathcal{F}$ to denote the class of R -modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, $\mathcal{G}\mathcal{P}$ -precovers are always surjective, and $\tilde{\mathcal{G}}\mathcal{P}$ contains all modules with finite projective dimension.

We now consider the functor $\text{Hom}_R(-, -): \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{A}$, together with the categories

$$\mathcal{X} = \mathcal{G}\mathcal{P}, \tilde{\mathcal{X}} = \tilde{\mathcal{G}}\mathcal{P} \quad \text{and} \quad \mathcal{Y} = \mathcal{G}\mathcal{I}, \tilde{\mathcal{Y}} = \tilde{\mathcal{G}}\mathcal{I}.$$

In this case we define, in the sense of section 2.4,

$$\text{Ext}_{\mathcal{G}\mathcal{P}}^n(-, N) = \text{R}_{\mathcal{G}\mathcal{P}}^n \text{Hom}_R(-, N) \quad \text{and} \quad \text{Ext}_{\mathcal{G}\mathcal{I}}^n(M, -) = \text{R}_{\mathcal{G}\mathcal{I}}^n \text{Hom}_R(M, -),$$

for fixed R -modules M and N . We wish, of course, to apply Theorem 2.6 to this situation. Note that by [12, Proposition 2.18], we have:

Proposition 3.1. *If M is an R -module with $\text{Gpd}_R M < \infty$, then there exists a short exact sequence $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$, where $G \rightarrow M$ is a $\mathcal{G}\mathcal{P}$ -precover of M (please see Remark 2.3), and $\text{pd}_R K = \text{Gpd}_R M - 1$ (in the case where M is Gorenstein projective, this should be interpreted as $K = 0$).*

Consequently, every R -module with finite Gorenstein projective dimension has a proper left $\mathcal{G}\mathcal{P}$ -resolution (that is, there is an inclusion $\tilde{\mathcal{G}}\mathcal{P} \subseteq \text{LeftRes}_{\mathcal{M}}(\mathcal{G}\mathcal{P})$).

Furthermore, we will need the following from [12, Theorem 2.13]:

Theorem 3.2. *Let M be any R -module with $\text{Gpd}_R M < \infty$. Then*

$$\text{Gpd}_R M = \sup\{n \geq 0 \mid \text{Ext}_R^n(M, L) \neq 0 \text{ for some } R\text{-module } L \text{ with } \text{pd}_R L < \infty\}.$$

Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an R -module M is given by

$$\text{pd}_R M = \{n \geq 0 \mid \text{Ext}_R^n(M, L) \neq 0 \text{ for some } R\text{-module } L\}.$$

It also follows that if $\text{pd}_R M < \infty$, then every projective resolution of M is actually a proper left \mathcal{GP} -resolution of M .

Lemma 3.4. *Assume that M is an R -module with finite Gorenstein projective dimension, and let $\mathbf{G}^+ = \dots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ be an augmented proper left \mathcal{GP} -resolution of M (which exists by Proposition 3.1). Then $\text{Hom}_R(\mathbf{G}^+, H)$ is exact for all Gorenstein injective modules H .*

Proof. We split the proper resolution \mathbf{G}^+ into short exact sequences. Hence it suffices to show exactness of $\text{Hom}_R(\mathbf{S}, H)$ for all Gorenstein injective modules H and all short exact sequences

$$\mathbf{S} = 0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0,$$

where $G \rightarrow M$ is a \mathcal{GP} -precover of some module M with $\text{Gpd}_R M < \infty$ (recall that \mathcal{GP} -precovers are always surjective). By Proposition 3.1, there is a special short exact sequence,

$$\mathbf{S}' = 0 \longrightarrow K' \xrightarrow{\iota} G' \xrightarrow{\pi} M \longrightarrow 0,$$

where $\pi: G' \rightarrow M$ is a \mathcal{GP} -precover and $\text{pd}_R K' < \infty$.

It is easy to see (as in Proposition 2.2) that the complexes \mathbf{S} and \mathbf{S}' are homotopy equivalent, and thus so are the complexes $\text{Hom}_R(\mathbf{S}, H)$ and $\text{Hom}_R(\mathbf{S}', H)$ for every (Gorenstein injective) module H . Hence it suffices to show the exactness of $\text{Hom}_R(\mathbf{S}', H)$ whenever H is Gorenstein injective.

Now let H be any Gorenstein injective module. We need to prove the exactness of

$$\text{Hom}_R(G', H) \xrightarrow{\text{Hom}_R(\iota, H)} \text{Hom}_R(K', H) \longrightarrow 0.$$

To see this, let $\alpha: K' \rightarrow H$ be any homomorphism. We wish to find $\varrho: G' \rightarrow H$ such that $\varrho\iota = \alpha$. Now pick an exact sequence

$$0 \longrightarrow \tilde{H} \longrightarrow E \xrightarrow{g} H \longrightarrow 0,$$

where E is injective, and \tilde{H} is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines H). Since \tilde{H} is Gorenstein injective and $\text{pd}_R K' < \infty$, we get $\text{Ext}_R^1(K', \tilde{H}) = 0$ by [7, Lemma 1.3], and thus a lifting $\varepsilon: K' \rightarrow E$ with $g\varepsilon = \alpha$:

$$\begin{array}{ccc} & K' & \xrightarrow{\iota} G' \\ & \swarrow \alpha & \downarrow \varepsilon \\ H & \xleftarrow{g} E & \end{array} \quad \begin{array}{c} \nearrow \tilde{\varepsilon} \\ \downarrow \varepsilon \end{array}$$

Next, injectivity of E gives $\tilde{\varepsilon}: G' \rightarrow E$ with $\tilde{\varepsilon}\iota = \varepsilon$. Now $\varrho = g\tilde{\varepsilon}: G' \rightarrow H$ is the desired map. □

With a similar proof we get:

Lemma 3.5. *Assume that N is an R -module with finite Gorenstein injective dimension, and let $\mathbf{H}^+ = 0 \rightarrow N \rightarrow H^0 \rightarrow H^1 \rightarrow \dots$ be an augmented proper right \mathcal{GI} -resolution of N (which exists by the dual of Proposition 3.1). Then $\text{Hom}_R(G, \mathbf{H}^+)$ is exact for all Gorenstein projective modules G . \square*

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. *For all R -modules M and N with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$, we have isomorphisms*

$$\text{Ext}_{\mathcal{GP}}^n(M, N) \cong \text{Ext}_{\mathcal{GI}}^n(M, N),$$

which are functorial in M and N . \square

3.7 (Definition of GExt). Let M and N be R -modules with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. Then we write

$$\text{GExt}_R^n(M, N) := \text{Ext}_{\mathcal{GP}}^n(M, N) \cong \text{Ext}_{\mathcal{GI}}^n(M, N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare GExt with the classical Ext. This is done in:

Theorem 3.8. *Let M and N be any R -modules. Then the following conclusions hold:*

(i) *There are natural isomorphisms $\text{Ext}_{\mathcal{GP}}^n(M, N) \cong \text{Ext}_R^n(M, N)$ under each of the conditions*

$$(\dagger) \text{ pd}_R M < \infty \quad \text{or} \quad (\ddagger) M \in \text{LeftRes}_{\mathcal{M}}(\mathcal{GP}) \text{ and } \text{id}_R N < \infty.$$

(ii) *There are natural isomorphisms $\text{Ext}_{\mathcal{GI}}^n(M, N) \cong \text{Ext}_R^n(M, N)$ under each of the conditions*

$$(\dagger) \text{ id}_R N < \infty \quad \text{or} \quad (\ddagger) N \in \text{RightRes}_{\mathcal{M}}(\mathcal{GI}) \text{ and } \text{pd}_R M < \infty.$$

(iii) *Assume that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. If either $\text{pd}_R M < \infty$ or $\text{id}_R N < \infty$, then*

$$\text{GExt}_R^n(M, N) \cong \text{Ext}_R^n(M, N)$$

is functorial in M and N .

Proof. (i) Assume that $\text{pd}_R M < \infty$, and pick any projective resolution \mathbf{P} of M . By Remark 3.3, \mathbf{P} is also a proper left \mathcal{GP} -resolution of M , and thus

$$\text{Ext}_{\mathcal{GP}}^n(M, N) = \text{H}^n(\text{Hom}_R(\mathbf{P}, N)) = \text{Ext}_R^n(M, N).$$

In the case where $M \in \text{LeftRes}_{\mathcal{M}}(\mathcal{GP})$ and $\text{id}_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\text{Hom}_R(-, N)$, that is, $\text{Ext}_R^i(G, N) = 0$ (the usual Ext) for every Gorenstein projective module G , and every integer $i > 0$.

This is because, if G is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \rightarrow G \rightarrow Q^0 \rightarrow \dots \rightarrow Q^{m-1} \rightarrow C \rightarrow 0$, where Q^0, \dots, Q^{m-1} are projective modules. Breaking this exact sequence into short exact ones, and applying $\text{Hom}_R(-, N)$, we get $\text{Ext}_R^i(G, N) \cong \text{Ext}_R^{m+i}(C, N) = 0$, as claimed.

Therefore [11, Chapter III, Proposition 1.2A] implies that $\text{Ext}_R^n(-, N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of $\text{GExt}_R^n(-, -)$. \square

4. GORENSTEIN DERIVING $-\otimes_R-$

In dealing with the tensor product we need, of course, both left and right R -modules. Thus the following addition to Notation 1.1 is needed:

If \mathcal{C} is any of the categories in Notation 1.1 ($\mathcal{M}, \mathcal{GP}$, etc.), we write ${}_R\mathcal{C}$, respectively, \mathcal{C}_R , for the category of left, respectively, right, R -modules with the property describing the modules in \mathcal{C} .

Now we consider the functor $-\otimes_R-: \mathcal{M}_R \times {}_R\mathcal{M} \rightarrow \mathcal{A}$. For fixed $M \in \mathcal{M}_R$ and $N \in {}_R\mathcal{M}$ we define, in the sense of section 2.4:

$$\text{Tor}_n^{\mathcal{GP}_R}(-, N) := L_n^{\mathcal{GP}_R}(-\otimes_R N) \quad \text{and} \quad \text{Tor}_n^{{}_R\mathcal{GP}}(M, -) := L_n^{{}_R\mathcal{GP}}(M \otimes_R -),$$

together with

$$\text{Tor}_n^{\mathcal{GF}_R}(-, N) := L_n^{\mathcal{GF}_R}(-\otimes_R N) \quad \text{and} \quad \text{Tor}_n^{{}_R\mathcal{GF}}(M, -) := L_n^{{}_R\mathcal{GF}}(M \otimes_R -).$$

The first two Tors use proper left Gorenstein *projective* resolutions, and the last two Tors use proper left Gorenstein *flat* resolutions. In order to compare these different Tors, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of

$$(\mathcal{X}, \tilde{\mathcal{X}}) = (\mathcal{GP}_R, \tilde{\mathcal{GP}}_R) \quad \text{or} \quad (\mathcal{GF}_R, \tilde{\mathcal{GF}}_R),$$

and

$$(\mathcal{Y}, \tilde{\mathcal{Y}}) = ({}_R\mathcal{GP}, {}_R\tilde{\mathcal{GP}}) \quad \text{or} \quad ({}_R\mathcal{GF}, {}_R\tilde{\mathcal{GF}}),$$

namely, the covariant-covariant version of Theorem 2.6, instead of the stated contra-variant-covariant version. We will need the classical notion:

Definition 4.1. The *left finitistic projective dimension* $\text{LeftFPD}(R)$ of R is defined as

$$\text{LeftFPD}(R) = \sup\{\text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty\}.$$

The right finitistic projective dimension $\text{RightFPD}(R)$ of R is defined similarly.

Remark 4.2. When R is commutative and Noetherian, the dimensions $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ coincide and are equal to the Krull dimension of R , by [10, Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12, Proposition 3.3], [12, Theorem 3.5] and [12, Proposition 3.18], respectively:

Proposition 4.3. *If R is right coherent with finite $\text{LeftFPD}(R)$, then every Gorenstein projective left R -module is also Gorenstein flat. That is, there is an inclusion ${}_R\mathcal{GP} \subseteq {}_R\mathcal{GF}$. \square*

Theorem 4.4. *For any left R -module M , we consider the following three conditions:*

- (i) *The left R -module M is G -flat.*
- (ii) *The Pontryagin dual $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ (which is a right R -module) is G -injective.*
- (iii) *M has an augmented proper right resolution $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$ consisting of flat left R -modules, and $\text{Tor}_i^R(I, M) = 0$ for all injective right R -modules I , and all $i > 0$.*

The implication (i) \Rightarrow (ii) always holds. If R is right coherent, then also (ii) \Rightarrow (iii) \Rightarrow (i), and hence all three conditions are equivalent. \square

Proposition 4.5. *Assume that R is right coherent. If M is a left R -module with $\text{Gfd}_R M < \infty$, then there exists a short exact sequence $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$, where $G \rightarrow M$ is an ${}_R\mathcal{GF}$ -precover of M , and $\text{fd}_R K = \text{Gfd}_R M - 1$ (in the case where M is Gorenstein flat, this should be interpreted as $K = 0$).*

In particular, every left R -module with finite Gorenstein flat dimension has a proper left ${}_R\mathcal{GF}$ -resolution (that is, there is an inclusion ${}_R\widetilde{\mathcal{GF}} \subseteq \text{LeftRes}_R \mathcal{M}({}_R\mathcal{GF})$). \square

Our first result is:

Lemma 4.6. *Let M be a left R -module with $\text{Gpd}_R M < \infty$, and let $\mathbf{G}^+ = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ be an augmented proper left ${}_R\mathcal{GP}$ -resolution of M (which exists by Proposition 3.1). Then the following conclusions hold:*

- (i) $T \otimes_R \mathbf{G}^+$ is exact for all Gorenstein flat right R -modules T .
- (ii) If R is left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R \mathbf{G}^+$ is exact for all Gorenstein projective right R -modules T .

Proof. (i) By Theorem 4.4 above, the Pontryagin dual $H = \text{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left R -module. Hence $\text{Hom}_R(\mathbf{G}^+, H) \cong \text{Hom}_{\mathbb{Z}}(T \otimes_R \mathbf{G}^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition 3.4. Since \mathbb{Q}/\mathbb{Z} is a faithfully injective \mathbb{Z} -module, $T \otimes_R \mathbf{G}^+$ is exact too.

(ii) With the given assumptions on R , the dual of Proposition 4.3 implies that every Gorenstein projective right R -module also is Gorenstein flat. \square

Lemma 4.7. *Assume that R is right coherent with finite $\text{LeftFPD}(R)$. Let M be a left R -module with $\text{Gfd}_R M < \infty$, and let $\mathbf{G}^+ = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ be an augmented proper left ${}_R\mathcal{GF}$ -resolution of M (which exists by Proposition 4.5, since R is right coherent). Then the following conclusions hold:*

- (i) $\text{Hom}_R(\mathbf{G}^+, H)$ is exact for all Gorenstein injective left R -modules H .
- (ii) $T \otimes_R \mathbf{G}^+$ is exact for all Gorenstein flat right R -modules T .
- (iii) If R is also left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R \mathbf{G}^+$ is exact for all Gorenstein projective right R -modules T .

Proof. (i) Since $\text{Gfd}_R M < \infty$ and R is right coherent, Proposition 4.5 gives a special short exact sequence $0 \rightarrow K' \rightarrow G' \rightarrow M \rightarrow 0$, where $G' \rightarrow M$ is an ${}_R\mathcal{GF}$ -precover of M , and $\text{fd}_R K' < \infty$. Since R has $\text{LeftFPD}(R) < \infty$, [13, Proposition 6] implies that also $\text{pd}_R K' < \infty$. Now the proof of Lemma 3.4 applies.

(ii) If T is a Gorenstein flat right R -module, then the left R -module $H = \text{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (i), just proved, we have exactness of

$$\text{Hom}_R(\mathbf{G}^+, H) \cong \text{Hom}_{\mathbb{Z}}(T \otimes_R \mathbf{G}^+, \mathbb{Q}/\mathbb{Z}).$$

Since \mathbb{Q}/\mathbb{Z} is a faithfully injective \mathbb{Z} -module, we also have exactness of $T \otimes_R \mathbf{G}^+$, as desired.

(iii) Under the extra assumptions on R , the dual of Proposition 4.3 implies that every Gorenstein projective right R -module is also Gorenstein flat. Thus (iii) follows from (ii). \square

Theorem 4.8. *Assume that R is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right R -module M , and every left R -module N , the following conclusions hold:*

(i) If $\text{Gfd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then

$$\text{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \text{Tor}_n^{R\mathcal{GF}}(M, N).$$

(ii) If $\text{Gpd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then

$$\text{Tor}_n^{\mathcal{GP}_R}(M, N) \cong \text{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \text{Tor}_n^{R\mathcal{GF}}(M, N).$$

(iii) If $\text{Gfd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then

$$\text{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \text{Tor}_n^{R\mathcal{GP}}(M, N) \cong \text{Tor}_n^{R\mathcal{GF}}(M, N).$$

(iv) If $\text{Gpd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then

$$\text{Tor}_n^{\mathcal{GP}_R}(M, N) \cong \text{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \text{Tor}_n^{R\mathcal{GP}}(M, N) \cong \text{Tor}_n^{R\mathcal{GF}}(M, N).$$

All the isomorphisms are functorial in M and N .

Proof. Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6. \square

4.9 (Definition of $g\text{Tor}$ and GTor). Assume that R is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let M be a right R -module, and let N be a left R -module. If $\text{Gfd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then we write

$$g\text{Tor}_n^R(M, N) := \text{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \text{Tor}_n^{R\mathcal{GF}}(M, N)$$

for the isomorphic abelian groups in Theorem 4.8(i). If $\text{Gpd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then we write

$$\text{GTor}_n^R(M, N) := \text{Tor}_n^{\mathcal{GP}_R}(M, N) \cong \text{Tor}_n^{R\mathcal{GP}}(M, N)$$

for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8:

Theorem 4.10. *Assume that R is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right R -module M with finite $\text{Gpd}_R M$, and for every left R -module N with $\text{Gpd}_R N < \infty$, we have isomorphisms:*

$$g\text{Tor}_n^R(M, N) \cong \text{GTor}_n^R(M, N)$$

that are functorial in M and N .

Finally we compare $g\text{Tor}$ (and hence GTor) with the usual Tor .

Theorem 4.11. *Assume that R is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let M be a right R -module with $\text{Gfd}_R M < \infty$, and let N be a left R -module with $\text{Gfd}_R N < \infty$. If either $\text{fd}_R M < \infty$ or $\text{fd}_R N < \infty$, then there are isomorphisms*

$$g\text{Tor}_n^R(M, N) \cong \text{Tor}_n^R(M, N)$$

that are functorial in M and N .

Proof. If $\text{fd}_R M < \infty$, then we also have $\text{pd}_R M < \infty$ by [13, Proposition 6] (since $\text{RightFPD}(R) < \infty$). Let \mathbf{P} be any projective resolution of M . As noted in Remark 3.3, \mathbf{P} is also a proper left \mathcal{GP}_R -resolution of M . Hence, Theorem 4.8(ii) and the definitions give:

$$g\text{Tor}_n^R(M, N) = \text{Tor}_n^{\mathcal{GP}_R}(M, N) = \text{H}_n(\mathbf{P} \otimes_R N) = \text{Tor}_n^R(M, N),$$

as desired. \square

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