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# ON CLEAVABILITY OF CONTINUA OVER LOTS

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Abstract. It is shown that any continuum cleavable over a LOTS $L$ is embeddable into $L$.

## 1. Introduction

This work is devoted to cleavability over LOTS (linearly ordered topological spaces). The general concept of cleavability was introduced by A. V. Arhangelskii (see [Ar2]). Comprehensive surveys on this subject can be found in [Ar5] and [Ar6].

A space $X$ is *cleavable* over a space $Y$ *along* $A \subseteq X$ if there exists a continuous mapping $f$ from $X$ to $Y$ such that $f(A) \cap f(X \setminus A) = \emptyset$. A space $X$ is *cleavable* over a space $Y$ if $X$ is cleavable over $Y$ along each $A \subseteq X$. A space $X$ is said to *condense* into a space $Y$ if there exists a one-to-one continuous mapping of $X$ into $Y$.

The following are the main problems in this field.

**Problem 1.1.** *When does cleavability of $X$ over $Y$ imply that $X$ embeds (condenses) into $Y$?*

**Problem 1.2.** *Which properties of $Y$ are possessed by every space cleavable over $Y$?*

Interesting results related to these problems are obtained in [Ar2], [Ar3], [Ar4], [Bal], [Bu], [Tka], and [Yas]. In this article it will be shown that any continuum cleavable over a LOTS $L$ is embeddable into $L$. (Throughout the paper, all spaces are assumed to be Hausdorff, and by a continuum we mean a connected compact topological space, not necessarily linearly ordered.) Connectedness of a compactum is needed in our theorem, as shown in the following example.

**Example 1.3.** Let $X, Y$ be three-point and two-point discrete compacta, respectively. It is easy to see that $Y$ is a LOTS and $X$ is cleavable over $Y$. (To cleave $X$ over $Y$ along $A$, just collapse two points that are both in $A$ or its complement.) Obviously, $X$ is not embeddable in $Y$, since $Y$ lacks one extra point. However, $X$ is a LOTS!

Thus, the following question remains of interest.

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**Question 1.4** (A. V. Arhangelskii)**.** *Let $X$ be a compactum cleavable over a LOTS. Is $X$ a LOTS?*

In [Ar3], Arhangelskii answered this question for cleavability over metric spaces. He proved that any compactum cleavable over a metric LOTS is a metric LOTS itself.

For a space $X$ cleavable over a LOTS, a set $A \subset X$ is called *absolute for $X$* if any continuous mapping that cleaves $X$ over a LOTS along $A$ is one-to-one. Let $X$ be a LOTS. The character of $x \in X$ from the right, denoted $\chi_r(x, X)$, is the smallest cardinal number $\tau$ such that there exists a set $\{x_\alpha : x_\alpha > x, \alpha < \tau\}$ whose closure contains $x$. Similarly, we define the character from the left, $\chi_l(x, X)$. A *$\tau$-sequence* is a LOTS homeomorphic to the space of all ordinal numbers less than $\tau$ with the topology of linear order. Note that our definition of $\tau$-sequence is non-standard. An interval of a LOTS is a set in the form $[a, b], (a, b), [a, b)$, or $(a, b]$, where $a \leq b$. If $a = b$, an interval is called *trivial.* Let $f$ be a continuous mapping of a LOTS $X$. We say that elements $x, y \in X$ are *collapsed (glued up)* under $f$ if $f(x) = f(y)$. If $I$ is an interval in $X$ and $f(I)$ is a single-element set, we say that $I$ is *squeezed* into a point under $f$. We will use the following simple known statements.

**Statement 1.5.** *If $X$ is a compact LOTS and $\chi_l(x, X) = \tau$, then there exists a $\tau$-sequence converging to $x$ from the left, and $cf(\tau) = \tau$.*

**Statement 1.6.** *If $cf(\tau) > \omega_0$ and $s_1, s_2$ are two $\tau$-sequences converging to $x$ from the left, then $s_1 \cap s_2 \neq \emptyset$.*

We will also use the following obvious fact about absolute sets.

**Statement 1.7.** *If $X$ is a connected infinite LOTS and $A$ is an absolute set for $X$, then $A$ is dense in $X$.*

*Proof.* If $A$ is not dense, then there exists a nontrivial closed interval $I \subset X \setminus A$. Squeeze $I$ into a point. This operation defines a continuous mapping $f$ of $X$ onto a LOTS. Obviously, $f$ cleaves $X$ along $A$, but $f$ is not one-to-one, since $I$ is infinite. $\square$

**Statement 1.8.** *If $X = [a, b]$ is a connected infinite LOTS, then there exists a countable $Y \subset X$ such that $|\overline{Y}| = 2^{\omega_0}$.*

*Proof.* The argument is the same as in the classical construction of the Cantor set. To avoid repetition, let us do only the first two steps. Take arbitrary distinct $y_{1/3}, y_{2/3} \in (a, b)$. Remove $(y_{1/3}, y_{2/3})$ from $[a, b]$. What is left consists of the two components $[a, y_{1/3}]$ and $[y_{2/3}, b]$. Now take arbitrary distinct $y_{1/9}, y_{2/9} \in (a, y_{1/3})$ and $y_{7/9}, y_{8/9} \in (y_{2/3}, b)$. Again remove central thirds and repeat the above-described process. The set of all end-points of the removed intervals is countable, and its closure has cardinality $2^{\omega_0}$. $\square$

## 2. SOME GENERAL FACTS

**Proposition 2.1.** *Let $X = [a, b]$ be a linearly ordered continuum. Let $f$ be a continuous mapping of $X$ onto a LOTS. Suppose that no element in $(a, b)$ collapses with $a$ under $f$. Then, $f(a)$ is an end-point of $f([a, b])$.*

*Proof.* Assume the contrary. Then there exist $x, y \in (a, b]$ such that $f(x)$ and $f(y)$ are the two end-points of $f(X)$. Due to connectedness of $[x, y]$, $f([x, y]) = f(X)$.

Therefore, there exists $z \in (x, y)$ such that $f(z) = f(a)$, which contradicts the proposition's assumptions. □

**Proposition 2.2.** *Let $X$ be a connected LOTS. Let $f$ be a continuous non-one-to-one mapping of $X$ onto a LOTS. Then:*

*(1) There exist two nontrivial closed disjoint intervals in $X$ that have the same images under $f$.*

*(2) Suppose $S$ is a nowhere-dense set in $X$. Then there exist two nontrivial closed disjoint intervals in $X$ that do not intersect $S$ and have the same images under $f$.*

*Proof.* (1) Let $c$ be a point in $X$ that collapses with another point under $f$. Let $C$ be the set of all elements in $X$ that have the same image as $c$. If $C$ contains a nontrivial open interval, then we are done. Otherwise, there exist distinct $a, b \in C$ such that no element in $(a, b)$ collapses with $c$. Then, by Proposition 2.1, $f(a)$ is an end-point of $f([a, b])$ (suppose it is the left end-point). Take $d \in (a, b)$ such that $f(d)$ is the other end-point of $f([a, b])$. Due to connectedness, $f([a, b]) = f([a, d]) = f([d, b])$. Now, take an arbitrary $z \in (f(a), f(d))$. Let $a_1 \in (a, d)$ be the closest point to $a$ such that $f(a_1) = z$ and let $b_1 \in (d, b)$ be the closest point to $b$ such that $f(b_1) = z$. Then, by Proposition 2.1, $f([a, a_1]) = f([b_1, b])$.

(2) If $[x, y] \subseteq X \setminus S$ and $f$ is not a homeomorphism on $[x, y]$, then (1) for $X = [x, y]$ gives the required intervals. Thus, we can assume now that $f$ is a homeomorphism on any interval outside of $S$. By (1) there exist two disjoint nontrivial closed intervals $I, J$ such that $f(I) = f(J)$. Take a nontrivial $[a, b] \subseteq I \setminus S$. Now, take a nontrivial $[c, d] \subseteq f^{-1}(f([a, b])) \cap J$ that does not intersect $S$. Thus, $f([c, d]) \subseteq f([a, b])$. Since $f$ is a homeomorphism on both $[a, b]$ and $[c, d]$, there exists $[a', b'] \subseteq [a, b]$ such that $f([a', b']) = f([c, d])$. □

**Proposition 2.3.** *Let $f$ be a continuous mapping of a connected LOTS $X$ onto a LOTS. Let $\chi_l(x, X) = \tau$. If no nontrivial interval is squeezed into a point under $f$, then either $\chi_l(f(x), f(X)) = \tau$ or $\chi_r(f(x), f(X)) = \tau$.*

*Proof.* Since no nontrivial interval is squeezed into a point, there exists a strictly increasing sequence $\{x_\alpha\}$ of cardinality $\tau$ that converges to $x$ from the left such that $f(x_\alpha) < f(x)$ for each $x_\alpha$ (or $f(x_\alpha) > f(x)$). We will consider only the case when $f(x_\alpha) < f(x)$ for each $x_\alpha$. Then, due to continuity of $f$, $\chi_l(f(x), f(X)) \leq \tau$. Suppose $\chi_l(f(x), f(X)) = \lambda < \tau$. Then there exists a strictly decreasing family of closed intervals $\mathcal{J} = \{J_\alpha : f(x)$ is the right end-point of $J_\alpha, \alpha < \lambda\}$ such that $\bigcap \mathcal{J} = \{f(x)\}$. Since $\lambda < cf(\tau)$, $\bigcap f^{-1}(\mathcal{J})$ contains a nontrivial interval of the form $(y, x)$. This interval, by the construction of $\mathcal{J}$, must have the same image as $x$, which contradicts the proposition's assumptions. □

**Proposition 2.4.** *Let $f$ be a continuous mapping of a connected LOTS $X$ onto a LOTS. Suppose $x \in X$ is not an end-point of $X$ and $\chi_l(x, X) = \tau > \chi_r(x, X) = \lambda$. If $f$ squeezes no nontrivial interval into a point, then either $\chi_l(f(x), f(X)) = \tau > \chi_r(f(x), f(X)) = \lambda$ or $\chi_l(f(x), f(X)) = \lambda < \chi_r(f(x), f(X)) = \tau$.*

*Proof.* Apply Proposition 2.3 twice. □

**Proposition 2.5.** *Let $X = [a, b]$ be a linearly ordered continuum with $\chi(a, X) = \tau$ and $\chi(b, X) = \lambda > \tau$. Let $f$ be a continuous mapping of $X$ onto a LOTS such* *that $f(a) = f(b)$. Let $f(a)$ be an end-point of $f(X)$. Then there exists a nontrivial interval in $X$ whose image under $f$ is a single-element set.*

*Proof.* Assume that no nontrivial interval is squeezed into a point. Then, by Proposition 2.3, $\chi(f(a), f(X)) = \tau < \chi(f(b), f(X)) = \lambda$, which contradicts $f(a) = f(b)$. $\square$

**Proposition 2.6.** *Let $f$ be a continuous mapping of a connected LOTS $X$ onto a LOTS. Let $I = [a, b), J = (c, d]$ be two disjoint intervals such that $f|_I, f|_J$ are embeddings, and $J$ is to the right of $I$. Suppose $f(x) = f(y)$ for $x \in I$ and $y \in J$, and no element in $(x, y)$ collapses with $x$ under $f$. If $f$ preserves (reverses) the order of $I$, then $f$ reverses (preserves) the order of $J$.*

*Proof.* By Proposition 2.1, $f(x)$ is an end-point of $f([x, y])$. If the order of $I$ is preserved, $f(x)$ is the left end-point of $f([x, y])$. Since $J$ is open from the left, there exists $z \in J$ such that $x < z < y$. Since $f(y)$ is the left end-point of $f([x, y])$, $f(z) > f(y)$. Since $f|_J$ is an embedding and switches the order of two elements, it must reverse all of $J$ (apply connectedness of $J$).

If the order of $I$ is reversed, apply the above arguments but interchange "left" by "right", "preserves" by "reverses", etc. $\square$

**Proposition 2.7.** *Let $[a, b]$ be a connected LOTS. If $(a, b)$ is first countable, then $\chi_r(a, [a, b]) \leq \omega_1$ and $\chi_l(b, [a, b]) \leq \omega_1$.*

*Proof.* If $\chi_r(a, [a, b]) = \tau > \omega_1$, then there exists a $\tau$-sequence converging to $a$ from the right. This sequence contains an $\omega_1$-subsequence converging to some element in $(a, b)$, which contradicts first-countability of $(a, b)$. $\square$

**Proposition 2.8.** *Let $X = [a, b]$ be a linearly ordered continuum. Let $f$ be a continuous mapping of $X$ onto a LOTS such that $f(a) = f(b)$. Let $c, d$ be elements in $X$ whose images are the two end-points of $f(X)$. Then for any $x \in X \setminus \{c, d\}$ there exists $y \in X \setminus \{x\}$ such that $f(x) = f(y)$.*

*Proof.* We may assume that $|f(X)| > 1$ and $c < d$. Since $f(a) = f(b)$, $f([a, c]) \cup f([d, b]) = f(X)$. But, $f([c, d]) = f(X)$ as well. Thus, any element in $[a, c) \cup (d, b]$ collapses with some element in $[c, d]$. Also, any element in $(c, d)$ collapses with some element in $[a, c] \cup [d, b]$. Since $[a, c) \cup (c, d) \cup (d, b]$ covers all of $X$, except $\{c, d\}$, we are done. $\square$

**Proposition 2.9.** *Let $f$ be a continuous one-to-one mapping of a connected LOTS $X$ to a LOTS. Then $f$ is an embedding.*

*Proof.* Since $f$ is a one-to-one continuous mapping of a connected LOTS into a LOTS, $f$ is strictly monotonic. Since $f(X)$ is a connected subspace of a LOTS, it is a LOTS itself. Thus, $f$ is a strictly monotonic mapping of a LOTS $X$ onto a LOTS $f(X)$, and therefore it is a homeomorphism. To see this, notice that images and inverse images of intervals are intervals. $\square$

**Proposition 2.10.** *Let $f$ be a continuous mapping of a connected LOTS $X = [a, b]$ to a LOTS. If $f$ is one-to-one on $(a, b)$, then $f$ is an embedding on $[a, b]$ ($[a, b)$).*

*Proof.* By Proposition 2.2, $f$ is one-to-one on $[a, b]$ ($[a, b)$), and, by Proposition 2.9, $f$ is an embedding on $[a, b]$ ($[a, b)$). $\square$

## 3. Lemmas

In our first lemma, we will partially employ a technique developed by Tkachuk [Tka].

**Lemma 3.1.** *Let $X$ be a first-countable connected LOTS. Then:*

(1) *There exists an absolute set $A \subset X$ for $X$.*

(2) *For two disjoint nowhere dense sets $S_1, S_2 \subset X$, there exists an absolute set $A$ for $X$ that contains $S_1$ and does not intersect $S_2$.*

*Proof.* It is enough to prove (2). We may assume that $X$ is infinite (if $X$ is a single-element set, the empty set is absolute for $X$). First, notice that the cardinality of $X$ is $2^{\omega_0}$. Indeed, any connected LOTS is locally compact. By Proposition 2.7, $X$ can be covered by $\omega_1$ many first countable compacta. Now apply Arhangelskii's theorem [Ar1].

Let $\mathcal{J}$ be the set of all nonempty open intervals that have no common elements with $S_1 \cup S_2$. Enumerate $\mathcal{J} = \{J_\alpha : \alpha < 2^{\omega_0}\}$. Let $\mathcal{P}$ be the set of all possible pairs $(C, D)$, where $C$ and $D$ are countable sets in $X$ whose closures are disjoint and have no common elements with $S_1 \cup S_2$.

For each $(C, D) \in \mathcal{P}$, let $G_{C,D}$ be the set of all one-to-one mappings $g_{C,D}$ from $C$ onto $D$ that possess the following property.

*Property $g_{C,D}$. There exist $2^{\omega_0}$ mutually disjoint sets $\{c, d\}$, where $c \in \overline{C} \setminus C$ and $d \in \overline{D} \setminus D$, such that for some sequence $\{c_n : c_m \neq c_k$ for $m \neq k\} \subset C$ we have $c_n \to c$ and $d \in \overline{\{g_{C,D}(c_n)\}}$.*

Enumerate $G = \bigcup_{(C,D)\in\mathcal{P}} G_{C,D} = \{g_\alpha : \alpha < 2^{\omega_0}\}$. Inductively, we will construct two disjoint sets $A = S_1 \cup \{a_\alpha : \alpha < 2^{\omega_0}\} \cup \{a'_\alpha : \alpha < 2^{\omega_0}\}$ and $B = \{b_\alpha : \alpha < 2^{\omega_0}\} \cup \{b'_\alpha : \alpha < 2^{\omega_0}\}$ with the following two properties.

$(*)$ *For every $g_\alpha = g_{C,D} \in G$, there exists a sequence $\{c_n : c_m \neq c_k$ for $m \neq k\} \subset C$ such that $c_n \to a_\alpha \notin C$ and $b_\alpha \in \overline{\{g_{C,D}(c_n)\}} \setminus D$.*

$(**)$ *For every $\alpha$, both $a'_\alpha$ and $b'_\alpha$ belong to $J_\alpha$.*

**Construction.** Assume $a_\beta, b_\beta$ are defined for $\beta < \alpha < 2^{\omega_0}$.

*Step $\alpha$.* Consider $g_\alpha = g_{C,D} \in G$. Let $a_\alpha$ and $b_\alpha$ be any two elements in $X \setminus (S_1 \cup S_2)$, distinct from all $a_\beta, a'_\beta$ and $b_\beta, b'_\beta$ for $\beta < \alpha$, that play the role of $c$ and $d$ in Property $g_{C,D}$. Also, take arbitrary distinct $a'_\alpha, b'_\alpha \in J_\alpha$ that are distinct from all $a_\beta, b_\beta$ for $\beta \leq \alpha$ and from all $a'_\beta, b'_\beta$ for $\beta < \alpha$. (This can be done because the set of already-chosen points has cardinality less than $2^{\omega_0}$.)

The set $A$ contains $S_1$ by the definition. To see why $A \cap S_2 = \emptyset$, notice that any element of $A$ is either in $S_1$, or $J_\alpha$ for some $\alpha$, or $\overline{C}$ for some $(C, D) \in \mathcal{P}$. Every $J_\alpha$ has no common elements with $S_1 \cup S_2$ by the definition. By the lemma's assumptions, $S_1$ and $S_2$ are disjoint. Also, by the definition of $\mathcal{P}$ neither $\overline{C}$ nor $\overline{D}$ has common elements with $S_2$.

So now we must show that $A$ is absolute for $X$. Let a continuous mapping $f$ cleave $X$ over a LOTS along $A$. Assume $f$ is not one-to-one. By Proposition 2.2, there exist two disjoint nontrivial closed intervals $I$ and $J$ that do not intersect $S_1 \cup S_2$ and whose images under $f$ coincide. We may assume that both $I$ and $J$ are compacta.

Notice that due to $(**)$, both $A$ and $X \setminus A$ are dense. Since $f(A) \cap f(X \setminus A) = \emptyset$, we have that $|f(I)| = 2^{\omega_0}$.

By Statement 1.8, there exists a countable subset $Y \subset f(I)$ with closure of cardinality $2^{\omega_0}$. Take countable $C \subset I$ and $D \subset J$ such that $f(C) = f(D) = Y$. We can also require that $f$ is one-to-one on both $C$ and $D$. Define a correspondence $g : C \to D$ by $g(c) = f^{-1}(f(c)) \cap D$. Since $f$ is one-to-one on both $C$ and $D$ and $f(C) = f(D) = Y$, $f$ is a well-defined one-to-one function of $C$ onto $D$.

Let us show that $g \in G$. Take any $c \in \overline{C} \setminus C$ such that $f(c) \notin Y$. Since $X$ is first countable, there exists $\{c_n : c_m \neq c_k \ for\ m \neq k\} \subset C$ such that $c_n \to c$. Consider $\{g(c_n)\}$. Since $f(g(c_n)) = f(c_n)$ and $f(c_n) \to f(c)$, there exists $d \in \overline{\{g(c_n)\}}$ such that $f(d) = f(c)$ (here we use compactness of $J$). Since $f(c) \notin Y$, we have that $d \in \overline{D} \setminus D$. To finish verification of Property $g_{C,D}$, notice that there are $2^{\omega_0}$ elements in $\overline{C}$ whose images are not in $Y$.

Thus, $g = g_\alpha \in G$ for some $\alpha$. Therefore, there exists a sequence $\{c_n : c_m \neq c_k \ for\ m \neq k\} \subset C$ such that $c_n \to a_\alpha \notin C$ and $b_\alpha \in \overline{\{g(c_n)\}} \setminus D$. By the definition of $g$, $f(b_\alpha) \in \overline{\{f(c_n)\}}$. Since $f$ is one-to-one on $D$, $f(b_\alpha)$ is an accumulation point of $\{f(c_n)\}$. But the latter set has only one accumulation point, namely $f(c)$, since $\{f(c_n)\}$ is a non-constant convergent sequence. Therefore, $f(a_\alpha) = f(b_\alpha)$, which contradicts the choice of $f$. This concludes the proof of Lemma 3.1. ☐

**Lemma 3.2.** *Let $X$ be a linearly ordered continuum with an open dense set of points of countable character. Then there exists an absolute set $A \subset X$ for $X$.*

*Proof.* We say that a set $O$ is a countable character component if it is a maximal open interval containing points only of countable character. Let $\mathcal{O} = \{O_\alpha\}$ be the maximal family of mutually disjoint countable character components. Obviously, $\bigcup \mathcal{O}$ is open and dense in $X$. For each $O_\alpha \in \mathcal{O}$, we perform the following steps 1 and 2.

1. Let $a_\alpha, b_\alpha$ be the left and right end-points of $\overline{O}_\alpha$. By Proposition 2.7, $\chi_r(a_\alpha, \overline{O}_\alpha)$ does not exceed $\omega_1$. If $\chi_r(a_\alpha, \overline{O}_\alpha) = \omega_1$, fix an $\omega_1$-sequence $r_\alpha \subset O_\alpha \setminus \{b_\alpha\}$ converging to $a_\alpha$ from the right. Similarly, if $\chi_l(b_\alpha, \overline{O}_\alpha) = \omega_1$, fix an $\omega_1$-sequence $l_\alpha \subset O_\alpha \setminus \overline{r}_\alpha$ converging to $b_\alpha$ from the left. If $\chi_r(a_\alpha, \overline{O}_\alpha) < \omega_1$, let $r_\alpha = \emptyset$. Similarly, if $\chi_l(b_\alpha, \overline{O}_\alpha) < \omega_1$, let $l_\alpha = \emptyset$.

2. Fix an absolute for $O_\alpha$ set $A_\alpha \subset O_\alpha$ that does not intersect $l_\alpha$ but contains $r_\alpha$ (Lemma 3.1).

Define a set $B$ as follows: $b \in B$ if and only if $b$ is the right end-point for some $O \in \mathcal{O}$, and is an accumulation point for the family $\mathcal{O}$ (i.e., a limit point of $\bigcup \mathcal{O}$ that is not a limit point of any single element of $\mathcal{O}$). Let us show that $A = \bigcup\{A_\alpha\} \cup B$ is an absolute set for $X$.

Let a continuous mapping $f$ cleave $X$ over a LOTS along $A$. Assume $f$ is not one-to-one. By the choice of $A$, $f|_O$ is one-to-one for any $O \in \mathcal{O}$. Also, by Proposition 2.10, $f|_{\overline{O}}$ is an embedding.

By Proposition 2.2 and density of $\bigcup \mathcal{O}$, there exists $x \in A_\alpha \subset O_\alpha \in \mathcal{O}$ that collapses with some element from the right. Let $y = \inf\{z \in X : z > x,\ f(z) = f(x)\}$. Since $O_\alpha$ is open and $f$ is one-to-one on $O_\alpha$, $y$ is strictly greater than $x$. By the choice of $f$, $y \in O_\beta \in \mathcal{O}$ or $y$ is the right end-point of $O_\beta$ for some $\beta \neq \alpha$. Assume $f$ preserves the order of $O_\alpha$. By Proposition 2.6, $f$ reverses $O_\beta$.

Since $f$ is one-to-one on each $O \in \mathcal{O}$, no nontrivial interval is squeezed into a point. Also, by Proposition 2.3, $f$ preserves the character of each non-end-point of $X$. Since any neighborhood of an end-point of $O_\alpha$ or $O_\beta$ contains points of uncountable character, $b_\alpha(a_\beta)$ cannot collapse with any element in $O_\beta(O_\alpha)$. Thus,

$f(b_\alpha) = f(a_\beta)$. If $b_\alpha$ is an accumulation point for the family $\mathcal{O}$ then, by the latter equality, $f(A)$ intersects $f(X \setminus A)$, which contradicts the choice of $f$ (recall that $a_\beta$ is the left end-point of $O_\beta$ and, therefore, cannot be an accumulation point for $\mathcal{O}$ and the right end-point for some $O$ at the same time). If $b_\alpha$ is not an accumulation point, then it is the boundary point of two elements in $\mathcal{O}$, and, due to maximality of each $O$, cannot be of countable character. Therefore, there exists an $\omega_1$-sequence (see Proposition 2.7) that converges to $b_\alpha$ from the right or left. Assume this sequence is $l_\alpha$. Since $f$ reverses $O_\beta$ and preserves characters, there exists an $\omega_1$-sequence $r_\beta$ that converges to $a_\beta$ from the right. Obviously, under $f$, images of these sequences intersect. But $l_\alpha$ is not in $A$ while $r_\beta$ is in $A$. A similar argument works if the $\omega_1$-sequence (some $r_\gamma$) converges to $b_\alpha$ from the right. This contradiction implies that $f$ is one-to-one. □

**Lemma 3.3.** *Let $X$ be a connected LOTS, and let $A$ be a dense set that does not contain end-points and possesses the following property.*

**(*)** *$\chi_l(x, X) < \chi_r(x, X)$ for all $x \in A$.*

*Then:*

(1) *The set $A$ is absolute for $X$.*

(2) *A set $A^+$ which is the union of $A$ and an arbitrary discrete-in-itself set of elements of countable character in $X$ is absolute for $X$ as well.*

*Proof.* It is enough to prove (2) only. Let a continuous mapping $f$ cleave $X$ over a LOTS along $A^+$. Assume $f$ is not one-to-one. First, notice that no nontrivial open interval $I$ collapses into a point. Indeed, there is a decreasing $\omega_0$-sequence converging to $x \in I$. Since $x$ does not satisfy **(*)**, it is not in $A$. The set of such $x$'s is dense, while $A^+ \setminus A$ is not. Therefore, each nontrivial open interval contains elements from both $A^+$ and its complement.

By Proposition 2.2, there are two disjoint nontrivial intervals in $X$ whose images coincide. Since $A$ is dense, there exists $a \in A$ that is glued up under $f$ with an element to the right of $a$. Let $b = \inf\{x \in X : x > a,\ f(x) = f(a)\}$. Then $b \geq a$.

If $b = a$, then there exists a decreasing $\tau$-sequence $\{x_\alpha : f(x_\alpha) = f(a), \alpha < \tau\}$ that converges to $a$ from the right and $\tau > \omega_0$ (see **(*)**). Since $\tau > \omega_0$, the $\tau$-sequence contains an $\omega_0$-sequence $\{y_1 = x_{\omega_0}, y_2 = x_{\omega_0+\omega_0}, ...\}$ that converges to $y = x_{\omega_0^2}$. Because of right-cofinality, no term $y_n$ can satisfy **(*)**, and, therefore, is not in $A$. For the same reason, $y \notin A$ either. Since $A^+ \setminus A$ is discrete in itself, it cannot contain all $y_n$'s and $y$. So, either some $y_n$ or $y$ does not belong to $A^+$. But that is impossible, because $f(y_n) = f(y) = f(a)$ and $f$ cleaves $X$ along $A^+$.

If $b > a$, then, due to the choice of $f$, $b \in A^+$. By Proposition 2.1, $f(b)$ is an end-point of $f([a, b])$. Also, by Proposition 2.5, $\chi_r(a, X) = \chi_l(b, X)$. Then, $b \in A$. By Proposition 2.4, $\chi_l(a, X) = \chi_r(b, X)$. As a result of these two equalities, $\chi_l(b, X) > \chi_r(b, X)$, which contradicts **(*)**. □

**Lemma 3.4.** *Let $X$ be a connected LOTS with the following properties:*

1. *There exists an open dense $O \subseteq X$ such that $\chi_l(x, X) = \chi_r(x, X)$ for all $x \in O$.*

2. *The set of elements of uncountable character is dense in $X$.*

*Then:*

(1) *There exists a set $A \subset O$ with the following properties:*

A1. *The set of all $x \in A$ with $\chi_l(x, X) = \chi_r(x, X) = \omega_1$ is dense in $X$.*

A2. *All $x$ in $A$ are of character either $\omega_0$ or $\omega_1$.*

A3. *For each $x \in A$ of character $\omega_1$ there exist an $\omega_1$-sequence $r_x \subset A$ converging to $x$ from the right and an $\omega_1$-sequence $l_x \subset X \setminus A$ converging to $x$ from the left.*

(2) *The set $A$ in* (1) *is absolute for $X$.*

*Proof.* (1) First, notice that any nonempty open interval $I$ in $X$ contains an $x$ with $\chi_l(x, X) = \chi_r(x, X) = \omega_1$ ($\omega_0$). Indeed, due to property 2 and density of $O$, $I$ contains a $\tau$-sequence ($\tau > \omega_0$) whose closure is in $O \cap I$. This sequence contains an $\omega_1(\omega_0)$-subsequence converging to some $x \in I \cap O$ from the right (left). Then $\chi_r(x, X) = \omega_1(\omega_0)$. Due to the definition of $O$ we have the required equality.

Let $a, b$ be the end-points of a linear ordered compactification of $X$. Inductively, we will define the sets $A_\alpha$ and $B_\alpha$ such that $A = \bigcup A_\alpha$ and the set $B = \bigcup B_\alpha$ contains all the $l_x$'s defined in property $A3$.

Suppose $A_\beta$ and $B_\beta$ are defined for all $\beta < \alpha$.

*For each pair $\{c, d\}$, where $c < d$, satisfying the following requirements* 1, 2:

1. $\{c, d\} \in (\overline{\bigcup_{\beta<\alpha}(A_\beta \cup B_\beta)} \cup \{a, b\})$,

2. $(c, d) \cap (\overline{\bigcup_{\beta<\alpha}(A_\beta \cup B_\beta)} \cup \{a, b\}) = \emptyset$ (that is, $c, d$ are neighbors),

*pick up an arbitrary point $x \in (c, d) \cap O$ of character $\omega_1$. Choose two disjoint $\omega_1$-sequences $r_x, l_x \subset (c, d) \cap O$ converging to $x$ from the right and left, respectively, and consisting of points of countable character.*

If no such $c, d$ exist, construction stops.

Let $A_\alpha = (\bigcup_{\beta<\alpha} A_\beta) \cup (\bigcup\{\text{all } \overline{r_x}\})$ and $B_\alpha = (\bigcup_{\beta<\alpha} B_\beta) \cup (\bigcup\{\text{all } l_x\})$.

Let $A = \bigcup A_\alpha$ and $B = \bigcup B_\alpha$. Properties $A2, A3$ as well as the property of $B$ follow from the construction. Let us show that $A$ possesses property $A1$.

Take an arbitrary nonempty open interval $J$. Suppose, $\overline{J \cap (A \cup B)} = \emptyset$. Take $c \in \overline{A \cup B \cup \{a\}}$ the closest to $J$ from the left and $d \in \overline{A \cup B \cup \{b\}}$ the closest to $J$ from the right. The pair $\{c, d\}$ satisfies the construction requirements, and therefore, construction must continue. Thus, $A \cup B$ is dense. During construction, from each interval we pick up not more than a nowhere dense set of elements for $A_\alpha$ and $B_\alpha$. Once we pick up at least two points from $J$, at the next step there exists a pair $\{c, d\}$ satisfying the construction requirements. At that step we pick up an $x \in (c, d)$ for $A$ with $\chi_l(x, X) = \chi_r(x, X) = \omega_1$.

(2) Now, let us show that $A$ is absolute for $X$. Let $f$ cleave $X$ over a LOTS along $A$. Assume $f$ is not one-to-one. Due to $A1$ and Proposition 2.2, there exists $x \in A$ with $\chi_l(x, X) = \chi_r(x, X) = \omega_1$ that collapses with some element from its left side. Let $y$ be the closest point from the left of $x$ that has the same image as $x$.

If no such $y$ exists, then some $\omega_1$-sequence converging to $x$ from the left, and, therefore, the most part of $l_x$ are squeezed into $f(x)$. But $l_x$ is not in $A$, while $x$ is.

If $y$ exists, then $y \in A$ (by the choice of $f$). Suppose $y$ is of countable character. By Proposition 2.1, $f(x)$ is an end-point of $f([y, x])$. Then, by Proposition 2.5, there exists an interval that is squeezed into a point under $f$. But any interval contains elements from both $A$ and its complement (this follows from the definition of $B$ and property $A1$). If $y$ is of character $\omega_1$, then the images of $r_y$ and $l_x$ intersect (recall that $r_y$, $l_x$ are both $\omega_1$-sequences). But $r_y \subset A$ while $l_x \subset B \subset X \setminus A$. Since all the possibilities bring us to contradictions, we conclude that $f$ is one-to-one. $\square$

## 4. THEOREM

**Theorem 4.1.** *Let $X$ be a linearly ordered continuum cleavable over a LOTS $L$. Then, $X$ is embeddable to $L$.*

*Proof.* Let us define families $\mathcal{B}, \mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ as follows. *Family* $\mathcal{B}$ consists of all maximal nontrivial closed intervals in $X$ containing an open dense set of elements of countable character in $X$. *Family* $\mathcal{C}$ consists of all maximal open intervals in $X$ containing a dense set of elements $x$ with $\chi_l(x, X) > \chi_r(x, X)$. *Family* $\mathcal{D}$ consists of all maximal open intervals in $X \setminus \overline{\bigcup \mathcal{C}}$ containing a dense set of elements $x$ with $\chi_l(x, X) < \chi_r(x, X)$.

Family $\mathcal{D}$ possesses the following properties:

*1. For each $D \in \mathcal{D}$, the set of all elements $x$ with $\chi_l(x, X) > \chi_r(x, X)$ is nowhere-dense in $D$.*

This follows from the maximality of Family $\mathcal{C}$.

*2. Each $D \in \mathcal{D}$ contains a dense set of elements of countable character.*

Indeed, take any open interval and a decreasing countable sequence inside the interval. Let the sequence converge to $x$. Due to property 1, we can pick up a sequence in such a manner that $\chi_l(x, X) \leq \chi_r(x, X) = \omega_0$.

*Family* $\mathcal{E}$ consists of all maximal open intervals in $X \setminus \overline{\bigcup(\mathcal{B} \cup \mathcal{C} \cup \mathcal{D})}$.

Family $\mathcal{E}$ possesses the following property.

*Each $E \in \mathcal{E}$ contains an open dense set of elements $x$ with $\chi_l(x, X) = \chi_r(x, X)$, and a dense set of elements of uncountable character.*

This follows from the maximality of the above families.

Let $\mathcal{S} = \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E}$. First, notice that elements in $\mathcal{S}$ are disjoint due to maximality of each family, and their union is dense in $X$. Also, the closures of two elements from the same family are disjoint.

For each element $B \in \mathcal{B}$, $C \in \mathcal{C}$, and $E \in \mathcal{E}$ fix absolute sets $A_B, A_C, A_E$ in accordance with Lemma 3.2, Lemma 3.3 (1), and Lemma 3.4 (2).

For each $D \in \mathcal{D}$, fix an absolute set $A_D$ from Lemma 3.3 (2), so that $A_D$ contains a discrete-in-itself set of points of countable character accumulating to endpoints of $\overline{D}$. This can be done, thanks to property 2 of family $\mathcal{D}$.

Let $A = \bigcup\{A_S : S \in \mathcal{S}\}$. Let us show that any continuous mapping $f$ that cleaves $X$ over $L$ along $A$ is an embedding.

By the choice of each $A_S$, $f|_S$ is one-to-one for each $S \in \mathcal{S}$. Also, by Proposition 2.10, $f|_{\overline{S}}$ is an embedding.

Suppose $f$ is not one-to-one. Then there exist two sets $S_1, S_2$ whose images intersect. This follows from the density of $\bigcup \mathcal{S}$ and Proposition 2.2.

We can assume that $S_1$ is not from $\mathcal{B}$. Indeed, let $x \in B \in \mathcal{B}$ collapse with some element. Take $y$ from the right (left) of $x$ that has the same image as $x$. By Proposition 2.8, any element in $[x, y]$, except maybe two points, is glued up with some other element. Since $f|_B$ is one-to-one, $[x, y] \setminus B$ is a nontrivial interval. Since no two elements of $\mathcal{B}$ are neighbors, there exists $S_1 \in \mathcal{S} \setminus \mathcal{B}$ such that $S_1 \cap (x, y) \neq \emptyset$ (recall that each $S \in \mathcal{S} \setminus \mathcal{B}$ is open). Fix this $S_1$.

Without loss of generality, $f$ preserves the order of $S_1$. There exists $S_2$ whose image intersects the image of $S_1$ by a nontrivial interval and $f$ reverses the order of $S_2$. Indeed, let $x \in A_{S_1}$ collapse with some element (such an $x$ exists, since $A_{S_1}$ is dense in $S_1$). Let $y$ be the closest element to $x$ from the right (left) such that $f(x) = f(y)$ (such a $y$ exists, since $S_1$ is open and $f$ is one-to-one on $S_1$). By the choice of $f$, $y$ belongs to some $S_2 \in \mathcal{S}$. Since $S_1$ is open, $f(S_1) \cap f(S_2)$ contains an interval. This means that $S_2$ cannot be from $\mathcal{B}$, since no interval of $S_1$ is first countable. Thus, $S_2 \in \mathcal{S} \setminus \mathcal{B}$. By Proposition 2.6, $f$ reverses $S_2$. Fix this $S_2$.

Now, we need only to show that the rest of the possible choices for $S_1$ and $S_2$ lead to contradictions.

**Case 1**. $S_1 \in \mathcal{E}$, $S_2 \in \mathcal{C} \cup \mathcal{D}$. As mentioned earlier, $f|_{S_1}$ and $f|_{S_2}$ are embeddings. But no open interval of $S_1$ is homeomorphic to any open part of $S_2$. This follows from the definitions of the families.

**Case 2**. $S_1, S_2 \in \mathcal{C}$. Due to density of the absolute set, there exist $a_1 \in A_{S_1}$ and $a_2 \in A_{S_2}$ with $f(a_1) = f(a_2)$. Since $f$ preserves the order of $S_1$, $\chi_l(f(a_1), f(X)) > \chi_r(f(a_1), f(X))$. Since the order of $S_2$ is reversed, $\chi_l(f(a_2), f(X)) < \chi_r(f(a_2), f(X))$. These two inequalities contradict $f(a_1) = f(a_2)$. (For the inequalities, see Lemma 3.3, (1).)

**Case 3**. $S_1, S_2 \in \mathcal{D}$. The argument is similar to Case 2.

**Case 4**. $S_1, S_2 \in \mathcal{E}$. Due to density of the absolute set, there exist $a_1 \in A_{S_1}$ and $a_2 \in A_{S_2}$ such that $f(a_1) = f(a_2)$ and $\chi_l(f(a_i), f(X)) = \chi_r(f(a_i), f(X)) = \omega_1$ (see Lemma 3.4, (1), Property A1). Suppose $a_1 < a_2$. Since the order of $S_1$ is preserved while that of $S_2$ is reversed, $f(r_{a_2}) \cap f(l_{a_1}) \neq \emptyset$ (recall that $r_{a_2}, l_{a_1}$ are $\omega_1$-sequences). But $r_{a_2}$ is in $A$ while $l_{a_1}$ is in the complement of $A$, which contradicts the choice of $f$ (see Lemma 3.4, (1), Property A3).

**Case 5**. $S_1 \in \mathcal{D}$, $S_2 \in \mathcal{C}$. Assume $S_2$ is to the right of $S_1$. By the choice of $S_1, S_2$, there exist $s_1 \in S_1$ and $s_2 \in S_2$ such that $f(s_1) = f(s_2)$. By Proposition 2.8, any element in $[s_1, s_2]$, except maybe two points, collapses with some other element. Since the right end-point of $\overline{S_1}$ is in $[s_1, s_2]$, there exists $a_1 \in A_{S_1} \cap [s_1, s_2]$ of countable character that collapses with some other element (see the definition of $A_D$). Take $a_2$ the closest to $a_1$ from the right (left) such that $f(a_1) = f(a_2)$. Such an $a_2$ exists, since $S_1$ is open and $f$ is an embedding on $S_1$. By the choice of $f$, $a_2$ must belong to $A_{S_3}$ for some $S_3 \in \mathcal{S}$. By Proposition 2.6, $f$ reverses $S_3$. Since $a_1$ is of countable character, $a_2$ must be of countable character as well. Therefore, $S_3$ is not in $\mathcal{C}$. The other cases have already been eliminated.

Thus, $f$ is one-to-one, and, therefore, an embedding (recall that $f$ is from a compactum). ☐

**Theorem 4.2.** *Any continuum $X$ cleavable over a LOTS $L$ is embeddable into $L$.*

*Proof.* By the theorem in [Bu], any continuum cleavable over a LOTS is a LOTS itself. Now, apply Theorem 4.1. ☐

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