

ON EMBEDDINGS OF FULL AMALGAMATED FREE PRODUCT C^* -ALGEBRAS

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ABSTRACT. We examine the question of when the $*$ -homomorphism $\lambda : A *_D B \rightarrow \tilde{A} *_{\tilde{D}} \tilde{B}$ of full amalgamated free product C^* -algebras, arising from compatible inclusions of C^* -algebras $A \subseteq \tilde{A}$, $B \subseteq \tilde{B}$ and $D \subseteq \tilde{D}$, is an embedding. Results giving sufficient conditions for λ to be injective, as well as classes of examples where λ fails to be injective, are obtained. As an application, we give necessary and sufficient conditions for the full amalgamated free product of finite-dimensional C^* -algebras to be residually finite dimensional.

1. INTRODUCTION

Given C^* -algebras A , B and D with injective $*$ -homomorphisms $\phi_A : D \rightarrow A$ and $\phi_B : D \rightarrow B$, the corresponding full amalgamated free product C^* -algebra (see [1] or [9, Chapter 5]) is the C^* -algebra \mathfrak{A} , equipped with injective $*$ -homomorphisms $\sigma_A : A \rightarrow \mathfrak{A}$ and $\sigma_B : B \rightarrow \mathfrak{A}$ such that $\sigma_A \circ \phi_A = \sigma_B \circ \phi_B$, such that \mathfrak{A} is generated by $\sigma_A(A) \cup \sigma_B(B)$ and satisfying the universal property that whenever \mathfrak{C} is a C^* -algebra and $\pi_A : A \rightarrow \mathfrak{C}$ and $\pi_B : B \rightarrow \mathfrak{C}$ are $*$ -homomorphisms satisfying $\pi_A \circ \phi_A = \pi_B \circ \phi_B$, there is a $*$ -homomorphism $\pi : \mathfrak{A} \rightarrow \mathfrak{C}$ such that $\pi \circ \sigma_A = \pi_A$ and $\pi \circ \sigma_B = \pi_B$. This situation is illustrated by the following commutative diagram:

(1)

$$\begin{array}{ccccc}
 & & D & & \\
 & \swarrow \phi_A & & \searrow \phi_B & \\
 A & \xrightarrow{\sigma_A} & \mathfrak{A} & \xleftarrow{\sigma_B} & B \\
 & \pi_A \downarrow & | \pi & \downarrow \pi_B & \\
 & & \mathfrak{C} & &
 \end{array}$$

The full amalgamated free product C^* -algebra \mathfrak{A} is commonly denoted by $A *_D B$, although this notation hides the dependence of \mathfrak{A} on the embeddings ϕ_A and ϕ_B .

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Question 1.1. Let $D, A, B, \tilde{D}, \tilde{A}$ and \tilde{B} be C^* -algebras and suppose there are injective $*$ -homomorphisms making the following diagram commute:

$$\begin{array}{ccccc} & & \tilde{A} & \xleftarrow{\phi_{\tilde{A}}} & \tilde{D} \\ & \lambda_A \uparrow & & & \downarrow \lambda_D \\ A & \xleftarrow{\phi_A} & D & \hookrightarrow & B \\ & & \lambda_B \uparrow & & \\ & & \tilde{B} & \xrightarrow{\phi_{\tilde{B}}} & \end{array}$$

Let $A *_D B$ and $\tilde{A} *_{\tilde{D}} \tilde{B}$ be the corresponding full amalgamated free product C^* -algebras, and let $\lambda : A *_D B \rightarrow \tilde{A} *_{\tilde{D}} \tilde{B}$ be the $*$ -homomorphism arising from λ_A and λ_B via the universal property. When is λ injective?

We prove in §2 that λ is injective when either (i) $D = \tilde{D}$ (or more precisely, when the $*$ -homomorphism λ_D is surjective), or (ii) there are conditional expectations $E_A : \tilde{A} \rightarrow A$ and $E_B : \tilde{B} \rightarrow B$ that send \tilde{D} onto D and agree on \tilde{D} . Injectivity in the case $D = \tilde{D}$ was previously proved by G. K. Pedersen [10]. (Moreover, earlier results of F. Boca [4] imply that the map λ is injective when $D = \tilde{D}$ and when there are conditional expectations

$$\tilde{A} \xrightarrow{E_{\tilde{A}}} A \xrightarrow{E_D} D \xleftarrow{E_{\tilde{D}}} B \xleftarrow{E_{\tilde{B}}} \tilde{B};$$

an argument for the case $D = \tilde{D} = \mathbf{C}$, which uses Boca's results, is outlined in [3, 4.7].) However, we include our proof because it is different from that found in [10] and because it contains the main idea of our proof of injectivity in case (ii). In §3, we consider some general conditions and give some concrete examples when λ fails to be injective. Finally, in §4, we apply this embedding result to extend a result from [5] about residual finite-dimensionality of full amalgamated free products of finite-dimensional C^* -algebras.

2. EMBEDDINGS OF FULL FREE PRODUCTS

The following result is of course well known. We include a proof for completeness.

Lemma 2.1. *Let A be a C^* -subalgebra of a C^* -algebra \tilde{A} and let $\pi : A \rightarrow B(\mathcal{H})$ be a $*$ -representation. Then there is a Hilbert space \mathcal{K} and a $*$ -representation $\tilde{\pi} : \tilde{A} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$ such that*

$$(2) \quad \tilde{\pi}(a)(h \oplus 0) = (\pi(a)h) \oplus 0 \quad (a \in A, h \in \mathcal{H}).$$

Proof. Since in general π is a direct sum of cyclic representations, we may without loss of generality assume that π is a cyclic representation with cyclic vector ξ . Let ϕ be the vector state $\phi(\cdot) = \langle \pi(\cdot)\xi, \xi \rangle$ of A . Then \mathcal{H} is identified with $L^2(A, \phi)$ and π is the associated GNS representation. Let $\tilde{\phi}$ be an extension of ϕ to a state of \tilde{A} , and let $\tilde{\mathcal{H}} = L^2(\tilde{A}, \tilde{\phi})$. Then the inclusion $A \hookrightarrow \tilde{A}$ gives rise to an isometry $\mathcal{H} \rightarrow \tilde{\mathcal{H}}$, and we may thus write $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{K}$ for a Hilbert space \mathcal{K} . If $\tilde{\pi} : \tilde{A} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$ is the GNS representation associated to $\tilde{\phi}$, then (2) holds. \square

The following result was first proved by G. K. Pedersen [10, Thm. 4.2]. We offer a new proof, which is perhaps more elementary. This proof contains essentially the same idea as our proof of Proposition 2.4 below.

Proposition 2.2. *Let*

$$\tilde{A} \supseteq A \supseteq D \subseteq B \subseteq \tilde{B}$$

be inclusions of C^* -algebras and let $A *_D B$ and $\tilde{A} *_D \tilde{B}$ be the corresponding full amalgamated free product C^* -algebras. Let $\lambda : A *_D B \rightarrow \tilde{A} *_D \tilde{B}$ be the $*$ -homomorphism arising via the universal property from the inclusions $A \hookrightarrow \tilde{A}$ and $B \hookrightarrow \tilde{B}$. Then λ is injective.

Proof. Let $\pi : A *_D B \rightarrow B(\mathcal{H})$ be a faithful $*$ -homomorphism. We will find a Hilbert space \mathcal{K} and a $*$ -homomorphism $\tilde{\pi} : \tilde{A} *_D \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$ such that

$$(3) \quad \tilde{\pi}(\lambda(x))(h \oplus 0) = (\pi(x)h) \oplus 0 \quad (x \in A *_D B, h \in \mathcal{H}).$$

This will imply that λ is injective.

Let $\pi_A : A \rightarrow B(\mathcal{H})$ and $\pi_B : B \rightarrow B(\mathcal{H})$ be the $*$ -representations obtained by composing π with the inclusions $A \hookrightarrow A *_D B$ and $B \hookrightarrow A *_D B$. Let

$$\begin{aligned} \sigma_{A,0} &: \tilde{A} \rightarrow B(\mathcal{H} \oplus \mathcal{K}_{A,0}), \\ \sigma_{B,0} &: \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K}_{B,0}) \end{aligned}$$

be $*$ -representations obtained from Lemma 2.1 such that

$$\sigma_{A,0}(a)(h \oplus 0) = (\pi_A(a)h) \oplus 0 \quad (a \in A, h \in \mathcal{H}),$$

and similarly with A replaced by B . Note that $0 \oplus \mathcal{K}_{A,0}$ is reducing for $\sigma_{A,0}(D)$. Using Lemma 2.1, we find Hilbert spaces $\mathcal{K}_{B,1}$ and $\mathcal{K}_{A,1}$ and $*$ -representations

$$\begin{aligned} \sigma_{B,1} &: \tilde{B} \rightarrow B(\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1}), \\ \sigma_{A,1} &: \tilde{A} \rightarrow B(\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1}) \end{aligned}$$

such that

$$\begin{aligned} \sigma_{B,1}(d)(k \oplus 0) &= \sigma_{A,0}(d)(0 \oplus k) \quad (d \in D, k \in \mathcal{K}_{A,0}), \\ \sigma_{A,1}(d)(k \oplus 0) &= \sigma_{B,0}(d)(0 \oplus k) \quad (d \in D, k \in \mathcal{K}_{B,0}). \end{aligned}$$

Proceeding recursively, for every integer $n \geq 2$ we find $*$ -representations

$$\begin{aligned} \sigma_{B,n} &: \tilde{B} \rightarrow B(\mathcal{K}_{A,n-1} \oplus \mathcal{K}_{B,n}), \\ \sigma_{A,n} &: \tilde{A} \rightarrow B(\mathcal{K}_{B,n-1} \oplus \mathcal{K}_{A,n}) \end{aligned}$$

such that

$$\begin{aligned} \sigma_{B,n}(d)(k \oplus 0) &= \sigma_{A,n-1}(d)(0 \oplus k) \quad (d \in D, k \in \mathcal{K}_{A,n-1}), \\ \sigma_{A,n}(d)(k \oplus 0) &= \sigma_{B,n-1}(d)(0 \oplus k) \quad (d \in D, k \in \mathcal{K}_{B,n-1}). \end{aligned}$$

We now define the Hilbert spaces

$$(4) \quad \begin{aligned} \tilde{\mathcal{H}}_A &= \overbrace{\mathcal{H} \oplus \mathcal{K}_{A,0}}^{\sigma_{A,0}} \oplus \overbrace{\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1}}^{\sigma_{A,1}} \oplus \overbrace{\mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2}}^{\sigma_{A,2}} \oplus \cdots, \\ \tilde{\mathcal{H}}_B &= \underbrace{\mathcal{H} \oplus \mathcal{K}_{B,0}}_{\sigma_{B,0}} \oplus \underbrace{\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1}}_{\sigma_{B,1}} \oplus \underbrace{\mathcal{K}_{A,1} \oplus \mathcal{K}_{B,2}}_{\sigma_{B,2}} \oplus \cdots, \end{aligned}$$

where the brackets indicate where the constructed representations act, and we let $\sigma_{\tilde{A}} : \tilde{A} \rightarrow B(\tilde{\mathcal{H}}_A)$ and $\sigma_{\tilde{B}} : \tilde{B} \rightarrow B(\tilde{\mathcal{H}}_B)$ be the $*$ -representations

$$\begin{aligned} \sigma_{\tilde{A}} &= \sigma_{A,0} \oplus \sigma_{A,1} \oplus \sigma_{A,2} \oplus \cdots, \\ \sigma_{\tilde{B}} &= \sigma_{B,0} \oplus \sigma_{B,1} \oplus \sigma_{B,2} \oplus \cdots, \end{aligned}$$

where the summands act as indicated by brackets in (4). Consider the unitary $U : \tilde{\mathcal{H}}_A \rightarrow \tilde{\mathcal{H}}_B$ mapping the summands in $\tilde{\mathcal{H}}_A$ identically to the corresponding summands in $\tilde{\mathcal{H}}_B$ as indicated by the arrows below:

$$\begin{array}{ccccccc} \tilde{\mathcal{H}}_A & = & \mathcal{H} & \oplus & \mathcal{K}_{A,0} & \oplus & \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2} \oplus \cdots \\ U \downarrow & & \downarrow & & \searrow & & \searrow & & \searrow \\ \tilde{\mathcal{H}}_B & = & \mathcal{H} & \oplus & \mathcal{K}_{B,0} & \oplus & \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,2} \oplus \cdots. \end{array}$$

Let $\mathcal{K} = \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \cdots$ and identify $\mathcal{H} \oplus \mathcal{K}$ with $\tilde{\mathcal{H}}_A$. Then we have the $*$ -representations $\tilde{\pi}_{\tilde{A}} = \sigma_{\tilde{A}} : \tilde{A} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$ and $\tilde{\pi}_{\tilde{B}} : \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$, the latter defined by $\tilde{\pi}_{\tilde{B}}(\cdot) = U^* \sigma_{\tilde{B}}(\cdot) U$. By construction, the restrictions of $\tilde{\pi}_{\tilde{A}}$ and $\tilde{\pi}_{\tilde{B}}$ to D agree, and we have

$$\begin{aligned} \tilde{\pi}_{\tilde{A}}(a)(h \oplus 0) &= (\pi_A(a)h) \oplus 0 & (a \in A, h \in \mathcal{H}), \\ \tilde{\pi}_{\tilde{B}}(b)(h \oplus 0) &= (\pi_B(b)h) \oplus 0 & (b \in B, h \in \mathcal{H}). \end{aligned}$$

Letting $\tilde{\pi} : \tilde{A} *_D \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$ be the $*$ -homomorphism obtained from $\tilde{\pi}_{\tilde{A}}$ and $\tilde{\pi}_{\tilde{B}}$ via the universal property, we have that (3) holds. \square

For a C^* -algebra A , unital or not, let A^u denote the unitization of A . Thus, as a vector space, $A^u = A \oplus \mathbf{C}$ with multiplication defined by $(a, \mu) \cdot (a', \mu') = (aa' + \mu a + \mu' a, \mu\mu')$. We identify A with the ideal $A \oplus 0$ of A^u , which has codimension 1.

Lemma 2.3. *Let $A \supseteq D \subseteq B$ be inclusions of C^* -algebras. Consider the unitizations and corresponding inclusions*

$$\begin{array}{ccccc} A^u & \xleftarrow{\quad} & D^u & \xrightarrow{\quad} & B^u \\ \uparrow & & \uparrow & & \uparrow \\ A & \xleftarrow{\quad} & D & \xrightarrow{\quad} & B. \end{array}$$

Let $\lambda : A *_D B \rightarrow A^u *_D B^u$ be the resulting $*$ -homomorphism between full amalgamated free products. Then there is an isomorphism $\pi : A^u *_D B^u \rightarrow (A *_D B)^u$ such that $\pi \circ \lambda : A *_D B \rightarrow (A *_D B)^u$ is the canonical embedding arising in the definition of the unitization.

Proof. Since any $*$ -representations of A and B that agree on D extend to $*$ -representations of A^u and B^u that agree on D^u , the $*$ -homomorphism λ is injective. Let $e \in A^u *_D B^u$ be the unit of A^u , which is of course identified with the units of B^u and D^u . Clearly, $A^u *_D B^u$ is generated by the image of λ together with e . One easily sees that

$$(\lambda(x) + \mu e)(\lambda(x') + \mu' e) = \lambda(xx') + \mu\lambda(x') + \mu'\lambda(x) + \mu\mu'e.$$

Moreover, if $\rho : A^u *_D B^u \rightarrow \mathbf{C}$ is the $*$ -homomorphism arising from the unital $*$ -homomorphisms $A^u \rightarrow \mathbf{C}$ and $B^u \rightarrow \mathbf{C}$, then $\rho(e) = 1$ and $\lambda(A *_D B) \subseteq \ker \rho$. Hence $\lambda(A *_D B)$ has codimension 1 in $A^u *_D B^u$. Now π can be defined by $\pi(\lambda(x) + \mu e) = (x, \mu)$. \square

Proposition 2.4. *Suppose*

$$(5) \quad \begin{array}{ccccc} & \tilde{A} & \hookleftarrow & \tilde{D} & \hookrightarrow \tilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array}$$

is a commuting diagram of inclusions of C^* -algebras. Let $\lambda : A *_D B \rightarrow \tilde{A} *_{\tilde{D}} \tilde{B}$ be the resulting $*$ -homomorphism of full free product C^* -algebras. Suppose there are conditional expectations $E_A : \tilde{A} \rightarrow A$, $E_D : \tilde{D} \rightarrow D$ and $E_B : \tilde{B} \rightarrow B$ onto A , D and B , respectively, such that the diagram

$$(6) \quad \begin{array}{ccccc} & \tilde{A} & \hookleftarrow & \tilde{D} & \hookrightarrow \tilde{B} \\ \downarrow E_A & & \downarrow E_D & & \downarrow E_B \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array}$$

commutes. Then λ is injective.

Proof. By appealing to Lemma 2.3, we may without loss of generality assume that all the algebras and $*$ -homomorphisms in (5) are unital. Let $\pi : A *_D B \rightarrow B(\mathcal{H})$ be a faithful, unital $*$ -representation. As in the proof of Proposition 2.2, in order to show λ is injective, we will find a Hilbert space \mathcal{K} and a $*$ -homomorphism $\tilde{\pi} : \tilde{A} *_{\tilde{D}} \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$ such that

$$(7) \quad \tilde{\pi}(\lambda(x))(h \oplus 0) = (\pi(x)h) \oplus 0 \quad (x \in A *_D B, h \in \mathcal{H}).$$

Let $\pi_A : A \rightarrow B(\mathcal{H})$ and $\pi_B : B \rightarrow B(\mathcal{H})$ be the $*$ -representations obtained by composing π with the inclusions $A \hookrightarrow A *_D B$ and $B \hookrightarrow A *_D B$, and let $\pi_D : D \rightarrow B(\mathcal{H})$ be their common restriction to D . Consider the canonical left action of \tilde{D} on the right Hilbert D -module $L^2(\tilde{D}, E_D)$, which is obtained from \tilde{D} by separation and completion with respect to the D -valued inner product $\langle \tilde{d}_1, \tilde{d}_2 \rangle = E_D(\tilde{d}_1^* \tilde{d}_2)$. Consider the Hilbert space $L^2(\tilde{D}, E_D) \otimes_D \mathcal{H}$, where the left action of D on \mathcal{H} is via π_D . Since π_D is unital, \mathcal{H} embeds as a subspace, and we can write

$$(8) \quad L^2(\tilde{D}, E_D) \otimes_D \mathcal{H} = \mathcal{H} \oplus \mathcal{K}_D.$$

Consider the left action of \tilde{D} on the Hilbert space $\mathcal{H} \oplus \mathcal{K}_D$. The subspace \mathcal{H} is reducing for the restriction of σ_D to D , and we have $\sigma_D(d)(h \oplus 0) = (\pi_D(d)h) \oplus 0$ for every $d \in D$ and $h \in \mathcal{H}$.

In a similar way, consider the Hilbert spaces

$$(9) \quad L^2(\tilde{A}, E_A) \otimes_A \mathcal{H}, \quad L^2(\tilde{B}, E_B) \otimes_B \mathcal{H}$$

and the associated left actions $\sigma_{A,0}$ of \tilde{A} , respectively $\sigma_{B,0}$ of \tilde{B} . Since the diagram (6) commutes, the Hilbert space (8) embeds canonically as a subspace of both spaces (9). We may thus write

$$\begin{aligned} L^2(\tilde{A}, E_A) \otimes_A \mathcal{H} &= \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{A,0}, \\ L^2(\tilde{B}, E_B) \otimes_B \mathcal{H} &= \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{B,0}, \end{aligned}$$

the subspace $\mathcal{H} \oplus \mathcal{K}_D \oplus 0$ is reducing for the restrictions of $\sigma_{A,0}$ and $\sigma_{B,0}$ to \tilde{D} , and we have $\sigma_{A,0}(\tilde{d})(\eta \oplus 0) = (\sigma_D(\tilde{d})\eta) \oplus 0 = \sigma_{B,0}(\tilde{d})(\eta \oplus 0)$ for every $\tilde{d} \in \tilde{D}$ and

$\eta \in \mathcal{H} \oplus \mathcal{K}_D$. Moreover, $\mathcal{H} \oplus 0 \oplus 0$ is reducing for the restrictions of $\sigma_{A,0}$ to A and $\sigma_{B,0}$ to B , and we have

$$\begin{aligned}\sigma_{A,0}(a)(h \oplus 0 \oplus 0) &= (\pi_A(a)h) \oplus 0 \oplus 0 & (a \in A, h \in \mathcal{H}), \\ \sigma_{B,0}(b)(h \oplus 0 \oplus 0) &= (\pi_B(b)h) \oplus 0 \oplus 0 & (b \in B, h \in \mathcal{H}).\end{aligned}$$

Let $\sigma_{A,0,\tilde{D}}$ denote the action of \tilde{D} on $\mathcal{K}_{A,0}$ obtained by restricting $\sigma_{A,0}$ to \tilde{D} and compressing, and similarly for $\sigma_{B,0,\tilde{D}}$.

We now proceed recursively as in the proof of Proposition 2.2. If Hilbert spaces $\mathcal{K}_{A,n-1}$ and $\mathcal{K}_{B,n-1}$ have been constructed with actions $\sigma_{A,n-1,\tilde{D}}$ and $\sigma_{B,n-1,\tilde{D}}$, respectively, of \tilde{D} , then use Lemma 2.1 to construct Hilbert spaces $\mathcal{K}_{B,n}$ and $\mathcal{K}_{A,n}$ and $*$ -homomorphisms

$$\begin{aligned}\sigma_{B,n} : \tilde{B} &\rightarrow B(\mathcal{K}_{A,n-1} \oplus \mathcal{K}_{B,n}), \\ \sigma_{A,n} : \tilde{A} &\rightarrow B(\mathcal{K}_{B,n-1} \oplus \mathcal{K}_{A,n}),\end{aligned}$$

such that

$$\begin{aligned}\sigma_{B,n}(\tilde{d})(k \oplus 0) &= (\sigma_{A,n-1,\tilde{D}}(\tilde{d})k) \oplus 0 & (\tilde{d} \in \tilde{D}, k \in \mathcal{K}_{A,n-1}), \\ \sigma_{A,n}(\tilde{d})(k \oplus 0) &= (\sigma_{B,n-1,\tilde{D}}(\tilde{d})k) \oplus 0 & (\tilde{d} \in \tilde{D}, k \in \mathcal{K}_{B,n-1}).\end{aligned}$$

Then let $\sigma_{B,n,\tilde{D}}$ be the action of \tilde{D} on $\mathcal{K}_{B,n}$ obtained from the restriction of $\sigma_{B,n}$ to \tilde{D} by compressing, and similarly define the action $\sigma_{A,n,\tilde{D}}$ of \tilde{D} on $\mathcal{K}_{A,n}$.

We may now define the Hilbert spaces

$$(10) \quad \begin{aligned}\widetilde{\mathcal{H}}_A &= \overbrace{\mathcal{H} \oplus \mathcal{K}_D}^{\sigma_D} \oplus \mathcal{K}_{A,0} \oplus \overbrace{\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1}}^{\sigma_{A,1}} \oplus \overbrace{\mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2}}^{\sigma_{A,2}} \oplus \cdots, \\ \widetilde{\mathcal{H}}_B &= \underbrace{\mathcal{H} \oplus \mathcal{K}_D}_{\sigma_{B,0}} \oplus \mathcal{K}_{B,0} \oplus \underbrace{\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1}}_{\sigma_{B,1}} \oplus \underbrace{\mathcal{K}_{A,1} \oplus \mathcal{K}_{B,2}}_{\sigma_{B,2}} \oplus \cdots,\end{aligned}$$

where the brackets indicate where the constructed representations act. We let $\sigma_{\tilde{A}} : \tilde{A} \rightarrow B(\widetilde{\mathcal{H}}_A)$ and $\sigma_{\tilde{B}} : \tilde{B} \rightarrow B(\widetilde{\mathcal{H}}_B)$ be the $*$ -representations

$$\begin{aligned}\sigma_{\tilde{A}} &= \sigma_{A,0} \oplus \sigma_{A,1} \oplus \sigma_{A,2} \oplus \cdots, \\ \sigma_{\tilde{B}} &= \sigma_{B,0} \oplus \sigma_{B,1} \oplus \sigma_{B,2} \oplus \cdots,\end{aligned}$$

where the summands act as indicated by brackets in (10). Consider the unitary $U : \widetilde{\mathcal{H}}_A \rightarrow \widetilde{\mathcal{H}}_B$ mapping the summands in $\widetilde{\mathcal{H}}_A$ identically to the corresponding summands in $\widetilde{\mathcal{H}}_B$ as indicated by the arrows below:

$$\begin{array}{ccccccc} \widetilde{\mathcal{H}}_A & = & \mathcal{H} & \oplus & \mathcal{K}_D & \oplus & \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2} \oplus \cdots \\ U \downarrow & & \downarrow & & \downarrow & & \\ \widetilde{\mathcal{H}}_B & = & \mathcal{H} & \oplus & \mathcal{K}_D & \oplus & \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,2} \oplus \cdots \end{array}$$

Let $\mathcal{K} = \mathcal{K}_D \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \cdots$ and identify $\mathcal{H} \oplus \mathcal{K}$ with $\widetilde{\mathcal{H}}_A$. Then we have the $*$ -representations $\tilde{\pi}_{\tilde{A}} = \sigma_{\tilde{A}} : \tilde{A} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$ and $\tilde{\pi}_{\tilde{B}} : \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$,

the latter defined by $\tilde{\pi}_{\tilde{B}}(\cdot) = U^* \sigma_{\tilde{B}}(\cdot) U$. By construction, the restrictions of $\tilde{\pi}_{\tilde{A}}$ and $\tilde{\pi}_{\tilde{B}}$ to \tilde{D} agree, and we have

$$\begin{aligned}\tilde{\pi}_{\tilde{A}}(a)(h \oplus 0) &= (\pi_A(a)h) \oplus 0 & (a \in A, h \in \mathcal{H}), \\ \tilde{\pi}_{\tilde{B}}(b)(h \oplus 0) &= (\pi_B(b)h) \oplus 0 & (b \in B, h \in \mathcal{H}).\end{aligned}$$

Letting $\tilde{\pi} : \tilde{A} *_{\tilde{D}} \tilde{B} \rightarrow B(\mathcal{H} \oplus \mathcal{K})$ be the $*$ -homomorphism obtained from $\tilde{\pi}_{\tilde{A}}$ and $\tilde{\pi}_{\tilde{B}}$ via the universal property, we have that (7) holds. \square

3. EXAMPLES OF NON-EMBEDDING

In this section, we give some examples when the map λ of Question 1.1 fails to be injective. (In contrast, it is known [2] that in the more stringent situation of *reduced amalgamated free products*, the map analogous to λ is always injective.)

We begin with a trivial class of examples.

Examples 3.1. Let A and B be C^* -subalgebras of a C^* -algebra E with $A \not\subseteq B$ and $B \not\subseteq A$. Let $D = A \cap B$, $\tilde{A} = E$ and $\tilde{D} = \tilde{B} = B$, equipped with the natural inclusions. Then the map $\lambda : A *_D B \rightarrow \tilde{A} *_{\tilde{D}} \tilde{B} = E$ is injective if and only if $A *_D B$ is exactly the C^* -subalgebra of E generated by A and B . This does not hold in general. Notice that in these examples, $B \cap \tilde{D} = B \supsetneq D$.

Proposition 3.2. Suppose

$$\begin{array}{ccccc} & \tilde{A} & \hookleftarrow & \tilde{D} & \hookrightarrow \tilde{B} \\ \uparrow & & & \uparrow & \\ A & \hookleftarrow & \tilde{D} & \hookrightarrow & B \end{array}$$

is a commutative diagram of inclusions of C^* -algebras, and let $\lambda : A *_D B \rightarrow \tilde{A} *_{\tilde{D}} \tilde{B}$ be the resulting $*$ -homomorphism of full free product C^* -algebras. Suppose there are conditional expectations $E_D^A : A \rightarrow D$ and $E_D^B : B \rightarrow D$ with E_D^B faithful. Suppose there are $\tilde{d} \in \tilde{D}$, $a \in A$ and $b \in B$ satisfying $a\tilde{d} \in A$, $\tilde{d}b \in B$,

$$(11) \quad D(\tilde{d}b) \cap Db = \{0\},$$

$$(12) \quad E_D^A(\tilde{d}^* a^* ad)b \neq 0.$$

Then λ is not injective.

Proof. Letting

$$(13) \quad \begin{aligned}\sigma_A : A &\hookrightarrow A *_D B, & \sigma_B : B &\hookrightarrow A *_D B, \\ \sigma_{\tilde{A}} : \tilde{A} &\hookrightarrow \tilde{A} *_{\tilde{D}} \tilde{B}, & \sigma_{\tilde{B}} : \tilde{B} &\hookrightarrow \tilde{A} *_{\tilde{D}} \tilde{B}\end{aligned}$$

be the embeddings as in (1), we have

$$\lambda(\sigma_A(a\tilde{d})\sigma_B(b)) = \sigma_{\tilde{A}}(a\tilde{d})\sigma_{\tilde{B}}(b) = \sigma_{\tilde{A}}(a)\sigma_{\tilde{B}}(\tilde{d}b) = \lambda(\sigma_A(a)\sigma_B(\tilde{d}b)).$$

Thus we need only show that

$$(14) \quad \sigma_A(a\tilde{d})\sigma_B(b) \neq \sigma_A(a)\sigma_B(\tilde{d}b).$$

We consider the reduced amalgamated free product of C^* -algebras (see [11] or [12]),

$$(A *_D^{\text{red}} B, E_D) = (A, E_D^A) *_D (B, E_D^B)$$

and the natural quotient $*$ -homomorphism $A *_D B \rightarrow A *_D^{\text{red}} B$. Let $L^2(A *_D^{\text{red}} B, E_D)$ be the right Hilbert D -module obtained by separation and completion from $A *_D^{\text{red}} B$ with respect to the D -valued inner product $\langle x, y \rangle = E_D(x^* y)$, and given $x \in A *_D^{\text{red}} B$, let \hat{x} denote the corresponding element in $L^2(A *_D^{\text{red}} B, E_D)$. Let $\mathcal{H}_A = L^2(A, E_D^A)$ and $\mathcal{H}_B = L^2(B, E_D^B)$ be similarly defined. Then in $L^2(A *_D^{\text{red}} B, E_D)$, the closure of the subspace spanned by elements of the form $(ab)^\wedge$ for $a \in A$ and $b \in B$ is isomorphic to the tensor product $\mathcal{H}_A \otimes_D \mathcal{H}_B$ of Hilbert D -modules. In order to show (14), it will suffice to show

$$(ad)^\wedge \otimes \hat{b} \neq \hat{a} \otimes (\tilde{d}b)^\wedge$$

in $\mathcal{H}_A \otimes_D \mathcal{H}_B$. Let $\zeta_B \in \mathcal{H}_B$. Then

$$(15) \quad \langle (ad)^\wedge \otimes \zeta_B, (ad)^\wedge \otimes \hat{b} \rangle = \langle \zeta_B, (E_D^A(\tilde{d}^* a^* ad)b)^\wedge \rangle,$$

$$(16) \quad \langle (ad)^\wedge \otimes \zeta_B, \hat{a} \otimes (\tilde{d}b)^\wedge \rangle = \langle \zeta_B, (E_D^A(\tilde{d}^* a^* a)\tilde{d}b)^\wedge \rangle.$$

From assumptions (11) and (12), we obtain $E_D^A(\tilde{d}^* a^* ad)b \neq E_D^A(\tilde{d}^* a^* a)\tilde{d}b$. Since E_D^B is faithful, there is $\zeta_B \in \mathcal{H}_B$ such that the right-hand sides of (15) and (16) are not equal. \square

Remark 3.3. From the above proof, one sees that the hypotheses of Proposition 3.2 can be weakened as follows: Assumptions (11) and (12) can be dropped, and E_D^B need not be assumed faithful, but instead one must assume

$$(17) \quad E_D^B(b^*(E_D^A(\tilde{d}^* a^* ad) - E_D^A(\tilde{d}^* a^* a)\tilde{d} - \tilde{d}^* E_D^A(a^* ad) + \tilde{d}^* E_D^A(a^* a)\tilde{d})b^*) \neq 0.$$

Note that the LHS of (17) is nothing other than

$$\langle (ad)^\wedge \otimes \hat{b} - \hat{a} \otimes (\tilde{d}b)^\wedge, (ad)^\wedge \otimes \hat{b} - \hat{a} \otimes (\tilde{d}b)^\wedge \rangle.$$

Corollary 3.4. *Suppose*

$$(18) \quad \begin{array}{ccccc} \tilde{A} & \xleftarrow{\quad} & \tilde{D}^C & \xrightarrow{\quad} & \tilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \xleftarrow{\quad} & D^C & \xrightarrow{\quad} & B \end{array}$$

is a commutative diagram of inclusions of C^* -algebras and let $\lambda : A *_D B \rightarrow \tilde{A} *_D^{\text{red}} \tilde{B}$ be the resulting $*$ -homomorphism of full free product C^* -algebras. Suppose one of the following holds:

- (1) $D = 0$,
- (2) $D = \mathbf{C}$, A and B are unital and the inclusions $D \hookrightarrow A$ and $D \hookrightarrow B$ are unital.

Suppose there are $\tilde{d} \in \tilde{D}$, $a \in A$ and $b \in B$ such that $a\tilde{d} \in A \setminus \{0\}$, $\tilde{d}b \in B$ and $\tilde{d}b \notin \mathbf{C}b$. Then λ is not injective.

Proof. We can reduce to the case in which (ii) holds by application of Lemma 2.3. We may without loss of generality assume A and B are separable. Letting $E_D^A : A \rightarrow \mathbf{C}$ and $E_D^B : B \rightarrow \mathbf{C}$ be faithful states, we find that the hypotheses of Proposition 3.2 are satisfied. \square

From this corollary, we have the following class of concrete examples, which shows that λ may be non-injective even if

$$(19) \quad B \cap \tilde{D} = D = A \cap \tilde{D}.$$

Example 3.5. Let \mathcal{H} be an infinite-dimensional, separable Hilbert space. Inside $B(\mathcal{H})$, let $D = \mathbf{C}1$ and let $A = B = D + K(\mathcal{H})$, where $K(\mathcal{H})$ is the set of compact operators. Let $u \in B(\mathcal{H})$ be a unitary operator that does not belong to D , and let $\tilde{D} = C^*(u)$, $\tilde{A} = \tilde{B} = \tilde{D} + K(\mathcal{H})$. Let $\lambda : A *_D B \rightarrow \tilde{A} *_{\tilde{D}} \tilde{B}$ be the $*$ -homomorphism arising from the inclusions (18). Then λ is not injective.

Proof. Take $\tilde{d} = u$ and $a \in K(\mathcal{H}) \setminus \{0\}$. Since $u \notin \mathbf{C}1$, there is $b \in K(\mathcal{H})$ such that $ub \notin \mathbf{C}b$. Now apply Corollary 3.4. One can choose u so that $C^*(u) \cap (\mathbf{C}1 + K(\mathcal{H})) = \mathbf{C}1$, in order to get (19). \square

Proposition 3.6. Suppose

$$(20) \quad \begin{array}{ccccc} & \tilde{A} & \xleftarrow{\quad} & \tilde{D} & \xhookrightarrow{\quad} \tilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \xleftarrow{\quad} & D & \xhookrightarrow{\quad} & B \end{array}$$

is a commutative diagram of inclusions of C^* -algebras, and let $\lambda : A *_D B \rightarrow \tilde{A} *_{\tilde{D}} \tilde{B}$ be the resulting $*$ -homomorphism of full free product C^* -algebras. Suppose one of the following holds:

- (1) $D = 0$,
- (2) $D = \mathbf{C}$, A and B are unital and the inclusions $D \hookrightarrow A$ and $D \hookrightarrow B$ are unital.

Suppose there are $\tilde{d} \in \tilde{D}$, $a_1, a_2 \in A$ and $b \in B \setminus D$ such that $a_1 \tilde{d}, \tilde{d} a_2 \in A$, $a_1 \tilde{d} \notin \mathbf{C}$ and $\tilde{d} b = b \tilde{d}$. Then λ is not injective.

Proof. We can reduce to the case in which (ii) holds by application of Lemma 2.3. We use the same notation as in (13). We have

$$\begin{aligned} \lambda(\sigma_A(a_1 \tilde{d}) \sigma_B(b) \sigma_A(a_2)) &= \sigma_{\tilde{A}}(a_1 \tilde{d}) \sigma_{\tilde{B}}(b) \sigma_{\tilde{A}}(a_2) \\ &= \sigma_{\tilde{A}}(a_1) \sigma_{\tilde{B}}(b) \sigma_{\tilde{A}}(\tilde{d} a_2) = \lambda(\sigma_A(a_1) \sigma_B(b) \sigma_A(\tilde{d} a_2)), \end{aligned}$$

and we must only show

$$(21) \quad \sigma_A(a_1 \tilde{d}) \sigma_B(b) \sigma_A(a_2) \neq \sigma_A(a_1) \sigma_B(b) \sigma_A(\tilde{d} a_2).$$

Without loss of generality, assume A and B are separable. Let $\phi_A : A \rightarrow \mathbf{C}$ and $\phi_B : B \rightarrow \mathbf{C}$ be faithful states. By adding a scalar multiple of the identity, if necessary, we may without loss of generality assume $\phi_B(b) = 0$. Let

$$(A *_{\mathbf{C}}^{\text{red}} B, \phi) = (A, \phi_A) *_{\mathbf{C}} (B, \phi_B)$$

be the reduced free product of C^* -algebras. Using arguments and notation as in the proof of Proposition 3.2, the closure of the subspace of $L^2(A *_{\mathbf{C}}^{\text{red}} B, \phi)$ spanned by elements of the form $(aba')^\wedge$ for $a, a' \in A$ is isomorphic to $\mathcal{H}_A \otimes (\mathbf{C}\hat{b}) \otimes \mathcal{H}_A$. To show (21), it will suffice to show

$$(a_1 \tilde{d})^\wedge \otimes \hat{b} \otimes \hat{a}_2 \neq \hat{a}_1 \otimes \hat{b} \otimes (\tilde{d} a_2)^\wedge$$

in $\mathcal{H}_A \otimes (\mathbf{C}\hat{b}) \otimes \mathcal{H}_A$. However, this follows from the assumptions. \square

From the above proposition, we get the following example, which requires only “bad” relations between A and \tilde{D} , not between B and \tilde{D} .

Example 3.7. Let D , \tilde{D} , A and \tilde{A} be as in Example 3.5. Let B be any unital C^* -algebra of dimension greater than 1, and let $\tilde{B} = B \otimes \tilde{D}$ (for the unique C^* -tensor norm). Then the $*$ -homomorphism $\lambda : A *_D B \rightarrow \tilde{A} *_{\tilde{D}} \tilde{B}$ arising from the inclusions (20) is not injective.

Remark 3.8. The problem with injectivity of λ in Examples 3.5 and 3.7 arises already at the algebraic level

$$(22) \quad A *_D^{\text{alg}} B \rightarrow \tilde{A} *_{\tilde{D}}^{\text{alg}} \tilde{B}.$$

On the other hand, in Examples 3.1, we can arrange that the map between algebras (22) is injective, while λ fails to be injective, e.g. by taking E to be a reduced free product. However, we do not know of an example where λ fails to be injective and where the algebraic map (22) is injective, but where $A \cap \tilde{D} = D = B \cap \tilde{D}$.

4. AN APPLICATION TO RESIDUAL FINITE-DIMENSIONALITY

A C^* -algebra is said to be residually finite dimensional (r.f.d.) if it has a separating family of finite-dimensional $*$ -representations. The first result linking full free products and residual finite dimensionality was M.-D. Choi's proof [6] that the full group C^* -algebras of nonabelian free groups are r.f.d. In [7], Exel and Loring proved that the full free product of any two r.f.d. C^* -algebras A and B with amalgamation over either the zero C^* -algebra or over the scalar multiples of the identity (if A and B are unital) is r.f.d. In [5], N. Brown and Dykema proved that a full amalgamated free product of matrix algebras $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$ over a unital subalgebra D is r.f.d. provided that the normalized traces on $M_k(\mathbf{C})$ and $M_\ell(\mathbf{C})$ restrict to the same trace on D . In this section, we observe that by applying Proposition 2.2, one obtains (as a corollary of the result from [5]) the analogous result for full amalgamated free products of finite-dimensional algebras.

Lemma 4.1. *Let $S = \{x \in \mathbf{R}^n \mid Ax = 0\}$, where A is an $m \times n$ matrix having only rational entries. Then vectors having only rational entries are dense in S .*

Proof. By considering the reduced row-echelon form of A , we see that there is a basis for S consisting of rational vectors. \square

Theorem 4.2. *Consider unital inclusions of C^* -algebras $A \supseteq D \subseteq B$ with A and B finite dimensional. Let $A *_D B$ be the corresponding full amalgamated free product. Then $A *_D B$ is residually finite dimensional if and only if there are faithful tracial states τ_A on A and τ_B on B whose restrictions to D agree.*

Proof. Since every separable r.f.d. C^* -algebra has a faithful tracial state, the necessity of the existence of τ_A and τ_B is clear.

Let us recall some well-known facts about a unital inclusion $D \subseteq A$ of finite-dimensional C^* -algebras (see e.g. Chapter 2 of [8]). Let p_1, \dots, p_m be the minimal central projections of A and q_1, \dots, q_n the minimal central projections of D . Then the inclusion matrix Λ_D^A is an $m \times n$ integer matrix whose (i, j) th entry is $\text{rank}(q_j p_i A q_j)/\text{rank}(q_j D)$, where the rank of a matrix algebra $M_k(\mathbf{C})$ is k . To a trace τ on A , we associate the column vector s of length m whose i th entry is the

trace of a minimal projection in $p_i A$. Then the restriction of τ to D has associated column vector $(\Lambda_D^A)^t s$, where the superscript t indicates transpose.

Thus, given $A \supseteq D \subseteq B$ as in the statement of the theorem, the existence of faithful tracial states τ_A and τ_B agreeing on D is equivalent to the existence of column vectors s_A and s_B , none of whose components are zero, such that $(\Lambda_D^A)^t s_A = (\Lambda_D^B)^t s_B$, i.e.,

$$(23) \quad \left[\begin{array}{cc} (\Lambda_D^A)^t & -(\Lambda_D^B)^t \end{array} \right] \left[\begin{array}{c} s_A \\ s_B \end{array} \right] = 0.$$

Supposing now that such traces τ_A and τ_B exist, by Lemma 4.1 there is a solution $\left[\begin{smallmatrix} s_A \\ s_B \end{smallmatrix} \right]$ to (23) whose entries are all strictly positive and rational. Therefore, the traces τ_A and τ_B agreeing on D can be chosen to take only rational values on minimal projections of A and, respectively, B . Hence there are unital inclusions into matrix algebras,

$$M_k(\mathbf{C}) \supseteq A \supseteq D \subseteq B \subseteq M_\ell(\mathbf{C}),$$

so that τ_A is the restriction of the tracial state on $M_k(\mathbf{C})$ to A and τ_B is the restriction of the tracial state on $M_\ell(\mathbf{C})$ to B . By Proposition 2.2, $A *_D B$ is a subalgebra of $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$. By Theorem 2.3 of [5], $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$ is r.f.d. Therefore, $A *_D B$ is r.f.d. \square

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