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ON EMBEDDINGS OF FULL AMALGAMATED FREE PRODUCT C*–ALGEBRAS

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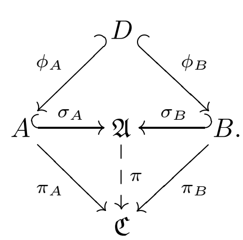
Abstract. We examine the question of when the $*$–homomorphism $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ of full amalgamated free product C*–algebras, arising from compatible inclusions of C*–algebras $A \subseteq \widetilde{A}$, $B \subseteq \widetilde{B}$ and $D \subseteq \widetilde{D}$, is an embedding. Results giving sufficient conditions for λ to be injective, as well as classes of examples where λ fails to be injective, are obtained. As an application, we give necessary and sufficient conditions for the full amalgamated free product of finite-dimensional C*–algebras to be residually finite dimensional.

1. Introduction

Given C*–algebras A, B and D with injective $*$–homomorphisms $\phi_A : D \to A$ and $\phi_B : D \to B$, the corresponding full amalgamated free product C*–algebra (see [1] or [9, Chapter 5]) is the C*–algebra $\mathfrak{A}$, equipped with injective $*$–homomorphisms $\sigma_A : A \to \mathfrak{A}$ and $\sigma_B : B \to \mathfrak{A}$ such that $\sigma_A \circ \phi_A = \sigma_B \circ \phi_B$, such that $\mathfrak{A}$ is generated by $\sigma_A(A) \cup \sigma_B(B)$ and satisfying the universal property that whenever $\mathfrak{C}$ is a C*–algebra and $\pi_A : A \to \mathfrak{C}$ and $\pi_B : B \to \mathfrak{C}$ are $*$–homomorphisms satisfying $\pi_A \circ \phi_A = \pi_B \circ \phi_B$, there is a $*$–homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\pi \circ \sigma_A = \pi_A$ and $\pi \circ \sigma_B = \pi_B$. This situation is illustrated by the following commutative diagram:

(1)



The full amalgamated free product C*–algebra $\mathfrak{A}$ is commonly denoted by $A *_D B$, although this notation hides the dependence of $\mathfrak{A}$ on the embeddings ϕ_A and ϕ_B.

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Question 1.1. Let D, A, B, $\widetilde{D}$, $\widetilde{A}$ and $\widetilde{B}$ be C*–algebras and suppose there are injective $*$–homomorphisms making the following diagram commute:

$$\begin{array}{ccccc} \widetilde{A} & \xhookleftarrow{\phi_{\widetilde{A}}} & \widetilde{D} & \xhookrightarrow{\phi_{\widetilde{B}}} & \widetilde{B} \\ {\scriptstyle\lambda_A}\big\uparrow & & {\scriptstyle\lambda_D}\big\uparrow & & {\scriptstyle\lambda_B}\big\uparrow \\ A & \xhookleftarrow{\phi_A} & D & \xhookrightarrow{\phi_B} & B. \end{array}$$

Let $A *_D B$ and $\widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the corresponding full amalgamated free product C*–algebras, and let $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the $*$–homomorphism arising from λ_A and λ_B via the universal property. When is λ injective?

We prove in §2 that λ is injective when either (i) $D = \widetilde{D}$ (or more precisely, when the $*$–homomorphism λ_D is surjective), or (ii) there are conditional expectations $E_A : \widetilde{A} \to A$ and $E_B : \widetilde{B} \to B$ that send $\widetilde{D}$ onto D and agree on $\widetilde{D}$. Injectivity in the case $D = \widetilde{D}$ was previously proved by G. K. Pedersen [10]. (Moreover, earlier results of F. Boca [4] imply that the map λ is injective when $D = \widetilde{D}$ and when there are conditional expectations

$$\widetilde{A} \overset{E^{\widetilde{A}}_A}{\to} A \overset{E^A_D}{\to} D \overset{E^B_D}{\leftarrow} B \overset{E^{\widetilde{B}}_B}{\leftarrow} \widetilde{B};$$

an argument for the case $D = \widetilde{D} = \mathbf{C}$, which uses Boca's results, is outlined in [3, 4.7].) However, we include our proof because it is different from that found in [10] and because it contains the main idea of our proof of injectivity in case (ii). In §3, we consider some general conditions and give some concrete examples when λ fails to be injective. Finally, in §4, we apply this embedding result to extend a result from [5] about residual finite-dimensionality of full amalgamated free products of finite-dimensional C*–algebras.

2. Embeddings of full free products

The following result is of course well known. We include a proof for completeness.

Lemma 2.1. *Let A be a C*–subalgebra of a C*–algebra $\widetilde{A}$ and let $\pi : A \to B(\mathcal{H})$ be a $*$–representation. Then there is a Hilbert space $\mathcal{K}$ and a $*$–representation $\tilde{\pi} : \widetilde{A} \to B(\mathcal{H} \oplus \mathcal{K})$ such that*

$$\tilde{\pi}(a)(h \oplus 0) = (\pi(a)h) \oplus 0 \qquad (a \in A,\ h \in \mathcal{H}). \tag{2}$$

Proof. Since in general π is a direct sum of cyclic representations, we may without loss of generality assume that π is a cyclic representation with cyclic vector ξ. Let ϕ be the vector state $\phi(\cdot) = \langle \pi(\cdot)\xi, \xi\rangle$ of A. Then $\mathcal{H}$ is identified with $L^2(A, \phi)$ and π is the associated GNS representation. Let $\tilde{\phi}$ be an extension of ϕ to a state of $\widetilde{A}$, and let $\widetilde{\mathcal{H}} = L^2(\widetilde{A}, \tilde{\phi})$. Then the inclusion $A \hookrightarrow \widetilde{A}$ gives rise to an isometry $\mathcal{H} \to \widetilde{\mathcal{H}}$, and we may thus write $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{K}$ for a Hilbert space $\mathcal{K}$. If $\tilde{\pi} : \widetilde{A} \to B(\mathcal{H} \oplus \mathcal{K})$ is the GNS representation associated to $\tilde{\phi}$, then (2) holds. □

The following result was first proved by G. K. Pedersen [10, Thm. 4.2]. We offer a new proof, which is perhaps more elementary. This proof contains essentially the same idea as our proof of Proposition 2.4 below.

Proposition 2.2. *Let*

$$\widetilde{A} \supseteq A \supseteq D \subseteq B \subseteq \widetilde{B}$$

be inclusions of C^**–algebras and let* $A *_D B$ *and* $\widetilde{A} *_D \widetilde{B}$ *be the corresponding full amalgamated free product* C^**–algebras. Let* $\lambda : A *_D B \to \widetilde{A} *_D \widetilde{B}$ *be the* $*$*–homomorphism arising via the universal property from the inclusions* $A \hookrightarrow \widetilde{A}$ *and* $B \hookrightarrow \widetilde{B}$*. Then* λ *is injective.*

Proof. Let $\pi : A *_D B \to B(\mathcal{H})$ be a faithful $*$–homomorphism. We will find a Hilbert space $\mathcal{K}$ and a $*$–homomorphism $\tilde{\pi} : \widetilde{A} *_D \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K})$ such that

$$\tilde{\pi}(\lambda(x))(h \oplus 0) = (\pi(x)h) \oplus 0 \qquad (x \in A *_D B,\ h \in \mathcal{H}). \tag{3}$$

This will imply that λ is injective.

Let $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ be the $*$–representations obtained by composing π with the inclusions $A \hookrightarrow A *_D B$ and $B \hookrightarrow A *_D B$. Let

$$\sigma_{A,0} : \widetilde{A} \to B(\mathcal{H} \oplus \mathcal{K}_{A,0}),$$
$$\sigma_{B,0} : \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K}_{B,0})$$

be $*$–representations obtained from Lemma 2.1 such that

$$\sigma_{A,0}(a)(h \oplus 0) = (\pi_A(a)h) \oplus 0 \qquad (a \in A,\ h \in \mathcal{H}),$$

and similarly with A replaced by B. Note that $0 \oplus \mathcal{K}_{A,0}$ is reducing for $\sigma_{A,0}(D)$. Using Lemma 2.1, we find Hilbert spaces $\mathcal{K}_{B,1}$ and $\mathcal{K}_{A,1}$ and $*$–representations

$$\sigma_{B,1} : \widetilde{B} \to B(\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1}),$$
$$\sigma_{A,1} : \widetilde{A} \to B(\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1})$$

such that

$$\sigma_{B,1}(d)(k \oplus 0) = \sigma_{A,0}(d)(0 \oplus k) \qquad (d \in D,\ k \in \mathcal{K}_{A,0}),$$
$$\sigma_{A,1}(d)(k \oplus 0) = \sigma_{B,0}(d)(0 \oplus k) \qquad (d \in D,\ k \in \mathcal{K}_{B,0}).$$

Proceeding recursively, for every integer $n \geq 2$ we find $*$–representations

$$\sigma_{B,n} : \widetilde{B} \to B(\mathcal{K}_{A,n-1} \oplus \mathcal{K}_{B,n}),$$
$$\sigma_{A,n} : \widetilde{A} \to B(\mathcal{K}_{B,n-1} \oplus \mathcal{K}_{A,n})$$

such that

$$\sigma_{B,n}(d)(k \oplus 0) = \sigma_{A,n-1}(d)(0 \oplus k) \qquad (d \in D,\ k \in \mathcal{K}_{A,n-1}),$$
$$\sigma_{A,n}(d)(k \oplus 0) = \sigma_{B,n-1}(d)(0 \oplus k) \qquad (d \in D,\ k \in \mathcal{K}_{B,n-1}).$$

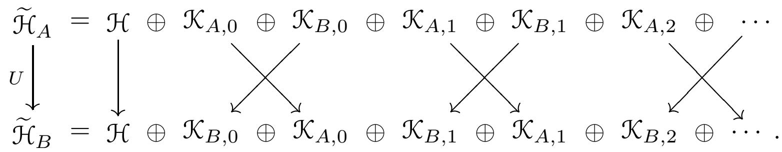
We now define the Hilbert spaces

$$\begin{aligned} \widetilde{\mathcal{H}}_A &= \overbrace{\mathcal{H} \oplus \mathcal{K}_{A,0}}^{\sigma_{A,0}} \oplus \overbrace{\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1}}^{\sigma_{A,1}} \oplus \overbrace{\mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2}}^{\sigma_{A,2}} \oplus \cdots, \\ \widetilde{\mathcal{H}}_B &= \underbrace{\mathcal{H} \oplus \mathcal{K}_{B,0}}_{\sigma_{B,0}} \oplus \underbrace{\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1}}_{\sigma_{B,1}} \oplus \underbrace{\mathcal{K}_{A,1} \oplus \mathcal{K}_{B,2}}_{\sigma_{B,2}} \oplus \cdots, \end{aligned} \tag{4}$$

where the brackets indicate where the constructed representations act, and we let $\sigma_{\widetilde{A}} : \widetilde{A} \to B(\widetilde{\mathcal{H}}_A)$ and $\sigma_{\widetilde{B}} : \widetilde{B} \to B(\widetilde{\mathcal{H}}_B)$ be the $*$–representations

$$\sigma_{\widetilde{A}} = \sigma_{A,0} \oplus \sigma_{A,1} \oplus \sigma_{A,2} \oplus \cdots,$$
$$\sigma_{\widetilde{B}} = \sigma_{B,0} \oplus \sigma_{B,1} \oplus \sigma_{B,2} \oplus \cdots,$$

where the summands act as indicated by brackets in (4). Consider the unitary $U : \widetilde{\mathcal{H}}_A \to \widetilde{\mathcal{H}}_B$ mapping the summands in $\widetilde{\mathcal{H}}_A$ identically to the corresponding summands in $\widetilde{\mathcal{H}}_B$ as indicated by the arrows below:



Let $\mathcal{K} = \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \cdots$ and identify $\mathcal{H} \oplus \mathcal{K}$ with $\widetilde{\mathcal{H}}_A$. Then we have the $*$–representations $\tilde{\pi}_{\widetilde{A}} = \sigma_{\widetilde{A}} : \widetilde{A} \to B(\mathcal{H} \oplus \mathcal{K})$ and $\tilde{\pi}_{\widetilde{B}} : \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K})$, the latter defined by $\tilde{\pi}_{\widetilde{B}}(\cdot) = U^* \sigma_{\widetilde{B}}(\cdot) U$. By construction, the restrictions of $\tilde{\pi}_{\widetilde{A}}$ and $\tilde{\pi}_{\widetilde{B}}$ to D agree, and we have

$$\tilde{\pi}_{\widetilde{A}}(a)(h \oplus 0) = (\pi_A(a)h) \oplus 0 \qquad (a \in A,\, h \in \mathcal{H}),$$
$$\tilde{\pi}_{\widetilde{B}}(b)(h \oplus 0) = (\pi_B(b)h) \oplus 0 \qquad (b \in B,\, h \in \mathcal{H}).$$

Letting $\tilde{\pi} : \widetilde{A} *_D \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K})$ be the $*$–homomorphism obtained from $\tilde{\pi}_{\widetilde{A}}$ and $\tilde{\pi}_{\widetilde{B}}$ via the universal property, we have that (3) holds. □

For a C*–algebra A, unital or not, let A^u denote the unitization of A. Thus, as a vector space, $A^u = A \oplus \mathbf{C}$ with multiplication defined by $(a, \mu) \cdot (a', \mu') = (aa' + \mu a + \mu' a, \mu\mu')$. We identify A with the ideal $A \oplus 0$ of A^u, which has codimension 1.

Lemma 2.3. *Let $A \supseteq D \subseteq B$ be inclusions of C*–algebras. Consider the unitizations and corresponding inclusions*

$$\begin{array}{ccccc} A^u & \hookleftarrow & D^u & \hookrightarrow & B^u \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookleftarrow & D & \hookrightarrow & B. \end{array}$$

Let $\lambda : A *_D B \to A^u *_{D^u} B^u$ be the resulting $*$–homomorphism between full amalgamated free products. Then there is an isomorphism $\pi : A^u *_{D^u} B^u \to (A *_D B)^u$ such that $\pi \circ \lambda : A *_D B \to (A *_D B)^u$ is the canonical embedding arising in the definition of the unitization.

Proof. Since any $*$–representations of A and B that agree on D extend to $*$–representations of A^u and B^u that agree on D^u, the $*$–homomorphism λ is injective. Let $e \in A^u *_{D^u} B^u$ be the unit of A^u, which is of course identified with the units of B^u and D^u. Clearly, $A^u *_{D^u} B^u$ is generated by the image of λ together with e. One easily sees that

$$(\lambda(x) + \mu e)(\lambda(x') + \mu' e) = \lambda(xx') + \mu\lambda(x') + \mu'\lambda(x) + \mu\mu' e.$$

Moreover, if $\rho : A^u *_{D^u} B^u \to \mathbf{C}$ is the $*$–homomorphism arising from the unital $*$–homomorphisms $A^u \to \mathbf{C}$ and $B^u \to \mathbf{C}$, then $\rho(e) = 1$ and $\lambda(A *_D B) \subseteq \ker \rho$. Hence $\lambda(A *_D B)$ has codimension 1 in $A^u *_{D^u} B^u$. Now π can be defined by $\pi(\lambda(x) + \mu e) = (x, \mu)$. □

Proposition 2.4. *Suppose*

$$
\begin{array}{ccccc}
\widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\
\uparrow & & \uparrow & & \uparrow \\
A & \hookleftarrow & D & \hookrightarrow & B
\end{array}
\tag{5}
$$

is a commuting diagram of inclusions of C^**–algebras. Let* $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ *be the resulting* $*$*–homomorphism of full free product* C^**–algebras. Suppose there are conditional expectations* $E_A : \widetilde{A} \to A$*,* $E_D : \widetilde{D} \to D$ *and* $E_B : \widetilde{B} \to B$ *onto* A*,* D *and* B*, respectively, such that the diagram*

$$
\begin{array}{ccccc}
\widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\
\downarrow{\scriptstyle E_A} & & \downarrow{\scriptstyle E_D} & & \downarrow{\scriptstyle E_B} \\
A & \hookleftarrow & D & \hookrightarrow & B
\end{array}
\tag{6}
$$

commutes. Then λ *is injective.*

Proof. By appealing to Lemma 2.3, we may without loss of generality assume that all the algebras and $*$–homomorphisms in (5) are unital. Let $\pi : A *_D B \to B(\mathcal{H})$ be a faithful, unital $*$–representation. As in the proof of Proposition 2.2, in order to show λ is injective, we will find a Hilbert space $\mathcal{K}$ and a $*$–homomorphism $\tilde{\pi} : \widetilde{A} *_{\widetilde{D}} \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K})$ such that

$$\tilde{\pi}(\lambda(x))(h \oplus 0) = (\pi(x)h) \oplus 0 \qquad (x \in A *_D B,\ h \in \mathcal{H}). \tag{7}$$

Let $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ be the $*$–representations obtained by composing π with the inclusions $A \hookrightarrow A *_D B$ and $B \hookrightarrow A *_D B$, and let $\pi_D : D \to B(\mathcal{H})$ be their common restriction to D. Consider the canonical left action of $\widetilde{D}$ on the right Hilbert D–module $L^2(\widetilde{D}, E_D)$, which is obtained from $\widetilde{D}$ by separation and completion with respect to the D–valued inner product $\langle \tilde{d}_1, \tilde{d}_2 \rangle = E_D(\tilde{d}_1^* \tilde{d}_2)$. Consider the Hilbert space $L^2(\widetilde{D}, E_D) \otimes_D \mathcal{H}$, where the left action of D on $\mathcal{H}$ is via π_D. Since π_D is unital, $\mathcal{H}$ embeds as a subspace, and we can write

$$L^2(\widetilde{D}, E_D) \otimes_D \mathcal{H} = \mathcal{H} \oplus \mathcal{K}_D. \tag{8}$$

Consider the left action of $\widetilde{D}$ on the Hilbert space $\mathcal{H} \oplus \mathcal{K}_D$. The subspace $\mathcal{H}$ is reducing for the restriction of σ_D to D, and we have $\sigma_D(d)(h \oplus 0) = (\pi_D(d)h) \oplus 0$ for every $d \in D$ and $h \in \mathcal{H}$.

In a similar way, consider the Hilbert spaces

$$L^2(\widetilde{A}, E_A) \otimes_A \mathcal{H}, \qquad L^2(\widetilde{B}, E_B) \otimes_B \mathcal{H} \tag{9}$$

and the associated left actions $\sigma_{A,0}$ of $\widetilde{A}$, respectively $\sigma_{B,0}$ of $\widetilde{B}$. Since the diagram (6) commutes, the Hilbert space (8) embeds canonically as a subspace of both spaces (9). We may thus write

$$
\begin{aligned}
L^2(\widetilde{A}, E_A) \otimes_A \mathcal{H} &= \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{A,0}, \\
L^2(\widetilde{B}, E_B) \otimes_B \mathcal{H} &= \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{B,0},
\end{aligned}
$$

the subspace $\mathcal{H} \oplus \mathcal{K}_D \oplus 0$ is reducing for the restrictions of $\sigma_{A,0}$ and $\sigma_{B,0}$ to $\widetilde{D}$, and we have $\sigma_{A,0}(\tilde{d})(\eta \oplus 0) = (\sigma_D(\tilde{d})\eta) \oplus 0 = \sigma_{B,0}(\tilde{d})(\eta \oplus 0)$ for every $\tilde{d} \in \widetilde{D}$ and

$\eta \in \mathcal{H} \oplus \mathcal{K}_D$. Moreover, $\mathcal{H} \oplus 0 \oplus 0$ is reducing for the restrictions of $\sigma_{A,0}$ to A and $\sigma_{B,0}$ to B, and we have

$$\sigma_{A,0}(a)(h \oplus 0 \oplus 0) = (\pi_A(a)h) \oplus 0 \oplus 0 \qquad (a \in A,\ h \in \mathcal{H}),$$
$$\sigma_{B,0}(b)(h \oplus 0 \oplus 0) = (\pi_B(b)h) \oplus 0 \oplus 0 \qquad (b \in B,\ h \in \mathcal{H}).$$

Let $\sigma_{A,0,\widetilde{D}}$ denote the action of $\widetilde{D}$ on $\mathcal{K}_{A,0}$ obtained by restricting $\sigma_{A,0}$ to $\widetilde{D}$ and compressing, and similarly for $\sigma_{B,0,\widetilde{D}}$.

We now proceed recursively as in the proof of Proposition 2.2. If Hilbert spaces $\mathcal{K}_{A,n-1}$ and $\mathcal{K}_{B,n-1}$ have been constructed with actions $\sigma_{A,n-1,\widetilde{D}}$ and $\sigma_{B,n-1,\widetilde{D}}$, respectively, of $\widetilde{D}$, then use Lemma 2.1 to construct Hilbert spaces $\mathcal{K}_{B,n}$ and $\mathcal{K}_{A,n}$ and $*$–homomorphisms

$$\sigma_{B,n} : \widetilde{B} \to B(\mathcal{K}_{A,n-1} \oplus \mathcal{K}_{B,n}),$$
$$\sigma_{A,n} : \widetilde{A} \to B(\mathcal{K}_{B,n-1} \oplus \mathcal{K}_{A,n}),$$

such that

$$\sigma_{B,n}(\tilde{d})(k \oplus 0) = (\sigma_{A,n-1,\widetilde{D}}(\tilde{d})k) \oplus 0 \qquad (\tilde{d} \in \widetilde{D},\ k \in \mathcal{K}_{A,n-1}),$$
$$\sigma_{A,n}(\tilde{d})(k \oplus 0) = (\sigma_{B,n-1,\widetilde{D}}(\tilde{d})k) \oplus 0 \qquad (\tilde{d} \in \widetilde{D},\ k \in \mathcal{K}_{B,n-1}).$$

Then let $\sigma_{B,n,\widetilde{D}}$ be the action of $\widetilde{D}$ on $\mathcal{K}_{B,n}$ obtained from the restriction of $\sigma_{B,n}$ to $\widetilde{D}$ by compressing, and similarly define the action $\sigma_{A,n,\widetilde{D}}$ of $\widetilde{D}$ on $\mathcal{K}_{A,n}$.

We may now define the Hilbert spaces

$$(10)\qquad \begin{aligned} \widetilde{\mathcal{H}}_A &= \overbrace{\overbrace{\mathcal{H} \oplus \mathcal{K}_D}^{\sigma_D} \oplus \mathcal{K}_{A,0}}^{\sigma_{A,0}} \oplus \overbrace{\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1}}^{\sigma_{A,1}} \oplus \overbrace{\mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2}}^{\sigma_{A,2}} \oplus \cdots, \\ \widetilde{\mathcal{H}}_B &= \underbrace{\underbrace{\mathcal{H} \oplus \mathcal{K}_D}_{\sigma_D} \oplus \mathcal{K}_{B,0}}_{\sigma_{B,0}} \oplus \underbrace{\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1}}_{\sigma_{B,1}} \oplus \underbrace{\mathcal{K}_{A,1} \oplus \mathcal{K}_{B,2}}_{\sigma_{B,2}} \oplus \cdots, \end{aligned}$$

where the brackets indicate where the constructed representations act. We let $\sigma_{\widetilde{A}} : \widetilde{A} \to B(\widetilde{\mathcal{H}}_A)$ and $\sigma_{\widetilde{B}} : \widetilde{B} \to B(\widetilde{\mathcal{H}}_B)$ be the $*$–representations

$$\sigma_{\widetilde{A}} = \sigma_{A,0} \oplus \sigma_{A,1} \oplus \sigma_{A,2} \oplus \cdots,$$
$$\sigma_{\widetilde{B}} = \sigma_{B,0} \oplus \sigma_{B,1} \oplus \sigma_{B,2} \oplus \cdots,$$

where the summands act as indicated by brackets in (10). Consider the unitary $U : \widetilde{\mathcal{H}}_A \to \widetilde{\mathcal{H}}_B$ mapping the summands in $\widetilde{\mathcal{H}}_A$ identically to the corresponding summands in $\widetilde{\mathcal{H}}_B$ as indicated by the arrows below:

$$\begin{array}{ccccccccccccccccc} \widetilde{\mathcal{H}}_A & = & \mathcal{H} & \oplus & \mathcal{K}_D & \oplus & \mathcal{K}_{A,0} & \oplus & \mathcal{K}_{B,0} & \oplus & \mathcal{K}_{A,1} & \oplus & \mathcal{K}_{B,1} & \oplus & \mathcal{K}_{A,2} & \oplus & \cdots \\ {\scriptstyle U}\downarrow & & \downarrow & & \downarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow \\ \widetilde{\mathcal{H}}_B & = & \mathcal{H} & \oplus & \mathcal{K}_D & \oplus & \mathcal{K}_{B,0} & \oplus & \mathcal{K}_{A,0} & \oplus & \mathcal{K}_{B,1} & \oplus & \mathcal{K}_{A,1} & \oplus & \mathcal{K}_{B,2} & \oplus & \cdots. \end{array}$$

Let $\mathcal{K} = \mathcal{K}_D \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \cdots$ and identify $\mathcal{H} \oplus \mathcal{K}$ with $\widetilde{\mathcal{H}}_A$. Then we have the $*$–representations $\tilde{\pi}_{\widetilde{A}} = \sigma_{\widetilde{A}} : \widetilde{A} \to B(\mathcal{H} \oplus \mathcal{K})$ and $\tilde{\pi}_{\widetilde{B}} : \widetilde{B} \to B(\mathcal{H} \oplus \mathcal{K})$, the latter defined by $\tilde\pi_{\widetilde B}(\cdot) = U^*\sigma_{\widetilde B}(\cdot)U$. By construction, the restrictions of $\tilde\pi_{\widetilde A}$ and $\tilde\pi_{\widetilde B}$ to $\widetilde D$ agree, and we have

$$\begin{aligned}\tilde\pi_{\widetilde A}(a)(h\oplus 0) &= (\pi_A(a)h)\oplus 0 \qquad (a\in A,\, h\in\mathcal H),\\ \tilde\pi_{\widetilde B}(b)(h\oplus 0) &= (\pi_B(b)h)\oplus 0 \qquad (b\in B,\, h\in\mathcal H).\end{aligned}$$

Letting $\tilde\pi : \widetilde A *_{\widetilde D} \widetilde B \to B(\mathcal H\oplus\mathcal K)$ be the $*$–homomorphism obtained from $\tilde\pi_{\widetilde A}$ and $\tilde\pi_{\widetilde B}$ via the universal property, we have that (7) holds. □

3. Examples of non-embedding

In this section, we give some examples when the map λ of Question 1.1 fails to be injective. (In contrast, it is known [2] that in the more stringent situation of *reduced* amalgamated free products, the map analogous to λ is always injective.)

We begin with a trivial class of examples.

Examples 3.1. Let A and B be C*-subalgebras of a C*-algebra E with $A \nsubseteq B$ and $B \nsubseteq A$. Let $D = A\cap B$, $\widetilde A = E$ and $\widetilde D = \widetilde B = B$, equipped with the natural inclusions. Then the map $\lambda : A *_D B \to \widetilde A *_{\widetilde D} \widetilde B = E$ is injective if and only if $A *_D B$ is exactly the C*-subalgebra of E generated by A and B. This does not hold in general. Notice that in these examples, $B\cap\widetilde D = B \supsetneqq D$.

Proposition 3.2. *Suppose*

$$\begin{array}{ccccc} \widetilde A & \hookleftarrow & \widetilde D & \hookrightarrow & \widetilde B\\ \uparrow & & \uparrow & & \uparrow\\ A & \hookleftarrow & D & \hookrightarrow & B\end{array}$$

is a commutative diagram of inclusions of C*–algebras, and let $\lambda : A *_D B \to \widetilde A *_{\widetilde D} \widetilde B$ be the resulting $*$–homomorphism of full free product C*–algebras. Suppose there are conditional expectations $E_D^A : A \to D$ and $E_D^B : B \to D$ with E_D^B faithful. Suppose there are $\tilde d \in \widetilde D$, $a\in A$ and $b \in B$ satisfying $a\tilde d\in A$, $\tilde d b\in B$,

$$D(\tilde d b)\cap Db = \{0\}, \tag{11}$$

$$E_D^A(\tilde d^* a^* a\tilde d)b \neq 0. \tag{12}$$

Then λ is not injective.

Proof. Letting

$$\begin{aligned}&\sigma_A : A \hookrightarrow A *_D B, \qquad \sigma_B : B\hookrightarrow A *_D B,\\ &\sigma_{\widetilde A} : \widetilde A \hookrightarrow \widetilde A *_{\widetilde D} \widetilde B, \qquad \sigma_{\widetilde B} : \widetilde B \hookrightarrow \widetilde A *_{\widetilde D}\widetilde B\end{aligned} \tag{13}$$

be the embeddings as in (1), we have

$$\lambda(\sigma_A(a\tilde d)\sigma_B(b)) = \sigma_{\widetilde A}(a\tilde d)\sigma_{\widetilde B}(b) = \sigma_{\widetilde A}(a)\sigma_{\widetilde B}(\tilde d b) = \lambda(\sigma_A(a)\sigma_B(\tilde d b)).$$

Thus we need only show that

$$\sigma_A(a\tilde d)\sigma_B(b) \neq \sigma_A(a)\sigma_B(\tilde d b). \tag{14}$$

We consider the reduced amalgamated free product of C*–algebras (see [11] or [12]),

$$(A *_D^{\mathrm{red}} B, E_D) = (A, E_D^A) *_D (B, E_D^B)$$

and the natural quotient $*$–homomorphism $A*_D B \to A*_D^{\mathrm{red}} B$. Let $L^2(A*_D^{\mathrm{red}} B, E_D)$ be the right Hilbert D–module obtained by separation and completion from $A*_D^{\mathrm{red}} B$ with respect to the D–valued inner product $\langle x, y\rangle = E_D(x^*y)$, and given $x \in A*_D^{\mathrm{red}} B$, let $\hat{x}$ denote the corresponding element in $L^2(A*_D^{\mathrm{red}} B, E_D)$. Let $\mathcal{H}_A = L^2(A, E_D^A)$ and $\mathcal{H}_B = L^2(B, E_D^B)$ be similarly defined. Then in $L^2(A *_D^{\mathrm{red}} B, E_D)$, the closure of the subspace spanned by elements of the form $(ab)\hat{}$ for $a \in A$ and $b \in B$ is isomorphic to the tensor product $\mathcal{H}_A \otimes_D \mathcal{H}_B$ of Hilbert D–modules. In order to show (14), it will suffice to show

$$(a\tilde{d})\hat{} \otimes \hat{b} \neq \hat{a} \otimes (\tilde{d}b)\hat{}$$

in $\mathcal{H}_A \otimes_D \mathcal{H}_B$. Let $\zeta_B \in \mathcal{H}_B$. Then

$$\langle (a\tilde{d})\hat{} \otimes \zeta_B, (a\tilde{d})\hat{} \otimes \hat{b}\rangle = \langle \zeta_B, (E_D^A(\tilde{d}^*a^*a\tilde{d})b)\hat{}\rangle, \tag{15}$$

$$\langle (a\tilde{d})\hat{} \otimes \zeta_B, \hat{a} \otimes (\tilde{d}b)\hat{}\rangle = \langle \zeta_B, (E_D^A(\tilde{d}^*a^*a)\tilde{d}b)\hat{}\rangle. \tag{16}$$

From assumptions (11) and (12), we obtain $E_D^A(\tilde{d}^*a^*a\tilde{d})b \neq E_D^A(\tilde{d}^*a^*a)\tilde{d}b$. Since E_D^B is faithful, there is $\zeta_B \in \mathcal{H}_B$ such that the right-hand sides of (15) and (16) are not equal. □

Remark 3.3. From the above proof, one sees that the hypotheses of Proposition 3.2 can be weakened as follows: Assumptions (11) and (12) can be dropped, and E_D^B need not be assumed faithful, but instead one must assume

$$E_D^B\big(b^*\big(E_D^A(\tilde{d}^*a^*a\tilde{d}) - E_D^A(\tilde{d}^*a^*a)\tilde{d} - \tilde{d}^*E_D^A(a^*a\tilde{d}) + \tilde{d}^*E_D^A(a^*a)\tilde{d}\big)b^*\big) \neq 0. \tag{17}$$

Note that the LHS of (17) is nothing other than

$$\langle (a\tilde{d})\hat{} \otimes \hat{b} - \hat{a} \otimes (\tilde{d}b)\hat{}, (a\tilde{d})\hat{} \otimes \hat{b} - \hat{a} \otimes (\tilde{d}b)\hat{}\rangle.$$

Corollary 3.4. *Suppose*

$$\begin{array}{ccccc} \widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array} \tag{18}$$

is a commutative diagram of inclusions of C^*–algebras and let $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the resulting $*$–homomorphism of full free product C^*–algebras. Suppose one of the following holds:

1. *$D = 0$,*
2. *$D = \mathbf{C}$, A and B are unital and the inclusions $D \hookrightarrow A$ and $D \hookrightarrow B$ are unital.*

Suppose there are $\tilde{d} \in \widetilde{D}$, $a \in A$ and $b \in B$ such that $a\tilde{d} \in A\backslash\{0\}$, $\tilde{d}b \in B$ and $\tilde{d}b \notin \mathbf{C}b$. Then λ is not injective.

Proof. We can reduce to the case in which (ii) holds by application of Lemma 2.3. We may without loss of generality assume A and B are separable. Letting $E_D^A : A \to \mathbf{C}$ and $E_D^B : B \to \mathbf{C}$ be faithful states, we find that the hypotheses of Proposition 3.2 are satisfied. □

From this corollary, we have the following class of concrete examples, which shows that λ may be non-injective even if

$$B \cap \widetilde{D} = D = A \cap \widetilde{D}. \tag{19}$$

Example 3.5. Let $\mathcal{H}$ be an infinite-dimensional, separable Hilbert space. Inside $B(\mathcal{H})$, let $D = \mathbf{C}1$ and let $A = B = D + K(\mathcal{H})$, where $K(\mathcal{H})$ is the set of compact operators. Let $u \in B(\mathcal{H})$ be a unitary operator that does not belong to D, and let $\widetilde{D} = C^*(u)$, $\widetilde{A} = \widetilde{B} = \widetilde{D} + K(\mathcal{H})$. Let $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the $*$–homomorphism arising from the inclusions (18). Then λ is not injective.

Proof. Take $\tilde{d} = u$ and $a \in K(\mathcal{H}) \backslash \{0\}$. Since $u \notin \mathbf{C}1$, there is $b \in K(\mathcal{H})$ such that $ub \notin \mathbf{C}b$. Now apply Corollary 3.4. One can choose u so that $C^*(u) \cap (\mathbf{C}1 + K(\mathcal{H})) = \mathbf{C}1$, in order to get (19). □

Proposition 3.6. *Suppose*

$$
\begin{array}{ccccc}
\widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\
\uparrow & & \uparrow & & \uparrow \\
A & \hookleftarrow & D & \hookrightarrow & B
\end{array}
\tag{20}
$$

is a commutative diagram of inclusions of C–algebras, and let* $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ *be the resulting $*$–homomorphism of full free product C*–algebras. Suppose one of the following holds:*

(1) $D = 0$,
(2) $D = \mathbf{C}$, *A and B are unital and the inclusions $D \hookrightarrow A$ and $D \hookrightarrow B$ are unital.*

Suppose there are $\tilde{d} \in \widetilde{D}$, $a_1, a_2 \in A$ *and* $b \in B \backslash D$ *such that* $a_1\tilde{d}, \tilde{d}a_2 \in A$, $a_1\tilde{d} \notin \mathbf{C}$ *and* $\tilde{d}b = b\tilde{d}$. *Then λ is not injective.*

Proof. We can reduce to the case in which (ii) holds by application of Lemma 2.3. We use the same notation as in (13). We have

$$
\begin{aligned}
\lambda(\sigma_A(a_1\tilde{d})\sigma_B(b)\sigma_A(a_2)) &= \sigma_{\widetilde{A}}(a_1\tilde{d})\sigma_{\widetilde{B}}(b)\sigma_{\widetilde{A}}(a_2) \\
&= \sigma_{\widetilde{A}}(a_1)\sigma_{\widetilde{B}}(b)\sigma_{\widetilde{A}}(\tilde{d}a_2) = \lambda(\sigma_A(a_1)\sigma_B(b)\sigma_A(\tilde{d}a_2)),
\end{aligned}
$$

and we must only show

$$
\sigma_A(a_1\tilde{d})\sigma_B(b)\sigma_A(a_2) \neq \sigma_A(a_1)\sigma_B(b)\sigma_A(\tilde{d}a_2). \tag{21}
$$

Without loss of generality, assume A and B are separable. Let $\phi_A : A \to \mathbf{C}$ and $\phi_B : B \to \mathbf{C}$ be faithful states. By adding a scalar multiple of the identity, if necessary, we may without loss of generality assume $\phi_B(b) = 0$. Let

$$
(A *_{\mathbf{C}}^{\text{red}} B, \phi) = (A, \phi_A) *_{\mathbf{C}} (B, \phi_B)
$$

be the reduced free product of C*–algebras. Using arguments and notation as in the proof of Proposition 3.2, the closure of the subspace of $L^2(A *_{\mathbf{C}}^{\text{red}} B, \phi)$ spanned by elements of the form $(aba')\hat{}$ for $a, a' \in A$ is isomorphic to $\mathcal{H}_A \otimes (\mathbf{C}\hat{b}) \otimes \mathcal{H}_A$. To show (21), it will suffice to show

$$
(a_1\tilde{d})\hat{} \otimes \hat{b} \otimes \hat{a}_2 \neq \hat{a}_1 \otimes \hat{b} \otimes (\tilde{d}a_2)\hat{}
$$

in $\mathcal{H}_A \otimes (\mathbf{C}\hat{b}) \otimes \mathcal{H}_A$. However, this follows from the assumptions. □

From the above proposition, we get the following example, which requires only "bad" relations between A and $\widetilde{D}$, not between B and $\widetilde{D}$.

Example 3.7. Let D, $\widetilde{D}$, A and $\widetilde{A}$ be as in Example 3.5. Let B be any unital C*–algebra of dimension greater than 1, and let $\widetilde{B} = B \otimes \widetilde{D}$ (for the unique C*–tensor norm). Then the $*$–homomorphism $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ arising from the inclusions (20) is not injective.

Remark 3.8. The problem with injectivity of λ in Examples 3.5 and 3.7 arises already at the algebraic level

$$A *_D^{\mathrm{alg}} B \to \widetilde{A} *_{\widetilde{D}}^{\mathrm{alg}} \widetilde{B}. \tag{22}$$

On the other hand, in Examples 3.1, we can arrange that the map between algebras (22) is injective, while λ fails to be injective, e.g. by taking E to be a reduced free product. However, we do not know of an example where λ fails to be injective and where the algebraic map (22) is injective, but where $A \cap \widetilde{D} = D = B \cap \widetilde{D}$.

4. An application to residual finite-dimensionality

A C*–algebra is said to be residually finite dimensional (r.f.d.) if it has a separating family of finite-dimensional $*$–representations. The first result linking full free products and residual finite dimensionality was M.-D. Choi's proof [6] that the full group C*–algebras of nonabelian free groups are r.f.d. In [7], Exel and Loring proved that the full free product of any two r.f.d. C*–algebras A and B with amalgamation over either the zero C*–algebra or over the scalar multiples of the identity (if A and B are unital) is r.f.d. In [5], N. Brown and Dykema proved that a full amalgamated free product of matrix algebras $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$ over a unital subalgebra D is r.f.d. provided that the normalized traces on $M_k(\mathbf{C})$ and $M_\ell(\mathbf{C})$ restrict to the same trace on D. In this section, we observe that by applying Proposition 2.2, one obtains (as a corollary of the result from [5]) the analogous result for full amalgamated free products of finite-dimensional algebras.

Lemma 4.1. *Let $S = \{x \in \mathbf{R}^n \mid Ax = 0\}$, where A is an $m \times n$ matrix having only rational entries. Then vectors having only rational entries are dense in S.*

Proof. By considering the reduced row-echelon form of A, we see that there is a basis for S consisting of rational vectors. □

Theorem 4.2. *Consider unital inclusions of C*–algebras $A \supseteq D \subseteq B$ with A and B finite dimensional. Let $A *_D B$ be the corresponding full amalgamated free product. Then $A *_D B$ is residually finite dimensional if and only if there are faithful tracial states τ_A on A and τ_B on B whose restrictions to D agree.*

Proof. Since every separable r.f.d. C*–algebra has a faithful tracial state, the necessity of the existence of τ_A and τ_B is clear.

Let us recall some well-known facts about a unital inclusion $D \subseteq A$ of finite-dimensional C*–algebras (see e.g. Chapter 2 of [8]). Let $p_1, \ldots, p_m$ be the minimal central projections of A and $q_1, \ldots, q_n$ the minimal central projections of D. Then the inclusion matrix Λ_D^A is an $m \times n$ integer matrix whose (i,j)th entry is $\mathrm{rank}\,(q_j p_i A q_j)/\mathrm{rank}\,(q_j D)$, where the rank of a matrix algebra $M_k(\mathbf{C})$ is k. To a trace τ on A, we associate the column vector s of length m whose ith entry is the trace of a minimal projection in p_iA. Then the restriction of τ to D has associated column vector $(\Lambda_D^A)^t s$, where the superscript t indicates transpose.

Thus, given $A \supseteq D \subseteq B$ as in the statement of the theorem, the existence of faithful tracial states τ_A and τ_B agreeing on D is equivalent to the existence of column vectors s_A and s_B, none of whose components are zero, such that $(\Lambda_D^A)^t s_A = (\Lambda_D^B)^t s_B$, i.e.,

$$\begin{bmatrix} (\Lambda_D^A)^t, & -(\Lambda_D^B)^t \end{bmatrix} \begin{bmatrix} s_A \\ s_B \end{bmatrix} = 0. \tag{23}$$

Supposing now that such traces τ_A and τ_B exist, by Lemma 4.1 there is a solution $\left[\begin{smallmatrix} s_A \\ s_B \end{smallmatrix}\right]$ to (23) whose entries are all strictly positive and rational. Therefore, the traces τ_A and τ_B agreeing on D can be chosen to take only rational values on minimal projections of A and, respectively, B. Hence there are unital inclusions into matrix algebras,

$$M_k(\mathbf{C}) \supseteq A \supseteq D \subseteq B \subseteq M_\ell(\mathbf{C}),$$

so that τ_A is the restriction of the tracial state on $M_k(\mathbf{C})$ to A and τ_B is the restriction of the tracial state on $M_\ell(\mathbf{C})$ to B. By Proposition 2.2, $A *_D B$ is a subalgebra of $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$. By Theorem 2.3 of [5], $M_k(\mathbf{C}) *_D M_\ell(\mathbf{C})$ is r.f.d. Therefore, $A *_D B$ is r.f.d. □

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