THE POSITIVITY
OF LINEAR FUNCTIONALS ON CUNTZ ALGEBRAS
ASSOCIATED TO UNIT VECTORS

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ABSTRACT. We study the linear functional $\rho$ on the Cuntz algebra $O_n$ associated
to a sequence $\eta_n$ of unit vectors in $\mathbb{C}^n$ that is a generalization of
the Cuntz state. We prove that $\rho$ is positive if and only if
$\eta_n$ is a constant sequence.

1. Introduction

For $n = 2, 3, \ldots$, the Cuntz algebra $O_n$ is a simple infinite $C^\star$-algebra generated
by isometries $s_1, s_2, \ldots, s_n$ satisfying the Cuntz relations $s_i^* s_j = \delta_{ij} I$ and $\sum s_is_i^* = I$ (see [2]). A UHF algebra is a uniformly hyperfinite algebra and a UHF$_n$ algebra
is a UHF algebra with Glimm type $n^\infty$. J. Glimm [3] showed that a UHF algebra is
a $C^\star$-algebra that is the norm closure of an increasing sequence of type $I_{\infty}$-factors.

In other words, UHF can be identified with $\bigotimes_{m=1}^\infty M_{n_m}$, where $M_{n_m}$ is an $n_m \times n_m$
matrix algebra, and UHF$_n$ is the UHF algebra $\bigotimes_{m=1}^\infty M_{n_m}$. Let $D_n$ be the canonical
diagonal subalgebra of UHF$_n$. We consider UHF$_n$ as a subalgebra of $O_n$ and $D_n$ as
a maximal abelian subalgebra of $O_n$. Since $O_n$ is the closure of the linear span of
$s_1, s_2, \ldots, s_k, s_{k+1}, \ldots, s_1$, it follows that a subalgebra UHF$_n$ is the closure of the linear
span of $s_1, s_2, \ldots, s_k, s_{k+1}, \ldots, s_1$, and a subalgebra $D_n$ is the closure of the linear
span of $s_1, s_2, \ldots, s_k, s_{k+1}, \ldots, s_1$.

One part of the study of the Cuntz algebra is representations of $O_n$. Representations
of a $C^\star$-algebra are related to states of it, and irreducible representations
are pure states by GNS constructions. As is known, states are linear
functionals, and linear functionals on abelian algebras and nonabelian algebras
have been studied (see [3], [4], [5]). On the other hand, product pure states of
UHF$_n$ can be described by sequences of unit vectors in $\mathbb{C}^n$, and extensions of these
states to $O_n$ become linear functionals on $O_n$.

In this work, we deal with a natural extension of product pure states of UHF$_n$ associated
to sequences of unit vectors in $\mathbb{C}^n$ to $O_n$. To do this, for a sequence $\{\eta_n\}$
of unit vectors $\eta_n$ in $\mathbb{C}^n$, we will define the associated linear functional $\rho$ on $O_n$.
Then it is natural to ask whether the linear functional \( \rho \) is a state. In fact, states of C*-algebras are positive linear functionals of norm one, and so our concern is the positivity of the associated linear functional \( \rho \) on \( \mathcal{O}_n \). We give a simple proof that if \( \rho \) is positive, then \( \langle \eta_m \rangle \) is a constant sequence.

2. Preliminaries

A state \( \rho \) of a UHF algebra is a product state if \( \rho \) satisfies \( \rho(xy) = \rho(x)\rho(y) \) for \( x \in I \otimes \cdots \otimes I \otimes \mathcal{M}_{n_1} \otimes I \otimes \cdots \) and \( y \in I \otimes \cdots \otimes I \otimes \mathcal{M}_{n_j} \otimes I \otimes \cdots \) \((i \neq j)\). R. T. Powers \[5\] studied product states of UHF. We note that product pure states of a state \( \mathcal{M} \) are monomials in \( O_n \)'s and \( Q_n \)'s and are expressed by \( PQ_n \) with \( P = s_{i_1}s_{i_2} \cdots s_{i_k} \) and \( Q = s_{j_1}s_{j_2} \cdots s_{j_l} \), and the set of all finite sums \( \sum \lambda_i P_i Q_i^* \), where the \( P_i \)'s and \( Q_i \)'s are monomials in \( O_n \) and \( \lambda_i \in \mathbb{C} \), is dense in \( \mathcal{O}_n \). For a matrix \( A \in \mathcal{M}_k \), \( \det A \) denotes the determinant of \( A \), and \( A \) is said to be positive if \( A = C^*C \) for some matrix \( C \), where the adjoint matrix \( C^* = (d_{ij}) \) of \( C = (c_{ij}) \)
is given by $d_{ij} = c_{ji}$. Furthermore, $\mathcal{M}_k(\mathcal{O}_n)$ denotes the set of all $k \times k$ matrices whose elements are in the Cuntz algebra $\mathcal{O}_n$. In addition, we recall that the inner product $\langle \xi, \eta \rangle$ of two vectors $\xi = (\xi^1, \xi^2, \ldots, \xi^n)$ and $\eta = (\eta^1, \eta^2, \ldots, \eta^n)$ in $\mathbb{C}^n$ is given by $\sum \xi^i \bar{\eta^i}$, and we get $|\xi|^2 = \langle \xi, \xi \rangle$.

3. The positivity of the associated linear functional on $\mathcal{O}_n$

In this section, we deal with the linear functional $\rho$ on $\mathcal{O}_n$ associated to a sequence of unit vectors in $\mathbb{C}^n$ and find the necessary and sufficient condition for the positivity of $\rho$. Note that the linear functional $\rho$ on $\mathcal{O}_n$ associated to a constant sequence is the Cuntz state, and so it is positive.

In the following theorem, we prove that the positivity of the linear functional on $\mathcal{O}_n$ associated to a sequence $\langle \eta_m \rangle$ in $\mathbb{C}^n$ implies that $\langle \eta_m \rangle$ is a constant sequence. But, since $\sum s_i s_i^* = I$ and so

$$\rho(I) = \rho(\sum s_i s_i^*) = \sum \rho(s_i s_i^*) = \sum \eta_i^* \bar{\eta_i} = |\eta_i|^2 = 1,$$

the positivity of $\rho$ is equivalent to $\rho$ being a state. Consequently, we show that, for the linear functional $\rho$ on $\mathcal{O}_n$ associated to a sequence of unit vectors in $\mathbb{C}^n$, if $\rho$ is a state, then the sequence is constant.

**Theorem 3.1.** For a sequence $\langle \eta_m \rangle$ of unit vectors $\eta_m \in \mathbb{C}^n$, let $\rho$ be the associated linear functional on $\mathcal{O}_n$. If $\rho$ is positive, then $\langle \eta_m \rangle$ is a constant sequence.

To prove our theorem we need the following lemma.

For two vectors

$$v = (v_1, v_2, \ldots, v_n), w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$$

and $a, b, c, d \in \mathbb{C}$, let us consider a matrix $A \in \mathcal{M}_{n+2}$ given by

$$A = \begin{pmatrix}
a & b & v_1 & v_2 & \cdots & v_n \\
c & d & w_1 & w_2 & \cdots & w_n \\
v_1 & w_1 & 1 & 0 & \cdots & 0 \\
v_2 & w_2 & 0 & 1 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
v_n & w_n & 0 & \cdots & 0 & 1
\end{pmatrix}.$$  

**Lemma 3.2.** In the above notation, the determinant $\det A$ of $A$ is

$$(a - |v|^2)(d - |w|^2) - (b - \langle v, w \rangle)(c - \langle w, v \rangle).$$
Proof. We have
\[
\begin{vmatrix}
a - v_n v_n & b - w_n v_n & v_1 & v_2 & \cdots & v_{n-1} \\
c - v_n w_n & d - w_n w_n & w_1 & w_2 & \cdots & w_{n-1} \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{vmatrix}
\]

Let
\[
\begin{vmatrix}
a - v_n v_n - v_{n-1} v_{n-1} & b - w_n v_n - w_{n-1} v_{n-1} & v_1 & v_2 & \cdots & v_{n-2} \\
c - v_n w_n - v_{n-1} w_{n-1} & d - w_n w_n - w_{n-1} w_{n-1} & w_1 & w_2 & \cdots & w_{n-2} \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{vmatrix}
\]

By the definition of \(\rho\), we have
\[
|\rho(P)|^2 = \rho(\sum_{P \in S_{k-1}} PP^*) = 1.
\]

Thus we can take a monomial \(P \in S_{k-1}\) with \(\rho(P) = \lambda (\neq 0)\). Now we consider \(n + 2\) monomials in \(O_n\) as follows:
\[
P_1 = I, \quad P_2 = P, \quad P_3 = Ps_1, \quad P_4 = Ps_2, \quad \ldots, \quad P_{n+2} = Ps_n.
\]

The fact that \(P_i P = I\) and the definition of \(\rho\) give that the matrix \((\rho(P_i^* P_j))\) in \(M_{k+2}\) is of the form
\[
\begin{pmatrix}
1 & \lambda & \lambda \eta_1^1 & \lambda \eta_1^2 & \cdots & \lambda \eta_1^n \\
\lambda & 1 & \eta_2^1 & \eta_2^2 & \cdots & \eta_2^n \\
\lambda \eta_1^1 & \eta_2^1 & 1 & 0 & \cdots & 0 \\
\lambda \eta_1^2 & \eta_2^2 & 0 & 1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\lambda \eta_1^n & \eta_2^n & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

On the other hand, the fact that
\[
(P_i^* P_j) = (P_1, \ldots, P_k)^*(P_1, \ldots, P_k) \in M_k(O_n)
\]
implies that the matrix \((P_i^* P_j)\) is positive. As is known, since a positive linear functional on a \(C^*\)-algebra is completely positive, the positivity of \(\rho\) gives that the
matrix $(\rho(P_i^*P_j)) = k((x_{ij})) = \rho(x_{ij}) \in \mathcal{M}_k$ for all $(x_{ij}) \in \mathcal{M}_k(\mathcal{O}_n).$

From the positivity of $(\rho(P_i^*P_j))$, we get $\det(\rho(P_i^*P_j)) \geq 0$, and by Lemma 3.2, we have the following inequality:

$$\det((\rho(P_i^*P_j))) = 0,$$

Since $|\eta_1| = 1$, the above inequality is equal to $\lambda - (\lambda \eta_k, \eta_1) \geq 0$. Thus we obtain $|\lambda|^2(1 - (\eta_k, \eta)) = 0$. Therefore $\eta_1$ and $\eta_k$ are two unit vectors with $(\eta_k, \eta_1) = 1$, which implies $\eta_1 = \eta_k$.

Hence, for any $k = 2, 3, \ldots$, we have $\eta_1 = \eta_k$, which implies that $\langle \eta_m \rangle$ is a constant sequence.

We have already noted that for a unit vector $\eta \in \mathbb{C}^n$ the Cuntz state $\omega_\eta$ is the linear functional on $\mathcal{O}_n$ associated to a constant sequence $\langle \eta \rangle$. Conversely, thanks to Theorem 3.1, we conclude that for the linear functional $\rho$ on $\mathcal{O}_n$ associated to a sequence of unit vectors in $\mathbb{C}^n$, if $\rho$ is a state, then $\rho$ must be a Cuntz state. Thus we obtain the following result.

**Corollary 3.3.** Let $\rho$ be an associated linear functional on $\mathcal{O}_n$. Then $\rho$ is a state if and only if $\rho$ is the Cuntz state.

**References**