

DECOMPOSITION OF AN ORDER ISOMORPHISM BETWEEN MATRIX-ORDERED HILBERT SPACES

YASUhide MIURA

(Communicated by David R. Larson)

ABSTRACT. The purpose of this note is to show that any order isomorphism between noncommutative L^2 -spaces associated with von Neumann algebras is decomposed into a sum of a completely positive map and a completely co-positive map. The result is an L^2 version of a theorem of Kadison for a Jordan isomorphism on operator algebras.

1. INTRODUCTION

In the theory of operator algebras, a notion of self-dual cones was studied by A. Connes [1], and he characterized a standard Hilbert space. In [9] L. M. Schmitt and G. Wittstock introduced a matrix-ordered Hilbert space to handle a non-commutative order and characterized it using the face property of the family of self-dual cones. From the point of view of the complete positivity of the maps, we shall consider a decomposition theorem of an order isomorphism on matrix-ordered Hilbert spaces.

Let \mathcal{H} be a Hilbert space over a complex number field \mathbb{C} , and let \mathcal{H}^+ be a self-dual cone in \mathcal{H} . A set of all $n \times n$ matrices is denoted by M_n . Put $\mathcal{H}_n = \mathcal{H} \otimes M_n (= M_n(\mathcal{H}))$ for $n \in \mathbb{N}$. Suppose that $(\mathcal{H}, \mathcal{H}_n^+, n \in \mathbb{N})$ and $(\tilde{\mathcal{H}}, \tilde{\mathcal{H}}_n^+, n \in \mathbb{N})$ are matrix-ordered Hilbert spaces. A linear map A of \mathcal{H} into $\tilde{\mathcal{H}}$ is said to be n -positive (resp. n -co-positive) when the multiplicity map $A_n (= A \otimes \text{id}_n)$ satisfies $A_n \mathcal{H}_n^+ \subset \tilde{\mathcal{H}}_n^+$ (resp. ${}^t(A_n \mathcal{H}_n^+) \subset \tilde{\mathcal{H}}_n^+$). Here ${}^t(\cdot)$ denotes a set of all transposed matrices. When A is n -positive (resp. n -co-positive) for all $n \in \mathbb{N}$, A is said to be completely positive (resp. completely co-positive). We refer mainly to [11] for standard results in the theory of operator algebras. We use the notation as introduced in [9] with respect to matrix-ordered standard forms.

2. RESULTS

We first generalize a theorem of A. Connes [1] for the polar decomposition of an order isomorphism to the case where a von Neumann algebra is non- σ -finite.

Received by the editors March 6, 2003.

2000 *Mathematics Subject Classification.* Primary 46L10, 46L40.

Key words and phrases. Order isomorphism, completely positive map, matrix-ordered Hilbert space.

This research was partially supported by the Grants-in-Aid for Scientific Research, The Ministry of Education, Culture, Sports, Science and Technology, Japan.

Proposition 1. *Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \tilde{J}, \tilde{\mathcal{H}}^+)$ be standard forms, and let A be a linear bijection of \mathcal{H} onto $\tilde{\mathcal{H}}$ satisfying $A\mathcal{H}^+ = \tilde{\mathcal{H}}^+$. Then for a polar decomposition $A = U|A|$ of A we obtain the following properties:*

- (1) *There exists a unique invertible operator B in \mathcal{M}^+ such that $|A| = BJB$ (cf. [4, Corollary II.3.2]).*
- (2) *There exists a unique Jordan $*$ -isomorphism α of \mathcal{M} onto $\tilde{\mathcal{M}}$ such that*

$$(\alpha(X)\xi, \xi) = (XU^{-1}\xi, U^{-1}\xi)$$

for all $X \in \mathcal{M}, \xi \in \tilde{\mathcal{H}}^+$.

Proof. (1) Let \mathcal{M} be non- σ -finite. Choose an increasing net $\{p_i\}_{i \in \mathbf{I}}$ of σ -finite projections in \mathcal{M} converging strongly to 1. Put $q_i = p_i J p_i J$. By [1, Theorem 4.2] $q_i \mathcal{H}^+$ is a closed face of \mathcal{H}^+ . Since A is an order isomorphism, $A(q_i \mathcal{H}^+)$ is a closed face of $\tilde{\mathcal{H}}^+$. There then exists a σ -finite projection $p'_i \in \tilde{\mathcal{M}}$ such that $A(q_i \mathcal{H}^+) = q'_i \tilde{\mathcal{H}}^+$ where q'_i denotes $p'_i J p'_i J$. Hence $q'_i A q_i$ is an order isomorphism of $q_i \mathcal{H}^+$ onto $q'_i \tilde{\mathcal{H}}^+$. These cones appear respectively in the reduced standard forms $(q_i \mathcal{M} q_i, q_i \mathcal{H}, q_i J q_i, q_i \mathcal{H}^+)$ and $(q'_i \tilde{\mathcal{M}} q'_i, q'_i \tilde{\mathcal{H}}, q'_i J q'_i, q'_i \tilde{\mathcal{H}}^+)$. Put $A_i = (q'_i A q_i)^* q'_i A q_i$. Then $A_i \in q_i \mathcal{M}^+ q_i$ is an order automorphism on $q_i \mathcal{H}^+$. By [3, Theorem 3.3] there exists a unique invertible operator $B_i \in q_i \mathcal{M}^+ q_i$ such that $A_i = B_i J_i B_i J_i$, where J_i denotes $q_i J q_i$. Taking a logarithm of both sides, we have $\log A_i = \log B_i + J_i(\log_{q_i} B_i) J_i$. Since $\{A_i\}$ is bounded, $\{\log B_i\}$ is bounded. Indeed, we have in a standard form that a map

$$X \mapsto \delta_X = \frac{1}{2}(X + JXJ)$$

is a Jordan isomorphism of a selfadjoint part of \mathcal{M} into a selfadjoint part of a set of all order derivations $D(\mathcal{H}^+)$ by [4, Corollary VI.2.3]. It is known that any isomorphism of a JB-algebra into another JB-algebra is an isometry (see [3, Proposition 3.4.3]). Hence

$$\|\delta_X\| = \|X\|, \quad X \in \mathcal{M}_{\text{s.a.}}$$

Thus $\{\log B_i\}$ is bounded. It follows that $\{p_i B_i p_i\}$ is bounded because $p_i \mathcal{M} p_i$ and $q_i \mathcal{M} q_i$ are $*$ -isomorphic. Therefore, one can find a subnet of $\{p_i \log B_i p_i\}$ that converges to some element $C \in \mathcal{M}^+$ in the σ -weak topology. We may index the subnet as the same $i \in \mathbf{I}$. We then have for $\xi, \eta \in \mathcal{H}$,

$$\begin{aligned} ((C + J C J) q_j \xi, q_j \eta) &= \lim_i ((p_i (\log B_i) p_i + J p_i (\log B_i) p_i J) q_j \xi, q_j \eta) \\ &= ((\log B_j + J_j (\log B_j) J_j) q_j \xi, q_j \eta) \\ &= \lim_i (\log A_i q_j \xi, q_j \eta) \\ &= (\log A^* A q_j \xi, q_j \eta), \end{aligned}$$

using the facts that $q_i X q_i J q_i X q_i J q_i = p_i X p_i J p_i X p_i J p_i$ for all $X \in \mathcal{M}$, and under the strong topology $\{A_i\}$ converges to $A^* A$; hence $\{q_i (\log A_i) q_i\}$ converges to $\log A^* A$. Since $\bigcup_{i \in \mathbf{I}} q_i \mathcal{H}$ is dense in \mathcal{H} , we obtain the equality $C + J C J = \log A^* A$. Therefore, $e^C J e^C J = A^* A$. Thus there exists an element $B \in \mathcal{M}^+$ such that $|A| = BJB$. Since, in addition, $q_i B q_i J q_i B q_i J q_i = q_i |A| q_i$, one easily sees the invertibility and the unicity of B using the same properties as in the σ -finite case.

(2) From (1) we have $U = AB^{-1}JB^{-1}J$. It follows that U is an isometry satisfying $U\mathcal{H}^+ = \tilde{\mathcal{H}}^+$. Let p_i and q_i be as in (1). There then exists a σ -finite projection $p'_i \in \tilde{\mathcal{M}}$ such that $U(q_i \mathcal{H}^+) = q'_i \tilde{\mathcal{H}}^+$ with $q'_i = p'_i J p'_i J$. Using also [1,

Theorem 3.3], one can find a unique Jordan $*$ -isomorphism α_i of $q_i\mathcal{M}q_i$ onto $q'_i\tilde{\mathcal{M}}q'_i$ such that

$$(\alpha_i(q_iXq_i)\xi, \xi) = (q_iXq_iU^{-1}\xi, U^{-1}\xi)$$

for all $X \in \mathcal{M}, \xi \in q'_i\tilde{\mathcal{H}}^+$. Now fix $X \in \mathcal{M}_{s.a.}$. Since $p'_i\tilde{\mathcal{M}}p'_i$ and $q'_i\tilde{\mathcal{M}}q'_i$ are $*$ -isomorphic, there exists a unique operator $Y_i \in p'_i\tilde{\mathcal{M}}_{s.a.}p'_i$ such that $Y_i|_{q'_i\tilde{\mathcal{H}}} = \alpha_i(q_iXq_i)$. Using an isometry between the Jordan algebras, one sees that $\{\alpha_i(q_iXq_i)\}$ is bounded, because $\|\alpha_i(q_iXq_i)\| = \|q_iXq_i\| \leq \|X\|, i \in \mathbf{I}$. Thus $\{Y_i\}$ is bounded. We may then say that $\{Y_i\}$ converges to some operator $Y \in \tilde{\mathcal{M}}_{s.a.}$ in the σ -weak topology. We then have for $\xi \in \tilde{\mathcal{H}}^+$,

$$\begin{aligned} (Yq'_j\xi, q'_j\xi) &= \lim_i (Y_iq'_j\xi, q'_j\xi) = \lim_i (\alpha_i(q_iXq_i)q'_j\xi, q'_j\xi) \\ &= \lim_i (q_iXq_iU^{-1}q'_j\xi, U^{-1}q'_j\xi) \\ &= (XU^{-1}q'_j\xi, U^{-1}q'_j\xi). \end{aligned}$$

Taking a limit with respect to j , we obtain

$$(Y\xi, \xi) = (XU^{-1}\xi, U^{-1}\xi)$$

for all $\xi \in \tilde{\mathcal{H}}^+$. It is known that any normal state on the von Neumann algebra $\tilde{\mathcal{M}}$ is represented by a vector state with respect to an element of $\tilde{\mathcal{H}}^+$ (see [2, Lemma 2.10 (1)]). Therefore, the above element Y is uniquely determined. Moreover, we have $q'_iYq'_i = \alpha_i(q_iXq_i)$. It follows that $\{\alpha_i(q_iXq_i)\}$ converges to Y in the strong topology. Hence one can define $\alpha(X) = Y$ for all $X \in \mathcal{M}$. It is now immediate that $\alpha(X^2) = \alpha(X)^2$ for all $X \in \mathcal{M}_{s.a.}$. Considering the inverse order isomorphism U^{-1} , we have $\alpha(\mathcal{M}) = \tilde{\mathcal{M}}$. This completes the proof. \square

In the following lemma we deal with a reduced matrix-ordered standard form by a completely positive projection.

Lemma 2. *With $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ a matrix-ordered standard form, let E be a completely positive projection on \mathcal{H} . Then $(EME, E\mathcal{H}, E_n\mathcal{H}_n^+)$ is a matrix-ordered standard form.*

Proof. The statement was shown in [8, Lemma 3] where \mathcal{M} is σ -finite. In the case where \mathcal{M} is not σ -finite, since E is a completely positive projection, there exists a von Neumann algebra \mathcal{N} such that $(\mathcal{N}, E\mathcal{H}, E_n\mathcal{H}_n^+)$ is a matrix-ordered standard form by [6, Lemma 3]. Hence $EM|_{E\mathcal{H}} = \mathcal{N}$ and $(EME, E\mathcal{H}, E_n\mathcal{H}_n^+)$ is a matrix-ordered standard form by using the same discussion as in the proof in [7]. \square

Now, we shall state the decomposition theorem for an order isomorphism between noncommutative L^2 -spaces.

Theorem 3. *Let $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \tilde{\mathcal{H}}_n^+)$ be matrix-ordered standard forms. Suppose that A is a 1-positive map of \mathcal{H} into $\tilde{\mathcal{H}}$ such that $A\mathcal{H}^+$ is a self-dual cone in the closed range of A . If both the support projection E and the range projection F of A are completely positive, then there exists a central projection P of EME such that AP is completely positive and $A(E - P)$ is completely co-positive.*

In particular, if A is an order isomorphism of \mathcal{H} onto $\tilde{\mathcal{H}}$, then there exists a central projection P of \mathcal{M} such that AP is completely positive and $A(1 - P)$ is completely co-positive.

Proof. We first consider the case where A is an order isomorphism. Let U, B and α be as in Proposition 1. It follows from a theorem of Kadison [5] that there exists a central projection P of \mathcal{M} satisfying

$$\alpha : \mathcal{M}_P \rightarrow \tilde{\mathcal{M}}_{\alpha(P)}, \text{ onto } *- \text{isomorphism}$$

and

$$\alpha : \mathcal{M}_{1-P} \rightarrow \tilde{\mathcal{M}}_{\alpha(1-P)}, \text{ onto } *- \text{anti-isomorphism.}$$

Indeed, $\alpha(P)$ is a central projection of $\tilde{\mathcal{M}}$. Since α preserves a $*$ -operation and power, $\alpha(P)$ is a projection. Suppose that Q is an arbitrary projection in \mathcal{M} . Since α is order preserving, we have $\alpha(QP) \leq \alpha(P)$ and $\alpha(Q(1-P)) \leq \alpha(1-P)$. It follows that two projections $\alpha(P)$ and $\alpha(QP)$ are commutative, and so are $\alpha(1-P)$ and $\alpha(Q(1-P))$. Hence, $\alpha(Q) = \alpha(QP + Q(1-P))$ and $\alpha(P)$ commute. Since α is bijective, a set $\alpha(Q)$ generates a von Neumann algebra $\tilde{\mathcal{M}}$. Therefore, $\alpha(P)$ belongs to a center of $\tilde{\mathcal{M}}$. Now, there then exists a unique completely positive isometry $u : P\mathcal{H} \rightarrow \alpha(P)\tilde{\mathcal{H}}$ such that

$$u(P\mathcal{H}^+) = \alpha(P)\tilde{\mathcal{H}}^+ \quad \text{and} \quad \alpha(x) = u x u^{-1}, \quad x \in \mathcal{M}_P$$

by [7, Proposition 2.4], which is also valid for the non- σ -finite case. Hence $(UxU^{-1}\xi, \xi) = (uxu^{-1}\xi, \xi), x \in \mathcal{M}_P, \xi \in \alpha(P)\tilde{\mathcal{H}}^+$. We have from the unicity of a completely positive isometry that $UP = u$. Note that $\alpha(P)UP = UP$. Indeed, we have for $\xi \in \alpha(1-P)\tilde{\mathcal{H}}^+$ the equality

$$\|PU^{-1}\xi\|^2 = (UPU^{-1}\xi, \xi) = (\alpha(P)\xi, \xi) = 0.$$

This yields $PU^{-1}\alpha(1-P) = O$, and so $PU^{-1} = PU^{-1}\alpha(P)$. Therefore, we obtain that $AP = UBJBJP = uBJBJP$ and AP is completely positive.

We next consider a $*$ -isomorphism $\alpha' : \mathcal{M}_{1-P} \rightarrow \tilde{\mathcal{M}}'_{1-\alpha(P)}$ defined by $\alpha'(X) = \tilde{J}\alpha(X)^*\tilde{J}, X \in \mathcal{M}_{1-P}$. There then exists a unique completely positive isometry $v : (1-P)\mathcal{H} \rightarrow \alpha(1-P)\tilde{\mathcal{H}}$ such that

$$v(1-P)\mathcal{H}^+ = (1-\alpha(P))\tilde{\mathcal{H}}^+ \quad \text{and} \quad \alpha'(x) = v x v^{-1}, \quad x \in \mathcal{M}_{1-P}.$$

Then we have $\alpha(x) = \tilde{J}v x^* v^{-1} \tilde{J}, x \in \mathcal{M}_{1-P}$. Note that the complete positivity above means $v_n(1-P)_n \mathcal{H}_n^+ = (1-\alpha(P))_n \tilde{\mathcal{H}}_n^{+t}$, where $\tilde{\mathcal{H}}_n^{+t}$ denotes the self-dual cones associated with $\tilde{\mathcal{M}}$ (cf. [10]). Hence v is a completely co-positive map under the setting $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \tilde{\mathcal{H}}_n^+)$. Hence

$$\begin{aligned} (UxU^{-1}\xi, \xi) &= (\tilde{J}v x^* v^{-1} \tilde{J}\xi, \xi) \\ &= (\tilde{J}\xi, v x^* v^{-1} \tilde{J}\xi) \\ &= (v x v^{-1}\xi, \xi) \end{aligned}$$

for all $x \in \mathcal{M}_{1-P}, \xi \in (1-P)\mathcal{H}^+$. It follows that $U(1-P) = v$. We conclude by the equality $A(1-P) = vBJB(1-P)$ that $A(1-P)$ is completely co-positive.

We now consider a general A . Since $A\mathcal{H}^+ \subset \tilde{\mathcal{H}}^+$, we have $A\mathcal{H}^+ \subset F\tilde{\mathcal{H}}^+$. Since F is a projection, $F\tilde{\mathcal{H}}^+$ is a self-dual cone in $F\tilde{\mathcal{H}}$. It follows from the self-duality of $A\mathcal{H}^+$ that $A\mathcal{H}^+ = F\tilde{\mathcal{H}}^+$. This yields from Lemma 2 that FAE is an order isomorphism of $E\mathcal{H}$ onto $F\tilde{\mathcal{H}}$ in the sense of matrix-ordered standard forms $(EME, E\mathcal{H}, E_n\mathcal{H}_n^+)$ and $(F\tilde{\mathcal{M}}F, F\tilde{\mathcal{H}}, F_n\tilde{\mathcal{H}}_n^+)$. Using the first part of the proof, we obtain the desired result. Indeed, there exists a central projection $P \in EME$

such that FAP is completely positive and $FA(E - P)$ is completely co-positive under the reduced matrix-ordered standard forms. We obtain the inclusion

$${}^t(A_n(E_n - P_n)\mathcal{H}_n^+) = {}^t(F_n A_n(E_n - P_n)\mathcal{H}_n^+) \subset F_n \tilde{\mathcal{H}}_n^+ \subset \tilde{\mathcal{H}}_n^+.$$

This completes the proof. \square

Finally, the author wishes to express his sincere gratitude to Professor Y. Katayama for having pointed out this problem to him. He also thanks the members of Sendai Seminar, especially Professors T. Okayasu and K. Saitô for their useful suggestions.

REFERENCES

- [1] A. Connes, *Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann*, Ann. Inst. Fourier (Grenoble) **24** (1974), 121–155. MR **51**:13705
- [2] U. Haagerup, *The standard form of von Neumann algebras*, Math. Scand. **37** (1975), 271–283. MR **53**:11387
- [3] H. Hanche-Olsen and E. Størmer, *Jordan Operator Algebras*, Monographs and Studies in Mathematics, vol. 21, Pitman, Boston-London-Melbourne, 1984. MR **86a**:46092
- [4] B. Iochum, *Cônes Autopolaires et Algèbres de Jordan*, Lecture Notes in Mathematics, 1049, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984. MR **86m**:46067
- [5] R. V. Kadison, *Isometries of operator algebras*, Ann. of Math. (2) **54** (1951), 325–338. MR **13**:256a
- [6] Y. Miura, *On a completely positive projection on a non-commutative L^2 -space*, Far East J. Math. Sci. **5** (1997), 521–530. MR **99j**:46077
- [7] Y. Miura, *Complete order isomorphisms between non-commutative L^2 -spaces*, Math. Scand. **87** (2000), 64–72. MR **2001i**:46092
- [8] Y. Miura and K. Nishiyama, *Complete orthogonal decomposition homomorphisms between matrix ordered Hilbert spaces*, Proc. Amer. Math. Soc. **129** (2001), 1137–1141. MR **2002a**:46085
- [9] L. M. Schmitt and G. Wittstock, *Characterization of matrix-ordered standard forms of W^* -algebras*, Math. Scand. **51** (1982), 241–260. MR **84i**:46062
- [10] ———, *Kernel representation of completely positive Hilbert-Schmidt operators on standard forms*, Arch. Math. **38** (1982), 453–458. MR **84a**:46132
- [11] M. Takesaki, *Theory of Operator Algebras, I*, Springer-Verlag, New York-Heidelberg-Berlin, 1979. MR **81e**:46038

DEPARTMENT OF MATHEMATICS, FACULTY OF HUMANITIES AND SOCIAL SCIENCES, IWATE UNIVERSITY, MORIOKA, 020-8550, JAPAN

E-mail address: ymiura@iwate-u.ac.jp