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THE RUSSO-DYE THEOREM IN A WEAKLY CLOSED $\mathcal{T}(\mathcal{N})$–MODULE

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ABSTRACT. Suppose that $\mathcal{N}$ is admissible. It is shown that the convex hull of unitary elements of a weakly closed $\mathcal{T}(\mathcal{N})$–module $\mathcal{U}$ contains the whole unit ball of $\mathcal{U}$ if and only if $\widetilde{I} = I$ and for any $N > 0$, $\widetilde{N} > 0$.

1. INTRODUCTION

One of the well-known results in the theory of C^*-algebras is the Russo-Dye Theorem [8]: the closed convex hull of the unitary group of any C^*-algebra is the closed unit ball. A number of refinements on this theorem led to the Kadison-Pederson's result [5] that every strict contraction of a C^*-algebra is the mean of a finite number of unitary elements of the C^*-algebra.

In non-selfadjoint algebras, the first surprising fact resulting from the Russo-Dye Theorem was proved by Anoussis and Katsoulis [1], that is: for any nest with no finite atoms, every strict contraction in the nest algebra is the average of unitary elements in the algebra. In [2], Davidson established that the Russo-Dye Theorem holds in a nest algebra if and only if $\mathcal{N}$ is admissible. The Russo-Dye Theorem has proved very useful. It provides one means of reducing the study of a non-normal element to that of normal (unitary) elements—and the device is reasonably sensitive to norm estimates. The main purpose of this paper is to establish the Russo-Dye Theorem in a weakly closed $\mathcal{T}(\mathcal{N})$–module. To our knowledge, this is the first result of this type for a non-algebraic operator system. Though the exposition and structure of this paper follows closely those in [2], the main result of this paper is unexpected.

Let us introduce some notation and terminology. $\mathcal{H}$ represents a complex separable infinite-dimensional Hilbert space, and $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. A nest $\mathcal{N}$ is a chain of closed subspaces of Hilbert space $\mathcal{H}$ containing (0) and $\mathcal{H}$ which is closed under intersection and closed span. If N is an element of a nest $\mathcal{N}$, then N_- denotes the immediate predecessor of N. Similarly we define N_+ as the immediate successor of N. Say that a nest is admissible if 0_+ and $I_-^\perp$ are either zero or of infinite rank. If $\mathcal{N}$ is a nest, then the nest algebra $\mathcal{T}(\mathcal{N})$ is the set of all operators T such that $TN \subseteq N$ for every element N in $\mathcal{N}$.

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Suppose that $N \to \widetilde{N}$ is an order homomorphism of $\mathcal{N}$ into itself (that is, $N \le N^{'}$ implies $\widetilde{N} \le \widetilde{N'}$). Then the set

$$\mathcal{U} = \{X \in \mathcal{B}(\mathcal{H}) : XN \subseteq \widetilde{N}, \forall N \in \mathcal{N}\}$$

is clearly a weakly closed (two sided) $\mathcal{T}(\mathcal{N})$–module. In [4], Erdos and Power show that every weakly closed $\mathcal{T}(\mathcal{N})$–module is of the above form, where the order homomorphism $N \to \widetilde{N}$ is left continuous and $\widetilde{0} = 0$. For each $N \in \mathcal{N}$, define

$$N_* = \bigwedge\{\widetilde{N'} : N^{'} > N\}$$

and

$$N_\sim = \bigvee\{N^{'} : \widetilde{N'} < N\}.$$

Since $\mathcal{N}$ is complete, N_* and $N_\sim$ are in $\mathcal{N}$.

An interval of $\mathcal{N}$ is a projection $E = N_2 - N_1$, where $N_1 < N_2$ belong to $\mathcal{N}$. The support $supp(e)$ in $\mathcal{N}$ of a vector e is the smallest interval E such that $Ee = e$. Similarly the support of a projection in the diagonal algebra $\mathcal{N}^{'}$ is the smallest interval supporting every vector of its range. If E, F are two intervals and $\mathcal{U}$ is a weakly closed $\mathcal{T}(\mathcal{N})$–module, then we say that $E \preceq F$ according to $\mathcal{U}$ if $E\mathcal{U}F = E\mathcal{B}(\mathcal{H})F$. Note that $N_* \preceq N^\perp$, $N \preceq N_\sim^\perp$ and $\widetilde{N} \preceq N_-^\perp$ according to $\mathcal{U}$.

The notation $x \otimes y$ denotes the rank one operator $(x \otimes y)(z) = (z, y)x$ for $z \in \mathcal{H}$.

Throughout this paper, let $\mathcal{T}(\mathcal{N})$ be the nest algebra associated with a nest $\mathcal{N}$. Let $\mathcal{U}$ be the weakly closed $\mathcal{T}(\mathcal{N})$–module determined by a left continuous order homomorphism $N \to \widetilde{N}$ with $\widetilde{0} = 0$.

2. THE RUSSO-DYE THEOREM

The main theorem can now be stated.

Theorem 1. *Suppose that $\mathcal{N}$ is admissible. Then the following statements are equivalent:*

1) *every element $A \in \mathcal{U}$ such that $\| A \| \le 1 - \frac{1}{n}$ is the mean of $16n^2$ unitary elements of $\mathcal{U}$;*

2) *the convex hull of these unitary elements in $\mathcal{U}$ contains the whole open unit ball of $\mathcal{U}$, thus the Russo-Dye Theorem holds in $\mathcal{U}$;*

3) $\widetilde{I} = I$ *and for any* $N > 0$, $\widetilde{N} > 0$.

Proof. We first prove 1) ⇒ 2) and 2) ⇒ 3).

1) ⇒ 2). Obvious.

2) ⇒ 3). Suppose that 3) is not true. Then $\widetilde{I} < I$ or there exists an element $N > 0$ in $\mathcal{N}$ and $\widetilde{N} = 0$. We deal with the two cases separately.

i) If $\widetilde{I} < I$, let U be a unitary element of $\mathcal{U}$. Thus $UI \subseteq \widetilde{I}$ and $U^*\widetilde{I}^\perp \subseteq I^\perp = 0$, which contradicts the fact that U^* is a unitary operator. This contradiction shows that there is no unitary operator in $\mathcal{U}$ if $\widetilde{I} < I$.

ii) If there is an element $N > 0$ and $\widetilde{N} = 0$, a similar argument shows that there exists no unitary operator in $\mathcal{U}$.

If 3) is not true, it follows from i) and ii) that there is no unitary operator in $\mathcal{U}$. This shows 2) does not hold.

The difficulties of the proof lie in 3) ⇒ 1). To prove this, we need to establish some auxiliary results first.

Lemma 2. *Suppose that* $\mathcal{U}$ *satisfies condition* 3) *in Theorem* 1 *and* Q *is a diagonal projection of* $\mathcal{T}(\mathcal{N})$*. Let* $\{f_k\}$ *be an orthonormal set such that infinitely many* f_k*'s belong to* N *for any every* $N > 0$ *in* $\mathcal{N}$*. Then there is a partial isometry* S *in* $\mathcal{U}$ *such that* $S^*S = Q$ *and its range is spanned by a subset of the* f_k*'s.*

Proof. If $E = 0_* = \bigwedge\{\widetilde{N} : N > 0\} \neq 0$, then $\mathcal{F} = \{f_k : f_k = Ef_k\}$ is infinite and E is of infinite rank. In this case, we construct a partial isometry S in $\mathcal{B}(\mathcal{H})$ with $S^*S = Q$ and range spanned by a subset of $\mathcal{F}$. For any $N \in \mathcal{N}\backslash(0)$, it follows from the definition of 0_* that $\widetilde{N} \geq 0_* = E$. Hence $SN \subseteq E \subseteq \widetilde{N}$, so $S \in \mathcal{U}$. Now we suppose that $0_* = 0$. If $0_+ \neq 0$, $0_* = \widetilde{0_+} > 0$ by condition 3) in Theorem 1. This shows that $0_+ = 0$. Thus there is a sequence $\{N_k\}$ in $\mathcal{N}$ strictly decreasing with limit 0. Choose an orthonormal basis for each $(N_k - N_{k+1})Q\mathcal{H}$ starting with $N_0 = I$, and combine them to form an orthonormal basis $\{e_k : k \geq 1\}$ for $Q\mathcal{H}$ with the property that $e_k = M_k^{\perp}e_k$ for some $M_k \in \mathcal{N}$ with $M_k > 0$. Since $M_k > 0$, $\widetilde{M_k} > 0$. Hence we can choose an increasing sequence $\{n_k\}$ such that f_{n_k} is contained in $\widetilde{M_k}$ for $k \geq 1$. The operator $S = \sum\limits_{k\geq 1} f_{n_k} \otimes e_k \in \mathcal{U}$ is the desired partial isometry. □

Lemma 3. *Suppose that* $\mathcal{U}$ *satisfies condition* 3) *in Theorem* 1 *and* P *is a diagonal projection in* $\mathcal{T}(\mathcal{N})$*. Let* $\{e_k\}$ *be an orthonormal set such that infinitely many* e_k*'s belong to* $N^{\perp}$ *for every* $N < I$ *in* $\mathcal{N}$*. Then there is a partial isometry* S *in* $\mathcal{U}$ *such that* $SS^* = P$ *and its domain is spanned by a subset of the* e_k*'s.*

Proof. If $I \neq I_{\sim} = \bigvee\{N : \widetilde{N} < I\}$, then $\mathcal{E} = \{e_k : I_{\sim}^{\perp}e_k = e_k\}$ is of infinite rank. Hence we can construct a partial isometry S in $\mathcal{B}(\mathcal{H})$ with $SS^* = P$ and its domain spanned by a subset of $\mathcal{E}$. For any $N \in \mathcal{N}$, if $\widetilde{N} = I$, then $S^*\widetilde{N}^{\perp} = (0) \subseteq N^{\perp}$; if $\widetilde{N} < I$, then $N \leq I_{\sim}$ and $I_{\sim}^{\perp} \leq N^{\perp}$. Since $Ran(S^*) = ker(S)^{\perp} \subseteq I_{\sim}^{\perp}$, $S^*\widetilde{N}^{\perp} \subseteq I_{\sim}^{\perp} \subseteq N^{\perp}$. Hence for any $N \in \mathcal{N}$, $S^*\widetilde{N}^{\perp} \subseteq N^{\perp}$ and $SN \subseteq \widetilde{N}$, so $S \in \mathcal{U}$. Now suppose that $I = I_{\sim}$. We first prove that $N < I$ is equivalent to $\widetilde{N} < I$. Suppose that $N < I$ and $\widetilde{N} = I$; thus for any $M \geq N$, $\widetilde{M} \geq \widetilde{N} = I$. So $I = I_{\sim} = \bigvee\{N' : \widetilde{N'} < I\} \leq N < I$; this contradiction shows $\widetilde{N} < I$. If $\widetilde{N} < I$ and $N = I$, we have $\widetilde{N} = \widetilde{I} = I$ by condition 3) in Theorem 1. This is a contradiction, so $N < I$. Hence $I = I_{\sim} = \bigvee\{N : \widetilde{N} < I\} = \bigvee\{N : N < I\} = I_{-}$. Thus there is a sequence N_k in $\mathcal{N}\backslash I$ strictly increasing with I. Since the order homomorphism $N \to \widetilde{N}$ is left order continuous, we have that $\{\widetilde{N_k}\}$ is also increasing with limit $\widetilde{I} = I$. Since $N_k < I$, $\widetilde{N_k} < I$. Hence by taking a subsequence, we can suppose that $\{\widetilde{N_k}\}$ is strictly increasing with limit I. Choose an orthonormal basis for each $(\widetilde{N_{k+1}} - \widetilde{N_k})P\mathcal{H}$ starting with $N_0 = 0 = \widetilde{0}$, and combine them to form an orthonormal basis $\{f_k : k \geq 1\}$ for $P\mathcal{H}$ with the property that each $f_k = \widetilde{M_k}f_k$ for some $M_k \in \mathcal{N}$ with $M_k < I$. Then choose an increasing sequence $\{n_k\}$ such that e_{n_k} is contained in $M_k^{\perp}$ for each $k \geq 1$. Then the operator $S = \sum\limits_{k\geq 1} f_k \otimes e_{n_k} \in \mathcal{U}$ does the job. □

Lemma 4. *Let* $\{e_k\}$ *be an orthonormal set such that each* $e_k = M_k^{\perp}e_k$ *for some* $M_k \in \mathcal{N}$ *with* $M_{k*} > 0$ *and infinitely many* e_k*'s belong to* $N^{\perp}$ *for every* $N < I$ *in* $\mathcal{N}$*. Also let* $\{f_k\}$ *be an orthonormal set such that each* $f_k = N_kf_k$ *for some* $N_k \in \mathcal{N}$ *with* $N_{k\sim} < I$ *and infinitely many* f_k*'s belong to* N *for some* $N > 0$ *in* $\mathcal{N}$*. Then* *there is a partial isometry in $\mathcal{U}$ with initial space span$\{e_k : k \geq 1\}$ and range space span$\{f_k : k \geq 1\}$.*

Proof. This proof is a slight modification of the proof of the last situation in [2], Lemma 1.

The partial isometry S is contructed using a well-known combinatoral device to determine an infinite permutation π of $\mathbf{N}$ such that

$$(1) \qquad supp(f_{\pi(k)}) \preceq supp(e_k) \text{ according to } \mathcal{U}, \quad \text{for all } k \in \mathbf{N}.$$

Indeed, at the kth stage, if $\pi(k)$ is not defined, choose it to be an integer $n = \pi(k)$ not yet in the range of π such that (1) holds. This is possible since $e_k = M_k^{\perp} e_k$ and infinitely many f_n's are supported on $M_{k*} > 0$. Then if k is not in the range of π, choose $n = \pi^{-1}(k)$ in the same manner to be an integer on which π is as yet undefined and for which $supp(f_k) \preceq supp(e_n)$ according to $\mathcal{U}$ (since $N_k f_k = f_k$ and infinitely many e_n's are supported on $N_{k\sim}^{\perp}$). In this way, a bijection satisfying (1) is obtained. Then the operator $S = \sum\limits_{k \geq 1} f_{\pi(k)} \otimes e_k \in \mathcal{U}$ is the desired partial isometry. □

Now suppose that $\mathcal{N}$ is admissible. In this case, it is easy to construct a projection $P = \sum\limits_{-\infty}^{+\infty} E_n$ in $\mathcal{N}'$ such that each E_n is an orthonormal projection in $\mathcal{N}'$, $E_n \preceq E_{n+1}$ according to $\mathcal{T}(\mathcal{N})$, $E_n \perp E_{n+1}$ and the smallest projection interval of $\mathcal{N}$ containing P is the identity I. Such a projection will also be called admissible. Note that $\mathcal{N}$ is admissible if and only if there are admissible projections in $\mathcal{T}(\mathcal{N})$. Any admissible projection P may be decomposed as a sum of countably many pairwise orthogonal admissible projections by splitting Z into countably many disjoint subsets which are not bounded above or below.

Lemma 5. *Suppose that $\mathcal{U}$ satisfies condition* 3) *in Theorem 1 and $P = \sum\limits_{-\infty}^{+\infty} E_n$ is an admissible projection in $\mathcal{T}(\mathcal{N})$. Choose orthonormal bases for each of these subspaces and combine them to form an orthonormal basis $\{h_k\}$ for $P\mathcal{H}$. Then*

1) *each $h_k = M_k^{\perp} h_k$ for some $M_k \in \mathcal{N}$ with $M_{k*} > 0$;*
2) *each $h_k = N_k h_k$ for some $N_k \in \mathcal{N}$ with $N_{k\sim} < I$.*

Proof. 1) We deal separately with two cases.

i) $0_* \neq 0$. For all $k \in \mathbf{N}$, $M_k = 0$ does the job.

ii) $0_* = 0$. It follows from a similar argument in Lemma 2 that $0_* = 0 = 0_+$. For any $N > 0$, $N_* = \bigwedge\{\widetilde{M} : M > N\} \geq \widetilde{N} > 0$. Thus for each h_k, since $0_+ = 0$ and h_k belong to some E_n, there exists some $M_k > 0$ in $\mathcal{N}$ such that $M_k^{\perp} h_k = h_k$ and $M_{k*} > 0$.

2) If $I_\sim \neq I$, for each h_k, we can take $N_k = I$. If $I_\sim = I$, by the proof of Lemma 3, $I = I_-$. We first prove the following assertion: for any $N < I$, $N_\sim < I$.

Indeed, if there exists an element $N < I$ and $N_\sim = \bigvee\{M : \widetilde{M} < N\} = I = I_-$, then there is a sequence $\{M_k'\}$ with $\widetilde{M_k'} < N$ and strictly increasing with limit I. It follows from the left continuity of the order homomorphism that the sequence $\{\widetilde{M_k'}\}$ is also increasing with limit $\widetilde{I} = I$. But this contradicts the fact $\widetilde{M_k'} < N < I$.

For each h_k, since $I = I_-$ and h_k belongs to some E_n, there exists an element $N_k < I$ in $\mathcal{N}$ such that $N_k h_k = h_k$ and $N_{k\sim} < I$. □

Proposition 6. *Suppose that* $\mathcal{U}$ *satisfies condition* 3) *in Theorem 1 and* $\mathcal{N}$ *is an adimissible nest with admissible projections* P *and* Q*. Let* A *be an element in* $\mathcal{U}$ *such that* $A = Q^\perp A P^\perp$ *and* $\| A \| < 1$*. Then there is a unitary* U *in* $\mathcal{U}$ *such that* $Q^\perp U P^\perp = A$*. Thus* A *is the average of four unitaries in* $\mathcal{U}$*.*

Proof. Let E_n be the subintervals of P ordered by Z in the partial order on subintervals of $\mathcal{N}$, and choose a unit vector e_n in the range of E_n for $n \in Z$. Similarly, let F_n be the subintervals of Q and let f_n be unit vectors in $F_n\mathcal{H}$ for $n \in Z$. Let P_0 and P_1 be the projections onto $span\{e_{2n} : n \in Z\}$ and $span\{e_{2n+1} : n \in Z\}$ respectively. Similarly define Q_0 and Q_1.

Since A is a strict contraction, $P^\perp - A^*A$ and $Q^\perp - AA^*$ are positive and invertible in $\mathcal{B}(P^\perp\mathcal{H})$ and $\mathcal{B}(Q^\perp\mathcal{H})$ respectively. Thus by [6], there exist invertible operators X in $\mathcal{T}(P^\perp\mathcal{N}) = P^\perp\mathcal{T}(\mathcal{N})P^\perp$ and Y in $\mathcal{T}(Q^\perp\mathcal{N}) = Q^\perp\mathcal{T}(\mathcal{N})Q^\perp$ such that

$$P^\perp - A^*A = X^*X \quad \text{and} \quad Q^\perp - AA^* = YY^*$$

(X^{-1}, Y^{-1} need not belong to the nest algebras). Using the polar decomposition for X and Y^*, we obtain

$$X = U_1(P^\perp - A^*A)^{\frac{1}{2}} \quad \text{and} \quad Y = (Q^\perp - AA^*)^{\frac{1}{2}}U_2,$$

where $U_1 = P^\perp U_1 P^\perp$ and $U_2 = Q^\perp U_2 Q^\perp$ are partial isometries in $\mathcal{B}(\mathcal{H})$. Set $T = -U_1A^*U_2 = P^\perp T Q^\perp$.

Fix a proper element M in $\mathcal{N}$ and use Lemma 2 to construct a partial isometry W_1 in $\mathcal{U}$ such that $W_1^*W_1 = P^\perp$ with range contained in $\widetilde{M}Q_0\mathcal{H}$ and spanned by a subset of the f_n's. Similarly construct another partial isometry W_2 in $\mathcal{U}$ such that $W_2W_2^* = Q^\perp$, with domain contained in $M^\perp P_0\mathcal{H}$ spanned by a subset of the e_n's. Since $W_1TW_2 = P(\widetilde{M})W_1TW_2P(M)^\perp$, $W_1TW_2 \in \mathcal{U}$. Decomposing $\mathcal{B}(\mathcal{H})$ as operators from $\mathcal{H} = P^\perp\mathcal{H} \oplus P\mathcal{H}$ into $\mathcal{H} = Q^\perp\mathcal{H} \oplus Q\mathcal{H}$, we set

$$S = A + YW_2 + W_1X + W_1TW_2 \simeq \begin{pmatrix} A & YW_2 \\ W_1X & W_1TW_2 \end{pmatrix}.$$

Since $X, Y \in \mathcal{T}(\mathcal{N})$ and $W_1, W_2 \in \mathcal{U}$, YW_2 and W_1X belong to $\mathcal{U}$. Hence $S \in \mathcal{U}$. Now we compute S^*S and SS^*:

$$S^*S \simeq \begin{pmatrix} A^*A + X^*W_1^*W_1X & A^*YW_2 + X^*W_1^*W_1TW_2 \\ W_2^*Y^*A + W_2^*T^*W_1^*W_1X & W_2^*Y^*YW_2 + W_2^*T^*W_1^*W_1TW_2 \end{pmatrix},$$

with

$$A^*A + X^*W_1^*W_1X = A^*A + X^*X = P^\perp;$$

$$\begin{aligned} A^*YW_2 + X^*W_1^*W_1TW_2 &= A^*YW_2 + X^*TW_2 \\ &= A^*(Q^\perp - AA^*)^{\frac{1}{2}}U_2W_2 - (P^\perp - A^*A)^{\frac{1}{2}}U_1^*U_1A^*U_2W_2 \\ &= [A^*(Q^\perp - AA^*)^{\frac{1}{2}} - (P^\perp - A^*A)^{\frac{1}{2}}A^*]U_2W_2 = 0; \end{aligned}$$

$$\begin{aligned} W_2^*Y^*YW_2 + W_2^*T^*W_1^*W_1TW_2 &= W_2^*(Y^*Y + T^*T)W_2 \\ &= W_2^*[U_2^*(Q^\perp - AA^*)U_2 + U_2^*AU_1^*U_1A^*U_2]W_2 \\ &= W_2^*[U_2^*(Q^\perp - AA^*)U_2 + U_2^*AA^*U_2]W_2 \\ &= W_2^*(U_2^*U_2)W_2 = W_2^*Q^\perp W_2 = W_2^*W_2. \end{aligned}$$

Hence $S^*S = P^\perp + W_2^*W_2$ and, by a similar argument, $SS^* = Q^\perp + W_1W_1^*$. So S is a partial isometry of $\mathcal{U}$. The projection $I - S^*S = P - W_2^*W_2$ dominates P_1 and is spanned by the properly supported projections E_n or $E_n - e_n \otimes e_n$ for $n \in Z$. Similarly, $I - SS^* = Q - W_1W_1^*$ dominates Q_1 and is spanned by the properly supported projections F_n or $F_n - f_n \otimes f_n$ for $n \in Z$. Thus $I - S^*S$ and $I - SS^*$ have infinite rank intersection with N and $N^\perp$ for every $N \in \mathcal{N} \backslash (0, I)$, and their ranges are spanned by subspaces which have proper support. We choose orthonormal bases for each of these subspaces and combine them to form orthonormal bases $\{g_k\}$ for $I - S^*S$ and $\{h_k\}$ for $I - SS^*$. It follows from the proof of Lemma 5 that $\{g_k\}$ and $\{h_k\}$ satisfy the hypothesis of Lemma 4. Hence there is a partial isometry S_1 in $\mathcal{U}$ such that $S_1^*S_1 = I - S^*S$ and $S_1S_1^* = I - SS^*$. It is easy to prove that $U = S + S_1$ is a unitary operator of $\mathcal{U}$ and $Q^\perp U P^\perp = Q^\perp S P^\perp = A$.

Finally, define unitary operators

$$U_k = (Q^\perp + i^k Q) U (P^\perp + i^k P) \in \mathcal{U}, \qquad 1 \le k \le 4.$$

By computing, we have

$$\frac{1}{4} \sum_{k=1}^{4} U_k = Q^\perp U P^\perp = A.$$

□

Now we are in the position to complete the proof of Theorem 1.

3) ⇒ 1) of **Theorem 1**: Let P and Q be admissible projections. Split P and Q into $2n$ orthonormal admissible projections $P_1, \cdots, P_{2n}$ and $Q_1, \cdots, Q_{2n}$, respectively. It follows that $X_i = Q_i^\perp - \frac{1}{2n} Q^\perp$ and $Y_i = P_i^\perp - \frac{1}{2n} P^\perp$ are contractions in $\mathcal{N}'$ such that

$$\sum_{i=1}^{2n} X_i = (2n-1)I = \sum_{i=1}^{2n} Y_i.$$

Moreover, $(1 - \frac{1}{2n})^2 > 1 - \frac{1}{n}$. Thus the operators

$$A_{ij} = (\frac{2n}{2n-1})^2 X_i A Y_j = Q_i^\perp A_{ij} P_j^\perp \quad \text{for } 1 \le i, j \le n$$

are strict contractions in $\mathcal{U}$. By Proposition 6, each A_{ij} is the mean of four unitaries $U_{ijk} \in \mathcal{U}$ for $1 \le k \le 4$. Hence

$$\begin{aligned} \frac{1}{16n^2} \sum_{i=1}^{2n} \sum_{j=1}^{2n} \sum_{k=1}^{4} U_{ijk} &= \frac{1}{4n^2} \sum_{i=1}^{2n} \sum_{j=1}^{2n} A_{ij} \\ &= \frac{1}{4n^2} (\sum_{i=1}^{2n} X_i)(\frac{2n}{2n-1})(\sum_{j=1}^{2n} Y_j) \\ &= \frac{1}{4n^2} (2n-1)^2 (\frac{2n}{2n-1})^2 A = A. \end{aligned}$$

□

Now as a corollary of Theorem 1, we give a sufficient condition for the Krein–Milman theorem to hold in $\mathcal{U}$.

Corollary 7. *If $\mathcal{N}$ is admissible and $\mathcal{U}$ satisfies condition* 3) *in Theorem* 1, *then $ext(\mathcal{U}_1) \neq \emptyset$ and $\overline{co}ext(\mathcal{U}_1) = \mathcal{U}_1$.*

Proof. It follows from Theorem 1 that the closed convex hull of the unitary elements in $\mathcal{U}$ equals the closed unit ball of $\mathcal{U}$, and Theorem 3.3 in [3] shows that each unitary element in $\mathcal{U}$ is an extreme point of $\mathcal{U}_1$. Then $ext(\mathcal{U}) \neq \emptyset$ and $\overline{co}ext(\mathcal{U}_1) = \mathcal{U}_1$. $\square$

Remark 8. Consider the case when $\mathcal{N}$ is not admissible.

If $\mathcal{N}$ is not admissible, it follows from [2] that the Russo-Dye Theorem does not hold in $\mathcal{T}(\mathcal{N})$. Furthermore, we can easily prove:

Suppose that $\mathcal{N}$ is not admissible. If $\widetilde{0_+} \leq 0_+$ or $\widetilde{I_-} \leq I_-$, then the Russo-Dye Theorem does not hold in $\mathcal{U}$.

In light of Theorem 1, it is natural to ask if $\widetilde{0_+} > 0_+$ and $\widetilde{I_-} = I$ is a sufficient condition for the Russo-Dye Theorem to hold in $\mathcal{U}$. This seems to be a challenging question. We shall continue the investigation.

References

1. M.Anoussis and E.Katsoulis, A nonself-adjoint Russo-Dye Theorem, Math. Ann. 304 (1996), 685–699. MR **97f:**47042
2. K.R.Davidson, The Russo-Dye theorem in nest algebras, Proc. Amer. Math. Soc. 126 (1998), 3055–3059. MR **2000f:**47109
3. Dong Zhe and Lu Shijie, Extreme points of weakly closed $\mathcal{T}(\mathcal{N})$-modules, Proc. Amer. Math. Soc. 130 (2002), 461–469. MR **2002m:**47086
4. J.A.Erdos and S.C.Power, Weakly closed ideals of nest algebras, J. Operator Theory 7 (1982), 219–235. MR **84a:**47056
5. R.V.Kadison and G.K.Pedersen, Means and convex combinations of unitary operators, Math. Scand. 57 (1985), 245–266. MR **87g:**47078
6. D.R.Larson, Nest algebras and similarity transforms, Ann. Math. 121 (1985), 409–427. MR **86j:**47061
7. R.Moore and T.Trent, Extreme points of certain operator algebras, Indiana U. Math. J. 36 (1987), 645–650. MR **89d:**47103
8. B.Russo and H.Dye, A note on unitary operators in C^*–algebras, Duke Math. J. 33 (1966), 413–416. MR **33:**1750

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