

Σ_n -BOUNDED AND Δ_n -INDUCTION

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(Communicated by Carl G. Jockusch, Jr.)

ABSTRACT. Working in the base theory of $\text{PA}^- + \text{I}\Sigma_0 + \text{exp}$, we show that for all $n \in \omega$, the bounding principle for Σ_n -formulas ($\text{B}\Sigma_n$) is equivalent to the induction principle for Δ_n -formulas ($\text{I}\Delta_n$). This partially answers a question of J. Paris.

1. INTRODUCTION

We begin with some background material on first-order arithmetic. However, in lieu of giving a detailed introduction to the subject, we settle for recommending the excellent texts by Kaye [3] and Hájek and Pudlák [2].

The language of first-order arithmetic consists of the usual symbols of first-order logic $\forall, \exists, (,), \neg, \wedge, \vee, \rightarrow, \leftrightarrow, =$, and variables x_1, x_2, \dots together with symbols from arithmetic: $0, 1, +, \cdot$, and $<$. Formulas and sentences are constructed as usual.

To fix some typographical notation, we use $x \leq y$ to indicate $x < y \vee x = y$. We also use boldface characters, such as \mathbf{p} and \mathbf{x} , to represent sequences of type p or x . The inequality $\mathbf{p} < r$ indicates that every element of the sequence \mathbf{p} is less than r .

PA^- consists of the axioms for the nonnegative part of a discretely ordered ring. It is the universal closures of the following formulas:

$$\begin{array}{ll} x + y = y + x, & x \cdot y = y \cdot x, \\ (x + y) + z = x + (y + z), & (x \cdot y) \cdot z = x \cdot (y \cdot z), \\ x \cdot (y + z) = x \cdot y + x \cdot z, & \\ x + 0 = x, & x \cdot 0 = 0, \\ x \cdot 1 = x, & \\ \neg(x < x), & (x < y \wedge y < z) \rightarrow x < z, \\ x < y \vee y < z \vee x = y, & \\ (x < y) \rightarrow (x + z < y + z), & (0 < z \wedge x < y) \rightarrow (x \cdot z < y \cdot z), \\ x < y \rightarrow (\exists z)(x + z = y), & \\ 0 < 1, & \\ 0 \leq x, & 0 < x \rightarrow 1 \leq x. \end{array}$$

Received by the editors November 20, 2002 and, in revised form, February 20, 2003.

2000 *Mathematics Subject Classification*. Primary 03F30, 03H15.

Key words and phrases. Σ_n -bounding, Δ_n -induction.

During the preparation of this paper, the author was partially supported by the Alexander von Humboldt Foundation and by the National Science Foundation Grant DMS-9988644. The author is grateful to Jan Krajíček for reading a preliminary version of this paper and suggesting improvements to it.

1.1. Bounding, least number, and induction principles for Σ_n and Π_n -formulas.

1.1.1. *Bounding for Σ_n -formulas* ($B\Sigma_n$). $B\Sigma_n$ consists of all sentences

$$(\forall \mathbf{p})(\forall a)[(\forall x < a)(\exists \mathbf{y})\varphi(x, \mathbf{y}, \mathbf{p}) \rightarrow (\exists b)(\forall x < a)(\exists \mathbf{y} < b)\varphi(x, \mathbf{y}, \mathbf{p})]$$

in which φ is Σ_n . That is, if for every number x less than a , there are numbers \mathbf{y} satisfying a Σ_n -property relative to x and parameters \mathbf{p} , then there is a bound b such that for each x less than a , there is such a \mathbf{y} all the elements of which are less than b .

1.1.2. *Least number principle for Σ_n -formulas* ($L\Sigma_n$). $L\Sigma_n$ consists of all the sentences

$$(\forall \mathbf{p}) [(\exists x)(\varphi(x, \mathbf{p}) \rightarrow (\exists x)[\varphi(x, \mathbf{p}) \wedge (\forall y < x)\neg\varphi(y, \mathbf{p})])]$$

in which φ is Σ_n . In other words, if A is defined by a Σ_n -formula relative to parameters and A is not empty, then A has a least element. $L\Pi_n$ is defined similarly.

1.1.3. *Induction for Σ_n -formulas* ($I\Sigma_n$). $I\Sigma_n$ consists of all the sentences

$$(\forall \mathbf{p})[(\varphi(0, \mathbf{p}) \wedge (\forall x)(\varphi(x, \mathbf{p}) \rightarrow \varphi(x + 1, \mathbf{p}))) \rightarrow (\forall x)\varphi(x, \mathbf{p})]$$

in which φ is Σ_n . In other words, if A is defined by a Σ_n -formula relative to parameters, $0 \in A$, and A is closed under the successor function, then every number is in A .

A *cut* in a model \mathfrak{M} is a subset J of \mathfrak{M} such that for every x and y , if $x \in J$ and $y < x$, then $y \in J$. A *proper* cut is a nonempty cut that does not include all of the elements of \mathfrak{M} . A *nonprincipal* cut is a cut that has no greatest element. One way in which Σ_n -induction could fail in \mathfrak{M} would be for there to be a proper nonprincipal cut that is definable by a Σ_n -formula relative to parameters in \mathfrak{M} .

1.1.4. *The Kirby and Paris theorem.* Building on Parsons [5], Kirby and Paris [4] proved the fundamental theorem connecting bounding, induction, and the least number principles for Σ_n and Π_n -formulas.

Theorem 1.1 (Kirby and Paris [4]). *Work in $PA^- + I\Sigma_0$. For all $n \geq 1$,*

- (1) $I\Sigma_n \iff III_n \iff L\Sigma_n \iff L\Pi_n$;
- (2) $B\Sigma_{n+1} \iff B\Pi_n$;
- (3) $I\Sigma_{n+1} \implies B\Sigma_{n+1} \implies I\Sigma_n$, and the implications are strict.

Consequently, the bounding principles are strictly intertwined with the equivalent induction and least number principles.

1.2. **Δ_n -formulas.** We form the principles of $L\Delta_n$ and $I\Delta_n$ by inserting a hypothesis of equivalence between Σ_n and Π_n -formulas.

1.2.1. *The least number principle for Δ_n -formulas* ($L\Delta_n$). $L\Delta_n$ consists of all the sentences

$$(\forall \mathbf{p}) \left[\begin{array}{c} (\forall x)(\varphi(x, \mathbf{p}) \leftrightarrow \neg\psi(x, \mathbf{p})) \rightarrow \\ ((\exists x)(\varphi(x, \mathbf{p}) \rightarrow (\exists x)[\varphi(x, \mathbf{p}) \wedge (\forall y < x)\neg\varphi(y, \mathbf{p})])]) \end{array} \right]$$

in which φ and ψ are Σ_n . In other words, if A is defined by a Σ_n -formula and also by a Π_n -formula relative to parameters and A is not empty, then A has a least element.

Theorem 1.2 (Gandy (unpublished), see [2]). *If $n \geq 1$ and \mathfrak{M} is a model of $\text{PA}^- + \text{I}\Sigma_0$, then*

$$\mathfrak{M} \models \text{B}\Sigma_n \iff \mathfrak{M} \models \text{L}\Delta_n.$$

Proof. $\text{B}\Sigma_n \implies \text{L}\Delta_n$. To sketch the proof, suppose that A is a Δ_n -set and suppose that A has an element less than a . Use $\text{B}\Sigma_n$ to bound the witnesses needed to determine for each x less than a , whether x belongs to A . $\text{B}\Sigma_n$ implies that the Π_{n-1} predicates are closed under bounded existential quantification; so the restriction of A to the numbers less than a can be defined by a Π_{n-1} -formula. Now, $\text{L}\Pi_{n-1}$ is provable from $\text{B}\Sigma_n$, and so there is a least element of A below a .

$\text{L}\Delta_n \implies \text{B}\Sigma_n$. We follow the account of Gandy's proof in [2]. We work in the theory $\text{L}\Delta_n$. Suppose that a, \mathbf{p} , and a Π_{n-1} -formula φ_0 are given so that $(\forall x < a)(\exists \mathbf{y})\varphi_0(x, \mathbf{y}, \mathbf{p})$. We must argue that $(\exists b)(\forall x < a)(\exists \mathbf{y} < b)\varphi_0(x, \mathbf{y}, \mathbf{p})$.

Define $\theta(z, a, \mathbf{p})$ to be the formula

$$z \leq a \wedge (\exists u)[(\forall x)(z \leq x < a \rightarrow (\exists \mathbf{y} < u)\varphi_0(x, \mathbf{y}, \mathbf{p})) \\ \wedge (\forall v < u)(\forall \mathbf{y} < v)\neg\varphi_0(z, \mathbf{y}, \mathbf{p})].$$

One can read $\theta(z, a, \mathbf{p})$ as saying that for any x with $z \leq x < a$, the least witness to $(\exists \mathbf{y})\varphi_0(x, \mathbf{y}, \mathbf{p})$ is less than or equal to the least witness to $(\exists \mathbf{y})\varphi_0(z, \mathbf{y}, \mathbf{p})$. Syntactically, $\theta(z, a, \mathbf{p})$ is equivalent to a formula of the form $(\exists u)\psi$, where ψ is obtained by the bounded quantification of a Π_{n-1} -formula. Now, $\text{L}\Delta_n \implies \text{I}\Sigma_{n-1}$, $\text{I}\Sigma_{n-1} \implies \text{B}\Sigma_{n-1}$, and $\text{B}\Sigma_{n-1}$ proves that the Π_{n-1} predicates are closed under bounded quantification. So, we can conclude that $\theta(z, a, \mathbf{p})$ is equivalent to a Σ_n -formula. Additionally, if the least witness for z is greater than or equal to some witness for x , then every witness for z is greater than or equal to some witness for x . So, $\theta(z, a, \mathbf{p})$ is equivalent to the following formula:

$$z \leq a \wedge (\forall u)[(\exists \mathbf{y} < u)\varphi_0(z, \mathbf{y}, \mathbf{p}) \rightarrow (\forall x)(z \leq x < a \rightarrow (\exists \mathbf{y} < u)\varphi_0(x, \mathbf{y}, \mathbf{p}))].$$

Applying $\text{B}\Sigma_{n-1}$ as above, this formula is equivalent to a Π_n -formula. Thus, $\text{L}\Delta_n$ implies that $\theta(z, a, \mathbf{p})$ is Δ_n , and we can apply the least number principle to $\theta(z, a, \mathbf{p})$. Let z_0 be the least number z such that $\theta(z, a, \mathbf{p})$, and let u_0 be the least number u such that $(\exists \mathbf{y} < u)\varphi_0(z_0, \mathbf{y}, \mathbf{p})$.

We claim that $(\forall x < a)(\exists \mathbf{y} < u_0)\varphi_0(x, \mathbf{y}, \mathbf{p})$. By the definition of z_0 , if x is greater than or equal to z_0 , then there is a \mathbf{y} such that $\mathbf{y} \leq u_0$ and $\varphi_0(x, \mathbf{y}, \mathbf{p})$. Suppose that there is an $x < a$ such that the least witness to $(\exists \mathbf{y})\varphi_0(x, \mathbf{y}, \mathbf{p})$ is greater than u_0 . Then, consider the set of x 's such that $(\forall \mathbf{y} < u_0)\neg\varphi_0(x, \mathbf{y}, \mathbf{p})$. Again by $\text{B}\Sigma_{n-1}$, this set is Π_{n-1} . By $\text{L}\Delta_n$, it has a greatest element z_1 . (Think of the least number of the form $a - x$ for such an x .) But z_1 would also satisfy $\theta(z, a, \mathbf{p})$ and be less than z_0 , a contradiction. \square

1.2.2. *The principle of induction for Δ_n -formulas* ($\text{I}\Delta_n$). $\text{I}\Delta_n$ consists of all the sentences

$$(\forall \mathbf{p}) \left[\begin{array}{c} (\forall x)(\varphi(x, \mathbf{p}) \leftrightarrow \neg\psi(x, \mathbf{p})) \rightarrow \\ ((\varphi(0, \mathbf{p}) \wedge (\forall x)(\varphi(x, \mathbf{p}) \rightarrow \varphi(x + 1, \mathbf{p}))) \rightarrow (\forall x)\varphi(x, \mathbf{p})) \end{array} \right]$$

in which φ and ψ are Σ_n .

Paris raised the question of determining the relationship between $\text{I}\Delta_n$ and $\text{B}\Sigma_n$; see the list of problems compiled by Clote and Krajíček [1].

As we will see in Section 2, the problem is to decide whether $\text{B}\Sigma_n$ (or equivalently $\text{L}\Delta_n$) follows from $\text{I}\Delta_n$. There is a natural approach to showing that it does, but

this approach does not lead to a correct proof. However, it does point out where the problem lies.

Let us attempt to prove $L\Delta_n$ from $I\Delta_n$. Suppose that we are given a nonempty Δ_n -set A , and let us attempt to show that it has a least element. We suppose that A has no least element, and we look for a failure of induction. Define the set

$$A^* = \{x : (\forall y)[y \leq x \rightarrow y \notin A]\}.$$

Clearly $0 \in A^*$, or 0 would be the least element of A . Equally clearly, if $a \in A^*$ and $a + 1 \notin A^*$, then $a + 1$ would be the least element of A^* . So, A^* satisfies the hypotheses needed to apply induction.

It would only remain to show that A^* is a Δ_n -set. Since A^* is explicitly Π_n , it would be sufficient to show that it is Σ_n . There is a standard argument to show that Σ_n -predicates are closed under bounded universal quantification. If φ is $(\exists w)\varphi_0$, where φ_0 is Π_{n-1} , then we would rewrite

$$(\forall y \leq x)(\exists w)\varphi_0$$

as

$$(\exists u)(\forall y \leq x)(\exists w < u)\varphi_0,$$

and then replace $(\exists w < u)\varphi_0$ by a formula that is Π_{n-1} . But look at the implication

$$(\forall y \leq x)(\exists w)\varphi_0 \rightarrow (\exists u)(\forall y \leq x)(\exists w < u)\varphi_0.$$

We have confronted an instance of $B\Sigma_n$, rather than an instance of $B\Sigma_{n-1}$ such as we found earlier. Of course, we cannot use an instance of $B\Sigma_n$ to prove $B\Sigma_n$, and the natural argument fails.

Even so, there is a less direct argument leading to the same conclusion. If we strengthen the base theory to include the assertion that exponentiation is a total function (exp), then we obtain the following partial answer to Paris's question:

$$PA^- + I\Sigma_0 + \text{exp} \implies (I\Delta_n \iff B\Sigma_n).$$

2. EQUATING BOUNDING AND INDUCTION

2.1. $B\Sigma_n$ **implies** $I\Delta_n$. For models of $PA^- + I\Sigma_0$, the implication

$$\mathfrak{M} \models B\Sigma_n \implies \mathfrak{M} \models I\Delta_n$$

is well known. However, the proof is short and so we include it here.

Suppose that $\mathfrak{M} \models B\Sigma_n$, that φ and ψ are Σ_n -formulas, and that there are parameters \mathbf{p} in \mathfrak{M} relative to which φ and $\neg\psi$ define the same subset J of \mathfrak{M} . We argue that J is not a counterexample to $I\Delta_n$ in \mathfrak{M} .

First, we can write φ as $\exists \mathbf{y}\varphi_0$ and ψ as $\exists \mathbf{y}\psi_0$, where φ_0 and ψ_0 are Π_{n-1} -formulas, which we may assume have the same number of free variables.

Now, suppose that a is strictly above some element of the complement of J in \mathfrak{M} . Since φ and ψ define complementary sets,

$$\mathfrak{M} \models (\forall x < a)(\exists \mathbf{y})[\varphi_0(x, \mathbf{y}, \mathbf{p}) \vee \psi_0(x, \mathbf{y}, \mathbf{p})].$$

By applying $B\Sigma_n$ in \mathfrak{M} , there is a b in \mathfrak{M} such that

$$\mathfrak{M} \models (\forall x < a)(\exists \mathbf{y} < b)[\varphi_0(x, \mathbf{y}, \mathbf{p}) \vee \psi_0(x, \mathbf{y}, \mathbf{p})].$$

But then, for the elements of \mathfrak{M} that are less than a , J is defined by the formula $(\exists \mathbf{y} < b)\varphi_0(x, \mathbf{y}, \mathbf{p})$ in \mathfrak{M} . Using $B\Sigma_n$ again, the intersection of J with the numbers below a is definable by a Π_{n-1} -formula relative to the parameters b , a , and \mathbf{p} . Since

$B\Sigma_n$ implies III_{n-1} and there is an element of the complement of J that is less than a , either $0 \notin J$ or J is not closed under the successor, as required.

2.2. $I\Delta_n$ implies $B\Sigma_n$. It is in our argument for the implication from $I\Delta_n$ to $B\Sigma_n$ that we make use of the exponential function. Ultimately, we take an element a in \mathfrak{M} , and use the base a representations of numbers less than a^a to code length a sequences of numbers less than a .

In the meantime, we make use of the existence of a standard coding within models \mathfrak{M} of $\text{PA}^- + I\Delta_1 + \text{exp}$ of sequences of elements of \mathfrak{M} by elements of \mathfrak{M} so that the relations “ c codes a sequence of length i ” and “ c_j is the j th element of the sequence coded by c ” are Δ_1 in \mathfrak{M} . We will write $\langle m_j : j < i \rangle$ to denote the code for the sequence of length i whose elements are the numbers m_j , for j less than i .

Theorem 2.1. *If $n \geq 1$ and \mathfrak{M} is a model of $\text{PA}^- + I\Sigma_0 + \text{exp}$, then*

$$\mathfrak{M} \models I\Delta_n \implies \mathfrak{M} \models B\Sigma_n.$$

We will prove Theorem 2.1 for the case when $n = 1$. The general case for $n > 1$ follows by the same argument relative to the complete Σ_{n-1} -subset of \mathfrak{M} .

Lemma 2.2. *Suppose that \mathfrak{M} is a model of $\text{PA}^- + I\Delta_1 + \text{exp}$ and \mathfrak{M} is not a model of $B\Sigma_1$. There are an element $a \in \mathfrak{M}$ and a function $f : [0, a) \rightarrow \mathfrak{M}$ such that the following conditions hold:*

- (1) f is injective;
- (2) the range of f is unbounded in \mathfrak{M} ;
- (3) the graph of f is Σ_0 relative to parameters in \mathfrak{M} .

Proof. By hypothesis, \mathfrak{M} is not a model of $B\Sigma_1$. Consequently, we may fix $a \in \mathfrak{M}$, a sequence of parameters \mathbf{p} from \mathfrak{M} , and a Σ_1 -formula $(\exists \mathbf{w})\varphi_0$ such that φ_0 is a Σ_0 -formula and the following conditions hold:

$$(2.1) \quad \mathfrak{M} \models (\forall x < a)(\exists \mathbf{y})(\exists \mathbf{w})\varphi_0(x, \mathbf{y}, \mathbf{w}, \mathbf{p}),$$

$$(2.2) \quad \mathfrak{M} \models (\forall s)(\exists x < a)(\forall \mathbf{y} < s)\neg(\exists \mathbf{w})\varphi_0(x, \mathbf{y}, \mathbf{w}, \mathbf{p}).$$

Define f so that for $x < a$, $f(x)$ is $\langle x, s_x \rangle$, where s_x is least s such that

$$(2.3) \quad \mathfrak{M} \models (\exists \mathbf{y} < s)(\exists \mathbf{w} < s)\varphi_0(x, \mathbf{y}, \mathbf{w}, \mathbf{p}).$$

There is an s satisfying (2.3) by application of (2.1), and there is a least such s by application of $I\Delta_1$ in \mathfrak{M} . Clearly, f is an injective function, since x is determined by the first coordinate of $f(x)$. The range of f is unbounded by (2.2). Finally, the graph of f is definable by a Σ_0 -formula essentially because the value of f at x is chosen to bound all of the quantifiers used to define it from x . Notice, we are invoking the properties of a well-behaved pairing function. \square

Lemma 2.3. *Suppose that \mathfrak{M} is a model of $\text{PA}^- + I\Delta_1 + \text{exp}$ and that \mathfrak{M} is not a model of $B\Sigma_1$. There are $a \in \mathfrak{M}$, a nonprincipal Σ_1 -cut I , and a function g such that the following conditions hold.*

- (1) $I \subset [0, a)$.
- (2) $g : I \rightarrow \mathfrak{M}$.
- (3) The graph of g is Σ_1 relative to parameters in \mathfrak{M} .
- (4) For each $i \in I$, $g(i)$ is the code for a sequence $\langle m_j : j < i \rangle$ such that for all unequal j_1 and j_2 less than i , $m_{j_1} \neq m_{j_2}$. (Note that the sequence coded by $g(i)$ has length i .)

- (5) For each $i_1 < i_2 \in I$ the sequence coded by $g(i_2)$ is an end extension of the one coded by $g(i_1)$. In other words, if $g(i_1)$ codes $\langle m_j : j < i_1 \rangle$ and $g(i_2)$ codes $\langle n_j : j < i_2 \rangle$, then for all $j < i_1$, $m_j = n_j$.
- (6) For each $m < a$, there is an $i \in I$ such that m appears in the sequence coded by $g(i)$.

Proof. Let f and a be as in Lemma 2.2. We view the set of numbers less than a , which is finite in the sense of \mathfrak{M} , as a recursively enumerable set in the sense of \mathfrak{M} . In this sense, m is enumerated into this set at stage $f(m)$. Since f is injective, there is at most one number enumerated during each stage. At each stage s , we can define the sequence of numbers that have been enumerated at earlier stages and order them according to the order in which they were enumerated by f . That is, we reorder the numbers n and m below a such that n comes before m if and only if $f(n)$ is less than $f(m)$.

We define g so that $g(i)$ is the code for sequence $\langle m_j : j < i \rangle$ such that for each $j < i$, m_j is the j th number enumerated by f . More formally, $g(i) = \langle m_j : j < i \rangle$ if and only if there is an s such that the following conditions hold:

- (i) $\{m : (\exists y < s)(f(m) = y)\} = \{m : (\exists j < i)(m = m_j)\}$,
 (ii) for all j_1 and j_2 less than i , $(j_1 < j_2) \leftrightarrow (f(m_{j_1}) < f(m_{j_2}))$.

We leave it to the reader to verify that g is well defined and Σ_1 in \mathfrak{M} , relative to the parameters needed to define f .

Let I denote the domain of g .

By definition, for each i in I , there is an s in \mathfrak{M} so that there are at least i many elements m of \mathfrak{M} such that $f(m) < s$. For such an s and any j less than i , there is a smaller s_j that is adequate to define g at j . So I is an initial segment of \mathfrak{M} . In fact, the map sending i to m_{i-1} is an order isomorphism between $I \setminus \{0\}$ and the range of f . Furthermore, for each $i \in I \setminus \{0\}$, the restriction of this map to $\{j : 0 < j \leq i\}$ is coded within \mathfrak{M} . Consequently, ordering $I \setminus \{0\}$ and the range of f by the ordering of \mathfrak{M} produces isomorphic order types.

Now, suppose that $i > a$ in \mathfrak{M} . If i were to be in I , then $g(i)$ would be the code for a sequence of length greater than a , with elements less than a , and with no repetitions. This is impossible, since every model of $\text{PA}^- + \text{I}\Delta_1 + \text{exp}$ satisfies the Σ_0 -pigeon hole principle; see [2]. Thus, I is a subset of $[0, a)$. Finally, since the range of f is unbounded in \mathfrak{M} and I has the same order type as the range of f , there can be no greatest element of I . Consequently, I is a proper nonprincipal cut in \mathfrak{M} . I is Σ_1 by virtue of being the domain of the Σ_1 function g .

Finally, for every m in the domain of f and every sufficiently large i in I , m appears in the sequence coded by $g(i)$. Since the domain of f is $[0, a)$, for each $m < a$, there is an $i \in I$ such that m appears in the sequence coded by $g(i)$. \square

Let g and I be fixed as in Lemma 2.3. Let $\mathbf{m}^* = \langle m_i^* : i \in I \rangle$ be the sequence of length I such that for all $i \in I$, m_i^* is equal to the i th element of $g(i+1)$. That is, \mathbf{m}^* is the sequence given by the limit of the range of g .

Lemma 2.4. *Suppose that $c \in \mathfrak{M}$, $\mathbf{n} = \langle n_j : j < c \rangle$ is coded in \mathfrak{M} , and \mathbf{n} is a sequence of elements of \mathfrak{M} that are less than a . Then, either c is not an upper bound for I or there is an $i \in I$ such that $n_i \neq m_i^*$.*

Proof. Let c and $\mathbf{n} = \langle n_i : i < c \rangle$ be given as above, and suppose that c is an upper bound for I . For the sake of a contradiction, suppose that for all $i \in I$, $m_i^* = n_i$.

But then consider the set

$$J = \{j : \text{for all } i < j, n_i \neq n_j\}.$$

For each i in I , the sequence coded by $g(i)$ has no repeated values, and so $I \subseteq J$. Conversely, every element of \mathfrak{M} that is less than a appears in \mathbf{m}^* . Consequently, if $j < a$ and $j \notin I$, then there is an i in I such that $m_i^* = n_j$. But then, $n_i = n_j$ and so j is not an element of J . Thus, J is equal to I .

But then J is a proper Σ_0 -cut, contrary to the assumption that \mathfrak{M} is a model of Δ_1 -induction. □

We can now present the proof of Theorem 2.1.

Proof. Suppose that \mathfrak{M} is a model of $\text{PA}^- + \text{I}\Delta_1 + \text{exp}$ and that \mathfrak{M} is not a model of $\text{B}\Sigma_1$. Let I, g , and \mathbf{m}^* be fixed as above.

We define a collection of intervals $[c_i, d_i]$, for $i \in I$. If $i = 0$, then $c_0 = 0$ and $d_0 = a^a$. For $i > 0$,

$$c_i = \sum_{j < i} m_j^* a^{a-(j+1)} \text{ and } d_i = c_i + a^{a-i}.$$

We calculate the first few values explicitly:

$$\begin{aligned} [c_0, d_0] &= [0, a^a], \\ [c_1, d_1] &= [m_0^* a^{a-1}, (m_0^* + 1)a^{a-1}], \\ [c_2, d_2] &= [m_0^* a^{a-1} + m_1^* a^{a-2}, m_0^* a^{a-1} + (m_1^* + 1)a^{a-2}]. \end{aligned}$$

Since each m_j^* is less than a , for each $i \in I$, $[c_{i+1}, d_{i+1}] \subseteq [c_i, d_i]$. Define J by

$$x \in J \iff (\exists i)(x \leq c_i),$$

and define K by

$$x \in K \iff (\exists i)(x \geq d_i).$$

Since the initial segments of \mathbf{m}^* are Σ_1 -definable relative to parameters in \mathfrak{M} , J and K are Σ_1 -definable relative to parameters in \mathfrak{M} . In addition, J is closed downward in \mathfrak{M} , and K is closed upward. It remains to show that J is a proper cut and that $J = \mathfrak{M} \setminus K$.

Since there is only one $i \in I$ such that m_i^* is equal to 0, for all but one i , $c_i < c_{i+1}$. Since I has no greatest element, it follows that neither does J . Consequently, J is a proper cut.

Now, suppose that n is an element of \mathfrak{M} that is neither an element of J nor one of K . Then, consider the base- a representation of n . Let n_i be the coefficient of the a^{a-i-1} term in this representation. For example, if n were $2a^{a-1} + 3a^{a-2}$, then n_0 would be 2, n_1 would be 3, and for every other i less than a , n_i would be 0. By the choice of n , for every i in I ,

$$\sum_{j < i} m_j^* a^{a-(j+1)} < n < \sum_{j < i} m_j^* a^{a-(j+1)} + a^{a-i}.$$

But then, for each i in I , n_i is equal to m_i^* . Since the base- a representation of n is definable from n by a bounded recursion, $\text{I}\Delta_1$ implies that the sequence $\langle n_j : j < a \rangle$ is coded by an element of \mathfrak{M} . This is a contradiction to Lemma 2.4.

Consequently, J is a Σ_1 -cut and the Σ_1 -set K is its complement in \mathfrak{M} . So, from the failure of $B\Sigma_1$ in \mathfrak{M} , we produced a failure of $I\Delta_1$ in \mathfrak{M} . Theorem 2.1 follows. \square

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