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ON c-SUPPLEMENTED MAXIMAL AND MINIMAL SUBGROUPS OF SYLOW SUBGROUPS OF FINITE GROUPS

HUAQUAN WEI, YANMING WANG, AND YANGMING LI

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ABSTRACT. This paper proves: Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$.

(1) If all maximal subgroups of any Sylow subgroup of $F^*(H)$ are c-supplemented in G, then $G \in \mathcal{F}$;

(2) If all minimal subgroups and all cyclic subgroups with order 4 of $F^*(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.

1. INTRODUCTION

All groups considered will be finite. We use conventional notions and notation, as in Huppert [14]. $M <\bullet\, G$ means that M is a maximal subgroup of group G.

Maximal and minimal subgroups of Sylow subgroups play an important role in determining the structure of a finite group. They have been studied by many scholars. For instance, Srinivasan [1] showed in 1980 that a finite group G is supersolvable if all maximal subgroups of any Sylow subgroup of G are normal in G. Buckley [2] proved in 1970 that a finite group G of odd order is supersolvable if all minimal subgroups of any Sylow subgroup of G are normal in G.

Following [1] and [2], extensions were found in [3], [4], [5], [6], [17]. By adding a saturated formation and conditions of complementation and normality, further extensions have been found: [7], [8], [9] for c-normality and [10], [11], [12] for c-complements. In 1996 Wang generalized the above two results by replacing the normality with the weaker c-normality [7]. By minimizing the number of c-normal maximal or minimal subgroups, Wei in 2001 extended the results further to a saturated formation containing the class of supersolvable groups [8]. Wang, Ballester-Bolinches and Guo in 2000 introduced the concept of c-supplementation of a finite group which is weaker than both c-normality and complementation [10], [11] and generalized the above two results as applications. In [12], Wang, Wei and Li extended the results further to a saturated formation containing the class of supersolvable groups by limiting the c-supplementation of maximal or minimal subgroups to the Fitting subgroup of a solvable group.

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It is meaningful to remove the solvability of a finite group. Since every simple non-Abelian group G has a trivial Fitting subgroup, one cannot expect a detailed structure if one only gives restrictions on maximal or minimal subgroups of Sylow subgroups of $F(G)$. So the solvability of G with respect to $F(G) \neq 1$ is usually assumed, as in [8], [12], so that the detailed structure of the group can be obtained. However, as we show in the present paper, we can obtain our results on the structure of G if we assume that the maximal or minimal subgroups of the generalized Fitting subgroup of some normal subgroup of G are c-supplemented in G. The results we mentioned above are special cases of our results, since the generalized Fitting subgroup is a Fitting subgroup when the group is solvable. In fact, we get:

Theorem 1.1. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F^*(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.*

Theorem 1.2. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F^*(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.*

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (i) if $G \in \mathcal{F}$ and $N \lhd G$, then $G/N \in \mathcal{F}$, and (ii) if $N_1, N_2 \lhd G$ such that $G/N_1, G/N_2 \in \mathcal{F}$, then $G/(N_1 \cap N_2) \in \mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation (see [10, p. 713, Satz 8.6]). For further discussions of formations, please refer to [13].

2. Preliminaries

Definition 2.1 ([10])**.** A subgroup H of a group G is said to be c-supplemented in G if there exists a subgroup N of G such that $G = HN$ and $H \cap N \leq H_G = \mathrm{Core}_G(H)$. We say that N is a c-supplement of H in G.

Recall that a subgroup H of G is c-normal in G if there exists a normal subgroup N of G such that $G = HN$ and $H \cap N \leq H_G$ ([7]). Also, a subgroup H of G is complemented in G if there exists a subgroup N of G such that $G = HN$ and $H \cap N = 1$. Hence c-supplementation is a generalization of c-normality and complementation. Two examples in [10] show that c-supplementation is more general than both c-normality and complementation.

Lemma 2.2 ([10, Lemma 2.1])**.** *Let G be a group. Then*

(1) *If H is c-supplemented in G, $H \leq K \leq G$, then H is c-supplemented in K.*

(2) *Let $K \lhd G$ and $K \leq H \leq G$; then H is c-supplemented in G iff H/K is c-supplemented in G/K.*

(3) *Let π be a set of primes, H a π-subgroup of G and N a normal π'-subgroup of G. If H is c-supplemented in G, then HN/N is c-supplemented in G/N. If, furthermore, N normalizes H, then the converse also holds.*

(4) *Let $H \leq G$ and $L \leq \Phi(H)$. If L is c-supplemented in G, then $L \lhd G$ and $L \leq \Phi(G)$.*

Let G be a group. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasi-nilpotent subgroup of G. Now $F^*(G)$ is an important subgroup of G and is a natural generalization of $F(G)$. Its definition and important properties can be found in [15, X 13]. We would like to give the following basic facts, which we will use in our proof.

Lemma 2.3. *Let G be a group and N a subgroup of G.*

(1) *If N is normal in G, then $F^*(N) \leq F^*(G)$.*

(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \mathrm{Soc}(F(G)C_G(F(G))/F(G))$.*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*

(4) *$C_G(F^*(G)) \leq F(G)$.*

(5) *If $K \leq Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

(6) *Suppose P is a normal subgroup of G contained in $O_p(G)$. Then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.*

Proof. (1)–(4) can be found in [15, X 13]. For (5), please refer to [17, Lemma 2.3(4)].

(6) Let $\bar{G} = G/\Phi(P)$. Then we get $F^*(\bar{G}) = F(\bar{G})E(\bar{G})$, where $E(\bar{G})$ is the layer of $\bar{G}$ (see [15, p. 128]. Since $\Phi(P) \leq \Phi(G)$, we have that $F(\bar{G}) = \overline{F(G)}$. Let $E/\Phi(P) = E(\bar{G})$. Now $E(\bar{G})/Z(E(\bar{G}))$ is a direct product of simple non-Abelian groups and $Z(E(\bar{G})) \leq Z(F(\bar{G}))$. So we may think that $E(\bar{G})$ acts on $F(\bar{G})$ the same way as $E(\bar{G})/Z(E(\bar{G}))$ does. Let $L/\Phi(P) = \bar{L}$ such that $\bar{L}Z(E(\bar{G}))/Z(E(\bar{G}))$ is a simple component of $E(\bar{G})/Z(E(\bar{G}))$ and Q is a Sylow q-subgroup of L with $q \neq p$. Since $\bar{L}$ acts trivially on $F(\bar{G})$, we have that $\bar{Q}$ acts trivially on $\bar{P}$. Therefore, $[Q, P] \leq \Phi(P)$. By a well-known application of the Burnside Basis Theorem, we have that Q acts trivially on P itself ([16], 7.3.12). For any Sylow r-subgroup $R\Phi(P)/\Phi(P)$ of $F(G)/\Phi(P)$ with $r \neq p$, it is clear that $[Q, R] \leq R \cap \Phi(P) = 1$. So we have that Q acts trivially on $F(G)$. Note that $\langle \bar{Q} | q \neq p \rangle Z(E(\bar{G}))/Z(E(\bar{G})) = \bar{L}Z(E(\bar{G}))/Z(E(\bar{G}))$ by the simplicity of $\bar{L}Z(E(\bar{G}))/Z(E(\bar{G}))$. We have that $[L, F(G)] = 1$. Therefore, $[E, F(G)] = 1$ and hence $E \leq C_G(F(G)) \leq F^*(G)$ (see (2)). Thus $F^*(G/\Phi(P)) \leq F^*(G)/\Phi(P)$ and so $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.

Lemma 2.4 ([10, Theorem 3.3])**.** *Let G be a finite group and H a normal subgroup of G such that G/H is supersolvable. If all maximal subgroups of any Sylow subgroup of H are c-supplemented in G, then G is supersolvable.*

Lemma 2.5 ([12, Theorem 4.5])**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.*

Lemma 2.6 ([8, Lemma 3])**.** *Let H be a solvable normal subgroup of G $(H \neq 1)$. If every minimal normal subgroup of G that is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G that are contained in H.*

Lemma 2.7 ([12, Theorem 4.1])**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.*

Lemma 2.8 ([11, Theorem 4.1])**.** *Let G be a group and let H be the supersolvable residual of G. If all minimal subgroups and all cyclic subgroups with order 4 of H are c-supplemented in G, then G is supersolvable.*

Lemma 2.9 ([9, Theorem 3.2])**.** *Let G be a finite group and H a normal subgroup of G such that G/H is supersolvable. If all minimal subgroups and all cyclic subgroups with order 4 of $F^*(H)$ are c-normal in G, then G is supersolvable.*

Lemma 2.10 ([12, Lemma 2.8])**.** *Let M be a maximal subgroup of G, P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then:*

(1) $P \cap M$ is a normal subgroup of G.

(2) If $p > 2$ and all minimal subgroups of P are normal in G, then M has index p in G.

3. Proofs

The Proof of Theorem 1.1.

Proof. We consider the following two cases:

Case 1. $\mathcal{F} = \mathcal{U}$.

Suppose that the theorem is false and let G be a counterexample of minimal order. Then we have:

(1) $H = G$, $F^*(G) = F(G) \neq 1$.

By Lemma 2.4, $F^*(H)$ is supersolvable. In particular, $F^*(H)$ is solvable and so $F^*(H) = F(H) \neq 1$ by Lemma 2.3(2), (3). If $H < G$, then the minimality of G implies that H is supersolvable, since H satisfies the hypothesis of the theorem. Then G is supersolvable by Lemma 2.5, a contradiction.

(2) Every proper normal subgroup N of G containing $F^*(G)$ is supersolvable.

By Lemma 2.3(1), (3), $F^*(G) = F^*(F^*(G)) \leq F^*(N) \leq F^*(G)$; so $F^*(N) = F^*(G)$. Thus all maximal subgroups of any Sylow subgroup of $F^*(N)$ are c-supplemented in N. Hence N is supersolvable by the minimal choice of G.

Now, let P be an arbitrary Sylow p-subgroup of $F(G)$. Then

(3) $\Phi(P) = 1$, i.e., P is an elementary Abelian p-group.

If $\Phi(P) \neq 1$, consider the factor group $G/\Phi(P)$. By Lemma 2.3(5), $F^*(G/\Phi(P)) = F^*(G)/\Phi(P) = F(G)/\Phi(P)$. If $P_1/\Phi(P)$ is a maximal subgroup of the Sylow p-subgroup $P/\Phi(P)$ of $F^*(G/\Phi(P))$, then P_1 is a maximal subgroup of P. So P_1 is c-supplemented in G by hypothesis. Thus $P_1/\Phi(P)$ is c-supplemented in $G/\Phi(P)$ by Lemma 2.2(2). If $Q^*/\Phi(P)$ is a maximal subgroup of the Sylow q-subgroup $Q\Phi(P)/\Phi(P)$ of $F^*(G/\Phi(P))$, where Q is a Sylow q-subgroup of $F(G)$ and $q \neq p$, then we can denote $Q^* = Q_1\Phi(P)$, where Q_1 is a maximal subgroup of Q. Now Q_1 is c-supplemented in G by hypothesis, and Lemma 2.2(3) implies that $Q^*/\Phi(P)$ is c-supplemented in $G/\Phi(P)$. In a word, we have proved that all maximal subgroups of any Sylow subgroup of $F^*(G/\Phi(P))$ are c-supplemented in $G/\Phi(P)$. By the minimality of G, $G/\Phi(P)$ is supersolvable. Since $\Phi(P) \leq \Phi(G)$, G is supersolvable, a contradiction.

(4) There is no subgroup of order p normal in G.

If not, let P_0 be such a subgroup of G. Then $P_0 \leq P$. Since $P_0 \leq Z(P) \leq Z(F(G))$, $F(G) \leq C_G(P_0) \leq G$. Note that $C_G(P_0)$ is normal in G, and $F^*(C_G(P_0)) = F^*(G) = F(G)$. If further $C_G(P_0) < G$, then $C_G(P_0)$ is supersolvable by (2). Since $G/C_G(P_0)$ is cyclic, G is supersolvable by Lemma 2.5, a contradiction. If $C_G(P_0) = G$, then $P_0 \leq Z(G)$. By applying Lemma 2.3(6), $F^*(G/P_0) = F^*(G)/P_0$. With the same argument in (3), we know that all maximal subgroups of any Sylow subgroup of $F^*(G/P_0)$ are c-supplemented in G/P_0. The minimal choice of G

implies that G/P_0 is supersolvable, and so G is supersolvable by Lemma 2.5, also a contradiction.

(5) Write $R = P \cap \Phi(G)$. Then $R \neq 1$.

If $R = 1$, then by Lemma 2.6, $P = L_1 \times \cdots \times L_s$, where L_i $(i = 1, \ldots, s)$ are minimal normal in G. Let L_1^* be a maximal subgroup of L_1. Then $L_1^* \times L_2 \times \cdots \times L_s = P_1$ is a maximal subgroup of P. Set $K_1 = L_2 \times \cdots \times L_s$. Then $(P_1)_G = K_1$. For if not, then $K_1 < (P_1)_G$, because $K_1 \leq (P_1)_G$. Thus $(P_1)_G = (P_1)_G \cap L_1^* K_1 = K_1((P_1)_G \cap L_1^*)$ implies that $(P_1)_G \cap L_1^* \neq 1$. Furthermore, $1 < (P_1)_G \cap L_1^* \leq (P_1)_G \cap L_1 < L_1$, where $(P_1)_G \cap L_1$ is normal in G. This is contrary to the minimal normality of L_1 in G. Hence $(P_1)_G = K_1$. Since P_1 is c-supplemented in G, there exists $N_1 \leq G$ such that $G = P_1 N_1$ and $P_1 \cap N_1 \leq K_1$. It follows that $G = P_1 N_1 = L_1^* K_1 N_1$. If further $L_1^* \cap K_1 N_1 \neq 1$, then $1 < L_1 \cap K_1 N_1 \leq L_1$, where $L_1 \cap K_1 N_1 \triangleleft \langle L_1, K_1 N_1 \rangle = G$. By the minimal normality of L_1, $L_1 \cap K_1 N_1 = L_1$, i.e., $L_1 \leq K_1 N_1$. Hence $G = K_1 N_1$ and $P_1 = P_1 \cap K_1 N_1 = K_1(P_1 \cap N_1) = K_1$, i.e., $L_1^* = 1$, contradicting $L_1^* \cap K_1 N_1 \neq 1$. Thus $G = L_1^* K_1 N_1$ with $L_1^* \cap K_1 N_1 = 1$, and so $L_1 \cap K_1 N_1 = L_1$ is a normal subgroup of G with order p. This is contrary to (4).

(6) $F(G) = P$, and G has a unique minimal normal subgroup contained in R, say L.

Let Q be the Sylow q-subgroup of $F(G)$, and let L be minimal normal subgroup of G contained in R, where $q \neq p$. Then Q is elementary Abelian by (3). By the properties of a generalized Fitting subgroup (see [15], p. 128), $F^*(G/L) = F(G/L)E(G/L)$ and $[F(G/L), E(G/L)] = 1$, where $E(G/L)$ is the layer of G/L. Since $L \leq \Phi(G)$, $F(G/L) = F(G)/L$. Now set $E/L = E(G/L)$. Since Q is normal in G and $[F(G)/L, E/L] = 1$, $[Q, E] \leq Q \cap L = 1$, i.e., $[Q, E] = 1$. Therefore, $F(G)E \leq C_G(Q)$. If $C_G(Q) < G$, then $C_G(Q)$ is supersolvable by (2). Thus $E(G/L) = E/L$ is supersolvable. The semisimplicity of $E(G/L)/Z(E(G/L))$ implies that $E(G/L) = Z(E(G/L))$. So $E(G/L) \leq F(G/L)$ and $F^*(G/L) = F(G)/L$. With the same argument in (3), we have that G/L satisfies the hypothesis of the theorem. By the minimality of G, G/L is supersolvable and so is G, a contradiction. If $C_G(Q) = G$, then $Q \leq Z(G)$. By Lemma 2.3(6), $F^*(G/Q) = F^*(G)/Q = F(G)/Q$. Similarly, G/Q is supersolvable and so is G by Lemma 2.5, a contradiction.

We know similarly that the minimal normal subgroup of G contained in R is unique.

(7) $P = \langle x_1 \rangle \times \cdots \times \langle x_m \rangle \times R$, where $\langle x_i \rangle \neq 1$ and $\langle x_i \rangle R \triangleleft G, i = 1, \ldots, m$.

If $R = P$, then by Lemma 2.2(4), all maximal subgroups of P are normal in G. This implies that there is a subgroup of order p normal in G since P is elementary Abelian, a contradiction. Hence $R \neq P$. Now $\Phi(G/R) = 1$. Then by Lemma 2.6, $P/R = (H_1/R) \times \cdots \times (H_m/R)$, where H_i/R $(i = 1, \ldots, m)$ are minimal normal in G/R. With the same argument as in (5), we know that H_i/R $(i = 1, \ldots, m)$ are all of order p because all maximal subgroups of P/R are c-supplemented in G/R. Again, P is an elementary Abelian p-group. So H_i is of the form $\langle x_i \rangle R, i = 1, \ldots, m$.

(8) For any maximal subgroup R^* of R, $(TR^*)_G \cap R \neq 1$, where $T = \langle x_1 \rangle \times \cdots \times \langle x_m \rangle$.

If not, $(TR^*)_G \cap R = 1$. Then $(TR^*)_G \cap \langle x_1 \rangle R$ is of order 1 or p. Note that $(TR^*)_G \cap \langle x_1 \rangle R$ is normal in G, and its order must not be p by (4). Thus $(TR^*)_G \cap \langle x_1 \rangle R = 1$. Similarly, $(TR^*)_G \cap \langle x_1 \rangle \langle x_2 \rangle R = \cdots = (TR^*)_G \cap TR = 1$, i.e., $(TR^*)_G = 1$. Since TR^* is maximal in P, by the hypothesis, there exists $N \leq G$ such that

$G = (TR^*)N$ with $TR^* \cap N = 1$. Thus $P \cap N$ is a normal subgroup of G of order p, also a contradiction.

(9) Final contradiction.

From (6) and (8), $L \leq (TR^*)_G \cap R$ for any maximal subgroup R^* of R. Now let N be the c-supplement of TR^* in G. Then $(TR^*)_G \cap R = TR^* \cap (TR^*)_G N \cap R = R^* \cap (TR^*)_G N$. Thus $L \leq \bigcap_{R^* <\cdot R} (R^* \cap (TR^*)_G N) = \Phi(R) \cap (\bigcap_{R^* <\cdot R} (TR^*)_G N) = 1$, a final contradiction.

Case 2. $\mathcal{F} \neq \mathcal{U}$.

By case 1, H is supersolvable. In particular, H is solvable and so $F^*(H) = F(H)$. Hence $G \in \mathcal{F}$ by Lemma 2.5.

This completes our proof.

The Proof of Theorem 1.2.

Proof. By Lemma 2.7, we need only prove that H is solvable. Suppose this is false, and let G be a counterexample of minimal order. Then

(1) $F^*(H) = F(H) \neq 1$.

In fact, $F^*(H)$ is supersolvable by Lemma 2.8. So $F^*(H) = F(H) \neq 1$ by Lemma 2.3(2), (3).

(2) $\Phi(H) \neq F(H)$.

If it is not so, then by Lemma 2.2(4), all minimal subgroups and all cyclic subgroups with order 4 of $F^*(H)$ are normal in G. By Lemma 2.9, H is supersolvable, a contradiction.

(3) For any proper normal subgroup K of H, K is solvable.

Since K is normal in H, $F^*(K) \leq F^*(H)$ by Lemma 2.3(1). Also, all minimal subgroups and all cyclic subgroups with order 4 of $F^*(K)$ are c-supplemented in K. The minimal choice of H implies that K is solvable.

(4) There exist $O_p(H)$ and $M <\bullet\ H$ such that $O_p(H) \cap M <\bullet\ O_p(H)$.

Since $\Phi(H) \neq F(H)$, there exist $O_p(H)$ and $M <\bullet\ H$ such that $O_p(H) \not\subseteq M$. Then $H = O_p(H)M$. To show $O_p(H) \cap M <\bullet\ O_p(H)$ we consider the following two cases:

(4.1) $p > 2$.

If $O_p(H)$ has at least one minimal subgroup $\langle x \rangle$ nonnormal in H, then by Lemma 2.2(1), $\langle x \rangle$ is c-supplemented in H; i.e., there is a subgroup N of H such that $H = \langle x \rangle N$ and $\langle x \rangle \cap N = 1$. Furthermore, N is a maximal subgroup of H and $O_p(H) \cap N$ is a normal subgroup of H by Lemma 2.10(1). Again, $O_p(H) = O_p(H) \cap \langle x \rangle N = \langle x \rangle (O_p(H) \cap N)$. If $O_p(H) \cap N \leq M$, then $H = O_p(H)M = \langle x \rangle M$ with $\langle x \rangle \cap M = 1$. This implies $|O_p(H) : O_p(H) \cap M| = |H : M| = |\langle x \rangle| = p$. Hence $O_p(H) \cap M <\bullet\ O_p(H)$. If $O_p(H) \cap N \not\subseteq M$, then $H = (O_p(H) \cap N)M$, where x is not in $O_p(H) \cap N$. With the same argument we may assume that all minimal subgroups of $O_p(H) \cap N$ are normal in H. By Lemma 2.10(2), $|O_p(H) : O_p(H) \cap M| = |H : M| = p$; so $O_p(H) \cap M <\bullet\ O_p(H)$.

(4.2) $p = 2$.

Let $\pi(H) = \{p_1, p_2, \ldots, p_n\}$, and let M_{p_i} be a Sylow p_i-subgroup of M, where $i = 1, 2, \ldots, n$ and $p_1 = 2$. Then we know easily that $O_2(H)M_2 = H_2$ is a Sylow 2-subgroup of H. Now, let P_1 be a maximal subgroup of H_2 containing M_2, and set $P_2 = P_1 \cap O_2(H)$. Then $P_1 = P_2 M_2$. Moreover, $P_2 \cap M_2 = O_2(H) \cap M_2$. So $|O_2(H) : P_2| = |O_2(H)M_2 : P_2 M_2| = |H_2 : P_1| = 2$, i.e., $P_2 <\bullet\ O_2(H)$. Again, for each $i \neq 1, O_2(H)M_{p_i}$ is supersolvable by Lemma 2.7, hence $O_2(H)M_{p_i} =$

$O_2(H) \times M_{p_i}$. Furthermore, $P_2 M_{p_i}$ forms a group, where $i = 1, 2, \ldots, n$. Hence $P_2\langle M_{p_1}, M_{p_2}, \cdots, M_{p_n}\rangle = P_2 M$ also forms a group. Since $|O_2(H) : P_2| = 2$ and $P_2 \cap M = O_2(H) \cap M$, it follows that $P_2 M < O_2(H)M = H$. By the maximality of M in $H, P_2 M = M$ and hence $P_2 \leq M$. Thus $O_2(H) \cap M = P_2 \cap M = P_2$. Therefore, $O_2(H) \cap M <\bullet\ O_2(H)$.

(5) Final contradiction.

From (4), $H = O_p(H)M$. Set $L = O_p(H) \cap M$. Then L is normal in H by Lemma 2.10(1) and $H/L = (O_p(H)/L)(M/L)$. Since $O_p(H)/L$ is a normal subgroup of H/L of order p, $F(H)/L \leq C_{H/L}(O_p(H)/L)$. Write $C_{H/L}(O_p(H)/L) = C/L$; then C is normal in H and $F(H) \leq C$. If $C < H$, then C is solvable by (3). Note that $H/C \cong (H/L)/(C/L)$ is cyclic; so H is solvable, a contradiction. If $C = H$, then $O_p(H/L) \leq Z(H/L)$. This implies that $O_p(H)/L$ centralizes M/L; so M/L is a normal subgroup of H/L. Thus M is a normal subgroup of H. Furthermore, M is solvable by (3). The solvability of H/M implies that H is solvable, a final contradiction.

This completes our proof.

Remarks. Theorems 1.1 and 1.2 are not true for non-saturated formations. Let $\mathcal{F}$ be the formation composed of all groups G such that $G^{\mathcal{U}}$, the supersolvable residual, is elementary Abelian. Clearly $\mathcal{U} \subseteq \mathcal{F}$, but $\mathcal{F}$ is not saturated. Set $G = SL(2,3), H = Z(G)$. Then $G/Z(G)$ is isomorphic to the alternating group of degree four and so $G/H \in \mathcal{F}$. The other hypotheses in Theorems 1.1 and 1.2 are satisfied since H is of order 2, but G does not belong to $\mathcal{F}$.

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References

[1] S. Srinivasan, Two sufficient conditions for supersolvability of finite groups, Israel J. Math. 35(3) (1980), 210-214. MR **81f:**20044

[2] J. Buckley, Finite groups whose minimal subgroups are normal, Math. Z. 116 (1970), 15-17. MR **41:**6967

[3] M. Asaad, On maximal subgroups of Sylow subgroups of finite groups, Communications in Algebra 26(11) (1998), 3647-3652. MR **99k:**20039

[4] M. Assad, M. Ramadan, and A. Shaalan, The influence of π-quasinormality of maximal subgroups of Sylow subgroups of Fitting subgroups of a finite group, Arch. Math. 56 (1991), 521-527. MR **92d:**20022

[5] M. Ramadan, Influence of normality on maximal subgroups of Sylow subgroups of a finite group, Acta Math. Hungar. 59(1-2) (1992), 107-110. MR **93c:**20042

[6] A. Ballester-Bolinches and M. C. Pedraza Aguilera, On minimal subgroups of finite groups, Acta Math. Hungar. 73 (1996), 335-342. MR **97j:**20019

[7] Y. Wang, c-Normality of finite groups and its properties, J. Algebra 180 (1996), 954-965. MR **97b:**20020

[8] H. Wei, On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups, Comm. Algebra 29(5) (2001), 2193-2200. MR **2002d:**20030

[9] Y. Wang, The influence of minimal subgroups on the structure of finite groups, Acta Math. Sinica, English Series 16(1) (2000), 63-70. MR **2001c:**20035

[10] Y. Wang, Finite groups with some subgroups of Sylow subgroups c-supplemented, J. Algebra 224 (2000), 467-478. MR **2001c:**20036

[11] A. Ballester-Bolinches, Y. Wang, and X. Guo, *c*-supplemented subgroups of finite groups, Glasgow Math. J. 42 (2000), 383-389. MR **2001k:**20031
[12] Y. Wang, H. Wei, and Y. Li, A generalization of a theorem of Kramer and its applications, Bull. Austral. Math. Soc. 65 (2002), 467-475. MR **2003f:**20025
[13] K. Doerk and T. O. Hawkes, Finite Soluble Groups, De Gruyter, Berlin, 1992. MR **93k:**20033
[14] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin and New York, 1967. MR **37:**302
[15] B. Huppert and N. Blackburn, Finite Groups III, Springer-Verlag, Berlin and New York, 1982. MR **84i:**20001b
[16] W. R. Scott, Group Theory, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964. MR **29:**4785
[17] Y. Li and Y. Wang, The influence of minimal subgroups on the structure of a finite group, Proc. Amer. Math. Soc. 131(2) (2003), 337-341. MR **2003h:**20032

Department of Mathematics, Zhongshan University, Guangzhou 510275, China and Department of Mathematics, Guangxi Teacher's College, Nanning, 530001, China
E-mail address: `weihuaquan@163.com`

Lingnan College and Department of Mathematics, Zhongshan University, Guangzhou, 510275, China
E-mail address: `stswym@zsu.edu.cn`

Department of Mathematics, Guangdong College of Education, Guangzhou, 510310, China
E-mail address: `liyangming@gdei.edu.cn`