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FALTINGS' THEOREM FOR THE ANNIHILATION OF LOCAL COHOMOLOGY MODULES OVER A GORENSTEIN RING

K. KHASHYARMANESH AND SH. SALARIAN

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Abstract. In this paper we study the Annihilator Theorem and the Local-global Principle for the annihilation of local cohomology modules over a (not necessarily finite-dimensional) Noetherian Gorenstein ring.

1. Introduction

Let A be a commutative Noetherian ring, let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A and let M be a finitely generated A-module. We first briefly recall the invariant $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M)$ defined in [3].

By the terminology of [3], the $\mathfrak{b}$-minimum $\mathfrak{a}$-adjusted depth $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M)$ of M is defined by

$$\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) = \inf\{\mathrm{depth} M_{\mathfrak{p}} + \mathrm{ht}(\mathfrak{a}+\mathfrak{p})/\mathfrak{p} : \mathfrak{p} \in \mathrm{Spec}(A)\backslash V(\mathfrak{b})\}$$

where $V(\mathfrak{b})$ denotes the set of prime ideals containing $\mathfrak{b}$.

We shall say that the Annihilator Theorem (for local cohomology modules) holds over A if $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) = f_{\mathfrak{a}}^{\mathfrak{b}}(M)$ for every choice of the finitely generated A-module M and for every choice of the ideals $\mathfrak{b}, \mathfrak{a}$ of A where $f_{\mathfrak{a}}^{\mathfrak{b}}(M)$ is the $\mathfrak{b}$-finiteness dimension of M relative to $\mathfrak{a}$. Also, we say that the Local-global Principle (for the annihilation of local cohomology modules) holds over A if

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M) = \inf\{f_{\mathfrak{a}A_{\mathfrak{p}}}^{\mathfrak{b}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(A)\}$$

for every choice of ideals $\mathfrak{a}, \mathfrak{b}$ of A and every choice of the finitely generated A-module M.

Faltings' Annihilator Theorem [5] states that if A is a homomorphic image of a regular ring or A has a dualizing complex, then the annihilator theorem (for local cohomology modules) holds over A. In [7], Raghavan deduced from Faltings' Annihilator Theorem [5] that if A is a homomorphic image of a regular ring, then the Local-global Principle (for the annihilation of local cohomology modules) holds over A. Recently, in [2], Brodmann, Rotthaus and Sharp showed, for every non-negative integer r, that if A is universally catenary and all formal fibres of all localizations of A satisfy Serre's Condition (S_r), then the Annihilator Theorem (for

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local cohomology modules) holds at level r over A if and only if the Local-global Principle (for the annihilation of local cohomology modules) holds at level r over A. They also established the Local-global Principle (for the annihilation of local cohomology modules) over an arbitrary commutative Noetherian ring of dimension not exceeding 4.

In this paper, we show that the Local-global Principle and the Annihilator Theorem (for local cohomology modules) hold over a (homomorphic image of) a commutative Noetherian Gorenstein ring. The proof of the main theorem relies heavily on ideas in Brodmann's proof of [1, Satz 3.12] and in Raghavan's proof of [8, 3.1]. But our method is based on the theory of Gorenstein dimension or G-dimension. To the best of our knowledge, this is the first application of this theory in the study of Faltings' Annihilator Theorem.

Throughout the paper, A will denote a commutative Noetherian ring (with non-zero identity), $\mathfrak{a}$, and $\mathfrak{b}$ will denote ideals of A, and M will denote a finitely generated A-module. We use $\mathbb{N}$ and $\mathbb{N}_0$ to denote the set of positive and nonnegative integers respectively. For any unexplained notation and terminology concerning G-dimension we refer the reader to [4].

2. Results

Let us, firstly, introduce $\mathfrak{b}$-finiteness dimension of M relative to $\mathfrak{a}$ (see [3, 9.1.5]):

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{b}^n H_{\mathfrak{a}}^i(M) \neq 0 \text{ for all } n \in \mathbb{N}\},$$

where, by convention, the infimum of the empty set of integers is interpreted by ∞. Note that if $\mathfrak{a} = \mathfrak{b}$, then $f_{\mathfrak{b}}^{\mathfrak{b}}(M) = f_{\mathfrak{a}}(M)$ is the finiteness dimension of M relative to $\mathfrak{a}$.

We begin by the following remark.

Remark 2.1. It follows from [2, 3.7] that $f_{\mathfrak{a}}^{\mathfrak{b}}(M) = f_{\mathfrak{a}+\mathfrak{b}}^{\mathfrak{b}}(M)$. So one can reduce the study of the Local-global Principle (for the annihilation of local cohomology modules) for arbitrary ideals $\mathfrak{b}$ and $\mathfrak{a}$ to the study of this principle in the case that $\mathfrak{b} \subseteq \mathfrak{a}$.

We now prove some preliminary lemmas and a proposition which help us to conclude the main theorem.

Lemma 2.2. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A such that $\mathfrak{b} \subseteq \mathfrak{a}$ and $0 :_A M/\Gamma_{\mathfrak{b}}(M) \subseteq \mathfrak{a}$. Let s be a positive integer. Suppose that $f_{\mathfrak{a}A_{\mathfrak{p}}}^{\mathfrak{b}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) > s$ for every prime ideal $\mathfrak{p}$ of A. Then $s < \mathrm{ht}_M \mathfrak{a}$.*

Proof. We suppose that $s \geq \mathrm{ht_M}\mathfrak{a}$, and look for a contradiction. Firstly, we can use the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{b}}(M) \longrightarrow M \longrightarrow M/\Gamma_{\mathfrak{b}}(M) \longrightarrow 0$$

to deduce the long exact sequence

$$\cdots \longrightarrow H_{\mathfrak{a}}^i(\Gamma_{\mathfrak{b}}(M)) \longrightarrow H_{\mathfrak{a}}^i(M) \longrightarrow H_{\mathfrak{a}}^i(M/\Gamma_{\mathfrak{b}}(M)) \longrightarrow \cdots .$$

Now, it follows from the assumption that $f_{\mathfrak{a}A_{\mathfrak{p}}}^{\mathfrak{b}A_{\mathfrak{p}}}(M/\Gamma_{\mathfrak{b}}(M))_{\mathfrak{p}} > s$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Since $\mathrm{ht}_{M/\Gamma_{\mathfrak{b}}(M)}\mathfrak{a} \leq \mathrm{ht}_M \mathfrak{a}$, we can thus replace M by $M/\Gamma_{\mathfrak{b}}(M)$ and assume, in our search for a contradiction, that there exists $x \in \mathfrak{b}$ that is a non-zero-divisor on M. Since $0 :_A M \subseteq \mathfrak{a}$, there exists a minimal prime ideal $\mathfrak{p}$ over $\mathfrak{a}$ such that

$\mathrm{ht}_M\mathfrak{p} \leq s$. Set $t := \mathrm{ht}_M\mathfrak{p}$. Then, by [3, 6.1.4], $H^t_{\mathfrak{a}A_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$ and by hypothesis, $x^{n(\mathfrak{p})}H^t_{\mathfrak{a}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for some $n(\mathfrak{p}) \in \mathbb{N}$. Now, using the exact sequence

$$0 \longrightarrow M \stackrel{x^{n(\mathfrak{p})}}{\longrightarrow} M \longrightarrow \frac{M}{x^{n(\mathfrak{p})}M} \longrightarrow 0,$$

we obtain the exact sequence

$$H^t_{\mathfrak{a}A_\mathfrak{p}}(M_\mathfrak{p}) \stackrel{x^{n(\mathfrak{p})}}{\longrightarrow} H^t_{\mathfrak{a}A_\mathfrak{p}}(M_\mathfrak{p}) \longrightarrow 0,$$

which, in turn, yields $H^t_{\mathfrak{a}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$; this is the required contradiction. □

Lemma 2.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A, and let M be a finitely generated A-module. Let $\mathfrak{c}$ be an ideal of A such that $\mathfrak{c} \subseteq 0 :_A M/\Gamma_\mathfrak{b}(M)$. Then $f^\mathfrak{b}_{\mathfrak{a}+\mathfrak{c}}(M) = f^\mathfrak{b}_\mathfrak{a}(M)$.*

Proof. Let n be a positive integer such that $\Gamma_\mathfrak{b}(M) = 0 :_M \mathfrak{b}^n$. Hence, it follows from our assumption that $\mathfrak{b}^n r H^i_\mathfrak{a}(M) = 0$, for all $r \in \mathfrak{c}$ and $i \geq 0$, and so $\mathfrak{b}^n(H^i_\mathfrak{a}(M))_r = 0$. Now, the claim follows from the exact sequence

$$\cdots \longrightarrow H^i_{\mathfrak{a}+Ar}(M) \longrightarrow H^i_\mathfrak{a}(M) \longrightarrow (H^i_\mathfrak{a}(M))_r \longrightarrow H^{i+1}_{\mathfrak{a}+Ar}(M) \longrightarrow \cdots .$$

□

For an A-module M the biduality map is the canonical map

$$\delta_M : M \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A),$$

defined by $\delta_M(x)(\psi) = \psi(x)$ for $\psi \in \mathrm{Hom}_A(M, A)$ and $x \in M$. It is a homomorphism of A-modules.

Definition 2.4. [4, 1.1.2] A finitely generated A-module M belongs to the G-class $G(A)$ if and only if

(1) $\mathrm{Ext}^t_A(M, A) = 0$ for $t > 0$;
(2) $\mathrm{Ext}^t_A(\mathrm{Hom}_A(M, A), A) = 0$ for $t > 0$; and
(3) the biduality map $\delta_M : M \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A)$ is an isomorphism.

Definition 2.5. [4, 1.2.1] A G-resolution of a finitely generated A-module M is a sequence of modules in $G(A)$,

$$\cdots \longrightarrow G_l \longrightarrow G_{l-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow 0,$$

which is exact at G_l for $l > 0$ and has $G_0/\mathrm{Im}(G_1 \longrightarrow G_0) \cong M$. That is, there is an exact sequence

$$\cdots \longrightarrow G_l \longrightarrow G_{l-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0.$$

The resolution is said to be of length n if $G_n \neq 0$ and $G_l = 0$ for $l > n$.

Note that every finitely generated A-module has a resolution by finitely generated free modules and, thereby, a G-resolution.

Definition 2.6. [4, 1.2.3] A finitely generated A-module M is said to have finite G-dimension if it has a G-resolution of finite length. We set $G - \dim_A 0 = -\infty$, and for $M \neq 0$ we define the G-dimension of M as follows: For any positive integer n we say that M has G-dimension at most n, and write $G - \dim_A M \leq n$ for short, if and only if M has a G-resolution of length n. If M has no G-resolution of finite length, then we say that it has infinite G-dimension and write $G - \dim_A M = \infty$.

Remark 2.7. By [4, 1.4.9], a commutative Noetherian local ring A is Gorenstein if and only if every finite A-module has a finite G-resolution.

Proposition 2.8. *Let M be a finitely generated A-module and let $G-\dim_A M = 0$. Then* $\operatorname{grade}(\mathfrak{a}, M) \geq \operatorname{grade}(\mathfrak{a}, A)$*, for every ideal* $\mathfrak{a}$ *of* A.

Proof. If $\mathfrak{a}M = M$, then $\operatorname{grade}(\mathfrak{a}, M) = \infty$ and the result is clear. We therefore suppose that $\mathfrak{a}M \neq M$. Set $g := \operatorname{grade}(\mathfrak{a}, A)$. We argue by induction on g. When $g = 0$ the result is clear. We therefore suppose inductively that $g > 0$ and that the result has been proved for smaller values of g. Since $G - \dim_A M = 0$, by [4, 1.1.8], we have that $\operatorname{Ass}M \subseteq \operatorname{Ass}A$. Hence there exists $x \in \mathfrak{a}$ such that x is an M- and A-sequence. Also, by [4, 1.3.6], $G - \dim_{A/xA} M/xM = 0$. Note that by our assumption $xM \neq M$. Therefore, by the inductive hypotheses, $\operatorname{grade}(\mathfrak{a}, M/xM) \geq \operatorname{grade}(\mathfrak{a}, A/xA)$ and so $\operatorname{grade}(\mathfrak{a}, M) \geq \operatorname{grade}(\mathfrak{a}, A)$. $\square$

Lemma 2.9. *Let M be a finitely generated A-module, and let $\mathfrak{p} \in \operatorname{Spec}(A)$ be such that $M_\mathfrak{p} \in G(A_\mathfrak{p})$. Then there exists $s \in A\backslash\mathfrak{p}$ such that $sH^i_\mathfrak{a}(M) = 0$ for all $i < \operatorname{grade}(\mathfrak{a}, A)$.*

Proof. We use an inductive argument on i. Let M be a finitely generated A-module, and let $\mathfrak{p} \in \operatorname{Spec}(A)$ be such that $M_\mathfrak{p} \in G(A_\mathfrak{p})$. If $i = 0$, since $H^0_\mathfrak{a}(M)$ is a finitely generated A-module and, in view of 2.8, $\mathfrak{p} \notin \operatorname{Supp}H^0_\mathfrak{a}(M)$, there exists $s \in A\backslash\mathfrak{p}$ such that $sH^0_\mathfrak{a}(M) = 0$. Now suppose that $1 \leq i < \operatorname{grade}(\mathfrak{a}, A)$ and that the result holds for smaller values of i. Consider the exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\delta_M} M^{**} \longrightarrow L \longrightarrow 0,$$

where M^{**} is the bidual of M. Since the A-modules K and L are finitely generated and $\mathfrak{p} \notin \operatorname{Supp}(K) \cup \operatorname{Supp}(L)$, there exists $t \in A\backslash\mathfrak{p}$ such that $tH^i_\mathfrak{a}(K) = 0 = tH^i_\mathfrak{a}(L)$ for all $i \geq 0$. Hence, by breaking the above exact sequence into short exact sequences, we need only to prove the result for M^{**}. Note that, by [4, 1.1.7], $M^{**}_\mathfrak{p} \in G(A_\mathfrak{p})$. Now, let

$$0 \longrightarrow W \longrightarrow F \longrightarrow M^* \longrightarrow 0$$

be a finite presentation of M^*. By applying the duality functor $-^* = \operatorname{Hom}_A(-, A)$, we obtain the exact sequence

$$0 \longrightarrow M^{**} \longrightarrow F \longrightarrow W^* \longrightarrow \operatorname{Ext}^1_A(M^*, A) \longrightarrow 0,$$

which breaks up into two short exact sequences

$$0 \longrightarrow M^{**} \longrightarrow F \longrightarrow D \longrightarrow 0 \ \text{ and } \ 0 \longrightarrow D \longrightarrow W^* \longrightarrow \operatorname{Ext}^1_A(M^*, A) \longrightarrow 0.$$

Since $\operatorname{grade}(\mathfrak{a}, A) > i$, we get from the induced long exact sequence of $H^i_\mathfrak{a}$,

(1) $H^i_\mathfrak{a}(M^{**}) \cong H^{i-1}_\mathfrak{a}(D)$, and

(2) $H^{i-2}_\mathfrak{a}(\operatorname{Ext}^1_A(M^*, A)) \longrightarrow H^{i-1}_\mathfrak{a}(D) \longrightarrow H^{i-1}_\mathfrak{a}(W^*)$ is exact.

On the other hand, since $\operatorname{Ext}^1_A(M^*, A)$ is a finitely generated A-module and $\mathfrak{p} \notin \operatorname{Supp}(\operatorname{Ext}^1_A(M^*, A))$, there exists $s \in A\backslash\mathfrak{p}$ such that $sH^i_\mathfrak{a}(\operatorname{Ext}^1_A(M^*, A)) = 0$ for all $i \in \mathbb{N}_0$ Also, by [4, 1.1.10(a)], we have that $W^*_\mathfrak{p} \in G(A_\mathfrak{p})$. Now the conclusion follows by applying the inductive hypotheses on W^* in conjunction with (1) and (2). $\square$

Theorem 2.10. *The Local-global Principle (for the annihilation of local cohomology modules) holds over a commutative Noetherian Gorenstein ring.*

Remark 2.11. Whenever A is the homomorphic image of a Gorenstein ring and M a finitely generated A-module then, by [2, 3.4(i)] and Faltings' Annihilator Theorem (in the local case), for every choice of the ideals $\mathfrak{b}$, $\mathfrak{a}$ of A, $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) = \inf\{f_{\mathfrak{a}_\mathfrak{p}}^{\mathfrak{b}_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}(M)\}$.

Proof of 2.10. We suppose, for ideals $\mathfrak{a}$ and $\mathfrak{b}$ of a Gorenstein ring A and a finitely generated A-module M, that $f_{\mathfrak{a}}^{\mathfrak{b}}(M) < f_{\mathfrak{a}A_\mathfrak{p}}^{\mathfrak{b}A_\mathfrak{p}}(M_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$ and look for a contradiction. We can (and do) assume that $\mathfrak{a}$ is a maximal member of the set $\{\mathfrak{c} : \mathfrak{c}$ is an ideal of A such that, for some finitely generated A-module $M', f_{\mathfrak{c}}^{\mathfrak{b}}(M') < f_{\mathfrak{c}A_\mathfrak{p}}^{\mathfrak{b}A_\mathfrak{p}}(M'_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(A)\}$. By [2, 3.2], it follows that $\mathfrak{a}$ is a prime ideal of A.

By 2.1 and 2.3, we can assume, in our search for a contradiction, that $\mathfrak{b} \subseteq \mathfrak{a}$, and $0 : M/\Gamma_{\mathfrak{b}}(M) \subseteq \mathfrak{a}$. Set $s := \inf\{f_{\mathfrak{a}A_\mathfrak{p}}^{\mathfrak{b}A_\mathfrak{p}}(M_\mathfrak{p}) : \mathfrak{p} \in \operatorname{Supp}(M)\}$. By 2.2, $s \leq \operatorname{ht} \mathfrak{a}$. There are two cases to consider.

Case 1. Suppose that $\operatorname{ht}\mathfrak{a} = s$. Let $\mathfrak{p}$ be a prime ideal of A such that $G - \dim_{A_\mathfrak{p}} M_\mathfrak{p} = 0$. Then, by 2.9, there exists $s_\mathfrak{p} \in A\backslash\mathfrak{p}$ such that $s_\mathfrak{p} H_{\mathfrak{a}}^i(M) = 0$ for all $i < s$. Set

$$S := \{\mathfrak{q} \in \operatorname{Spec}(A) : M_\mathfrak{q} \in G(A_\mathfrak{q})\},$$

and let g be the ideal generated by all $s_\mathfrak{p}$'s, where $\mathfrak{p}$ ranges in S. In case $S = \emptyset$, we set $g := 0$. Observe that if $g \subseteq \mathfrak{p}$, then $\mathfrak{p} \notin S$. Consequently, $G - \dim_{A_\mathfrak{p}} M_\mathfrak{p} > 0$. Also note that $gH_{\mathfrak{a}}^i(M) = 0$ for all $i < s$. If $\mathfrak{b} \subseteq \operatorname{Rad}(g)$, then we have nothing to do any more. So we may assume that $\mathfrak{b} \not\subseteq \operatorname{Rad}(g)$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal primes over g not containing $\mathfrak{b}$. It is easy to see that $\bigcap\limits_{i=1}^{t} \mathfrak{p}_i \not\subseteq \mathfrak{a}$. (If $\mathfrak{p}_i \subset \mathfrak{a}$, by the same argument as in [3, 9.4.8], one can deduce that $G - \dim M_{\mathfrak{p}_i} > 0$ and so, by the Auslander-Bridger formula (cf. [4, 1.4.8]), $\operatorname{depth} R_{\mathfrak{p}_i} - \operatorname{depth} M_{\mathfrak{p}_i} > 0$. This implies that $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) \leq \operatorname{depth} M_{\mathfrak{p}_i} + \operatorname{ht}\frac{\mathfrak{a}+\mathfrak{p}_i}{\mathfrak{p}_i} < \operatorname{depth} R_{\mathfrak{p}_i} + \operatorname{ht}\frac{\mathfrak{a}}{\mathfrak{p}_i} \leq \operatorname{ht}\mathfrak{a}$, which is a contradiction.) Let $x \in \bigcap\limits_{i=1}^{t} \mathfrak{p}_i \backslash \mathfrak{a}$. Note that $\mathfrak{b}_x \subseteq (\operatorname{Rad}(g))_x$. Now it follows from the exact sequence

$$\cdots \longrightarrow H_{\mathfrak{a}+Ax}^i(M) \longrightarrow H_{\mathfrak{a}}^i(M) \longrightarrow H_{\mathfrak{a}_x}^i(M_x) \longrightarrow \cdots$$

that $f_{(\mathfrak{a}+Ax)A_\mathfrak{p}}^{\mathfrak{b}A_\mathfrak{p}}(M_\mathfrak{p}) \geq s$. Thus by the "maximality" assumption about $\mathfrak{a}$, we have that $f_{\mathfrak{a}+Ax}^{\mathfrak{b}}(M) \geq s$. Now, we deduce from the above exact sequence that $f_{\mathfrak{a}}^{\mathfrak{b}}(M) \geq s$.

Case 2. Suppose that $\operatorname{ht}\,\mathfrak{a} > s$. Consider the finite presentation

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

of M to deduce the long exact sequence

$$(*) \qquad \cdots \longrightarrow H_{\mathfrak{a}}^i(M) \longrightarrow H_{\mathfrak{a}}^{i+1}(K) \longrightarrow H_{\mathfrak{a}}^{i+1}(F) \longrightarrow \cdots .$$

Therefore, $f_{\mathfrak{a}A_\mathfrak{p}}^{\mathfrak{b}A_\mathfrak{p}}(K_\mathfrak{p}) > s$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$. Hence, by case 1, $f_{\mathfrak{a}}^{\mathfrak{b}}(K) \geq s+1$. Now, by using the exact sequence $(*)$, we can deduce that $f_{\mathfrak{a}}^{\mathfrak{b}}(M) \geq s$. $\square$

Corollary 2.12. *The Annihilator Theorem (for local cohomology modules) holds over a commutative Noetherian Gorenstein ring.*

Proof. Let A be a Gorenstein ring, let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A, and let M be a finitely generated A-module. We must show that $f_{\mathfrak{a}}^{\mathfrak{b}}(M) = \lambda_{\mathfrak{a}}^{\mathfrak{b}}(M)$. By [2, 3.7], we may assume that $\mathfrak{b} \subseteq \mathfrak{a}$. Hence, by [3, 9.3.5], the only nontrivial point is the proof that $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) \leq f_{\mathfrak{a}}^{\mathfrak{b}}(M)$. By 2.10, there exists a prime ideal $\mathfrak{p}$ of A such that $f_{\mathfrak{a}}^{\mathfrak{b}}(M) = f_{\mathfrak{a}A_{\mathfrak{p}}}^{\mathfrak{b}A_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Also, by [5], $f_{\mathfrak{a}A_{\mathfrak{p}}}^{\mathfrak{b}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \lambda_{\mathfrak{a}A_{\mathfrak{p}}}^{\mathfrak{b}A_{\mathfrak{p}}}(M_{\mathfrak{p}})$. We now use [3, 9.2.5] to complete the proof. □

Corollary 2.13. *The Local-global Principle and the Annihilator Theorem (for local cohomology modules) hold over a homomorphic image of a commutative Noetherian Gorenstein ring.*

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Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran – and – Department of Mathematics, Damghan University, P.O. Box 36715-364, Damghan, Iran

E-mail address: `Khashyar@ipm.ir`

Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran – and – Department of Mathematics, Damghan University, P.O. Box 36715-364, Damghan, Iran

E-mail address: `Salarian@ipm.ir`