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A WOLD-TYPE DECOMPOSITION FOR COMMUTING ISOMETRIC PAIRS

DAN POPOVICI

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Abstract. We obtain a Wold-type decomposition theorem for an arbitrary pair of commuting isometries V on a Hilbert space. More precisely, V can be uniquely decomposed into the orthogonal sum between a bi-unitary, a shift-unitary, a unitary-shift and a weak bi-shift part, that is, a part $S = (S_1, S_2)$ that can be characterized by the condition that S_1S_2, $S_1|_{\bigcap_{n\geq 0} \ker S_2^* S_1^n}$ and $S_2|_{\bigcap_{n\geq 0} \ker S_1^* S_2^n}$ are shifts. Moreover, S contains bi-shift and modified bi-shift maximal parts.

1. Preliminaries

Wold introduced, in a probabilistic language, a remarkable decomposition for stationary stochastic processes ([Wo]). Separating the deterministic part by the part corrupted by noises, this decomposition becomes the cornerstone of prediction theory for this kind of process. Other famous mathematicians, including von Neumann, Kolmogorov, and Halmos, formulated Wold's result for isometric operators on Hilbert spaces: the study of such operators is reduced to the study of unitary operators (which are well understood: we have the associated spectral theory and functional calculus) and of unilateral shifts (which have a very simple geometrical structure).

A (*unilateral*) *shift* is an operator S on a Hilbert space $\mathcal{H}$ unitarily equivalent to multiplication by the independent variable z on a certain Hardy space on the torus. More precisely, there exist a Hilbert space $\mathcal{W}$ and a unitary operator $U : \mathcal{H} \to H^2(\mathbb{T}) \otimes \mathcal{W}$ such that $S = U^*(T_z \otimes I)U$ (the symbol "$\otimes$" denotes the Hilbertian tensor product). The following characterization illustrates the shift's geometrical structure: an isometry S on $\mathcal{H}$ is a shift if and only if there exists a wandering subspace $\mathcal{W}$ (i.e., $S^n\mathcal{W} \perp S^m\mathcal{W}$, $n, m \geq 0$, $n \neq m$) that generates $\mathcal{H}$, that is, $\mathcal{H} = \bigoplus_{n\geq 0} S^n\mathcal{W}$. $\mathcal{W}$ is unique ($\mathcal{W} = \ker S^*$) and is said to be the *defect* of S.

An operator V on $\mathcal{H}$ is said to be *reduced* by a (closed) subspace $\mathcal{H}_0 \subset \mathcal{H}$ if $\mathcal{H}_0$ is invariant under both V and V^*, i.e., $V\mathcal{H}_0 \subset \mathcal{H}_0$ and $V^*\mathcal{H}_0 \subset \mathcal{H}_0$.

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Wold's Theorem (see [NF1], Chapter 1)**.** *To any isometry V on $\mathcal{H}$ there corresponds a unique orthogonal decomposition of the form*

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$$

such that V is reduced by $\mathcal{H}_u$ to a unitary operator and by $\mathcal{H}_s$ to a shift. More exactly, $\mathcal{H}_u = \bigcap_{n\geq 0} V^n\mathcal{H}$ and $\mathcal{H}_s = \bigoplus_{n\geq 0} V^n\mathcal{W}$, with $\mathcal{W} = \ker V^*$.

We want to underline the importance of this decomposition in dilation theory, invariant subspace theory, prediction theory, and operator interpolation problems. It is expected that such a result for several commuting isometries would provide a larger class of applications. Some steps forward, which motivate our present work, have already been made: certain multi-dimensional Wold-type decompositions have been applied in index theory for $\mathbf{C}^*$-algebras ([BCL]) or invariant subspace theory ([Ko]).

One of the first attempts to extend the Wold theorem was made by Słociński [Sł], who provided conditions on a commuting isometric pair in order to obtain a fourfold Wold-type decomposition of the form unitary-unitary, unitary-shift, shift-unitary and shift-shift. In particular, his result applies for double commuting isometries, that is, isometric pairs (V_1, V_2) such that V_1 commutes not only with V_2 but also with the adjoint of V_2 ($V_1 \ker V_2^* \subset \ker V_2^*$ or, equivalently, $V_2 \ker V_1^* \subset \ker V_1^*$).

A natural extension for the unilateral shift is the notion of *bi-shift*, namely a pair $S = (S_1, S_2)$ on a Hilbert space $\mathcal{H}$ unitarily equivalent to the pair of multiplications by coordinate functions z_1 and z_2 on a certain Hardy space on the bitorus. More precisely, there exist a Hilbert space $\mathcal{W}$ and a unitary operator $U : \mathcal{H} \to H^2(\mathbb{T}^2)\otimes\mathcal{W}$ such that $S_i = U^*(T_{z_i}\otimes I)U,\ i = 1, 2$. The following characterization illustrates the geometrical structure of a bi-shift [Sł]: a commuting isometric pair $S = (S_1, S_2)$ on $\mathcal{H}$ is a bi-shift if and only if S is a doubly commuting shift pair if and only if there exists a wandering subspace $\mathcal{W}$ (i.e., $S^p\mathcal{W} \perp S^q\mathcal{W},\ p, q \in \mathbb{Z}_+^2,\ p \neq q$) that generates $\mathcal{H}$, that is, $\mathcal{H} = \bigoplus_{p\in\mathbb{Z}_+^2} S^p\mathcal{W}$ (if $p = (p_1, p_2) \in \mathbb{Z}^2$ and $T = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^2$ is a commuting pair we use the notation T^p for: $T_1^{p_1}T_2^{p_2}$ if $p_1, p_2 \geq 0$; $T_1^{*|p_1|}T_2^{p_2}$ if $p_1 < 0, p_2 \geq 0$; $T_2^{*|p_2|}T_1^{p_1}$ if $p_1 \geq 0, p_2 < 0$; or $T_1^{*|p_1|}T_2^{*|p_2|}$ if $p_1, p_2 < 0$). $\mathcal{W}$ is unique ($\mathcal{W} = \ker S_1^* \cap \ker S_2^*$) and is said to be the *defect* of S.

We are now able to formulate

Słociński's Theorem ([Sł])**.** *To any double commuting isometric pair $V = (V_1, V_2)$ there corresponds a unique orthogonal decomposition of the form*

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{ss}$$

such that $\mathcal{H}_{\alpha_1\alpha_2}$ reduces V_i to a unitary operator if $\alpha_i = u$ and to a unilateral shift if $\alpha_i = s$, $i = 1, 2$. More exactly, $\mathcal{H}_{uu} = \bigcap_{m\geq 0} V_1^m\mathcal{H} \cap \bigcap_{n\geq 0} V_2^n\mathcal{H}$, $\mathcal{H}_{us} = \bigoplus_{n\geq 0} V_2^n(\bigcap_{m\geq 0} V_1^m\mathcal{W}_2)$, $\mathcal{H}_{su} = \bigoplus_{m\geq 0} V_1^m(\bigcap_{n\geq 0} V_2^n\mathcal{W}_1)$ and $\mathcal{H}_{ss} = \bigoplus_{p\in\mathbb{Z}_+^2} V^p\mathcal{W}$, with $\mathcal{W}_1 = \ker V_1^*$, $\mathcal{W}_2 = \ker V_2^*$ and $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$.

It is our aim in this paper to obtain such a Wold-type decomposition for an arbitrary commuting isometric pair, the bi-shift part being replaced by what we will call a weak bi-shift. This weak bi-shift contains, in particular, bi-shift and modified bi-shift maximal parts.

2. A GENERALIZED WOLD-SŁOCIŃSKI DECOMPOSITION

The first step in our construction is an important observation made by I. Suciu [Su], namely the existence of a maximal bi-unitary part for every commuting pair of isometries (in short, a bi-isometry). Some years later, Berger, Coburn and Lebow [BCL] stated that this decomposition between bi-unitary and completely non-unitary parts is exactly the Wold decomposition for the product isometry, a result that clarifies the structure of the spaces:

Proposition 2.1 ([Su], [BCL])**.** *Let $V = (V_1, V_2)$ be a bi-isometry on $\mathcal{H}$. There exists a maximal subspace $\mathcal{H}_{uu}$ of $\mathcal{H}$ reducing V to a bi-unitary operator. Moreover, the orthogonal decomposition $\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{uu}^{\perp}$ coincides with the classical Wold decomposition attached to the isometry V_1V_2; that is,*

$$\mathcal{H}_{uu} = \bigcap_{n\geq 0} (V_1V_2)^n\mathcal{H}. \tag{1}$$

V is said to be *completely non-unitary* (*cnu*) if $\mathcal{H}_{uu} = \{0\}$.

In such a generality we identify a unitary-shift maximal subspace as in the Słociński decomposition theorem:

Proposition 2.2. *There is a maximal subspace $\mathcal{H}_{us}$ of $\mathcal{H}$ reducing V_1 to a unitary operator and V_2 to a shift. More precisely,*

$$\mathcal{H}_{us} = \bigoplus_{n\geq 0} V_2^n\Big(\bigcap_{m\geq 0} V_1^m\big(\bigcap_{i\geq 0} \ker V_2^*V_1^i\big)\Big). \tag{2}$$

Proof. Since $\mathcal{W}_2 := \bigcap_{m\geq 0} V_1^m\big(\bigcap_{i\geq 0}\ker V_2^*V_1^i\big) \subset \ker V_2^*$ is wandering for V_2, we can define $\mathcal{H}_{us}$ as the orthogonal sum in (2). Observe that, since the sequence $\{V_1^m\big(\bigcap_{i\geq 0}\ker V_2^*V_1^i\big)\}_{m\geq 0}$ is decreasing, its intersection remains unchanged if indexed by $m \geq 1$. Hence $V_1\mathcal{H}_{us} = \mathcal{H}_{us}$; that is, $\mathcal{H}_{us}$ reduces V_1 to a unitary operator. Moreover, $V_2\mathcal{H}_{us} = \mathcal{H}_{us} \ominus \mathcal{W}_2 \subset \mathcal{H}_{us}$ and $V_2^*\mathcal{H}_{us} = \mathcal{H}_{us}$. Consequently, by the structure of $\mathcal{H}_{us}$, $V_2|_{\mathcal{H}_{us}}$ is a shift having $\mathcal{W}_2$ as the defect subspace.

Suppose that $\mathcal{H}_0$ is a subspace of $\mathcal{H}$ reducing V_1 to a unitary operator and V_2 to a shift. We shall prove that $\mathcal{H}_0 \subset \mathcal{H}_{us}$. Since $V|_{\mathcal{H}_0}$ is double commuting, we can use the geometrical structure in the Słociński decomposition:

$$\mathcal{H}_0 = \bigoplus_{n\geq 0} V_2^n\Big(\bigcap_{m\geq 0} V_1^m\big(\ker V_2^* \cap \mathcal{H}_0\big)\Big).$$

Observe that $V_2^*V_1^ix = V_1^iV_2^*x = 0$ $(i \geq 0)$, for any $x \in \ker V_2^* \cap \mathcal{H}_0$. Then $\mathcal{H}_0 \subset \mathcal{H}_{us}$, and the maximality is proved. □

By symmetry we can also formulate

Proposition 2.3. *There is a maximal subspace $\mathcal{H}_{su}$ of $\mathcal{H}$ reducing V_1 to a shift and V_2 to a unitary operator. More precisely,*

$$\mathcal{H}_{su} = \bigoplus_{m\geq 0} V_1^m\Big(\bigcap_{n\geq 0} V_1^n\big(\bigcap_{j\geq 0} \ker V_1^*V_2^j\big)\Big). \tag{3}$$

Let $S = (S_1, S_2)$ be a bi-isometry on $\mathcal{H}$.

Remark 2.4. S is bi-shift if and only if S is double commuting and

$$S_1|_{\ker S_2^*},\ S_2|_{\ker S_1^*} \text{ and } S_1S_2 \text{ are shifts.}$$

The direct inclusion follows by the observation that the restriction of a shift to any of its reducing subspaces remains a shift. For the converse, observe that, under the given hypothesis $\ker S_2^* = \bigoplus_{m\geq 0} S_1^m(\ker S_1^* \cap \ker S_2^*)$ ($S_1|_{\ker S_2^*}$ is a shift) and $\ker S_1^* = \bigoplus_{n\geq 0} S_2^n(\ker S_1^* \cap \ker S_2^*)$ ($S_2|_{\ker S_1^*}$ is a shift), we have

$$\bigoplus_{p\in\mathbb{Z}_+^2} S^p(\ker S_1^* \cap \ker S_2^*) = \bigoplus_{m\geq 0} S_1^m \ker S_1^* = \bigoplus_{n\geq 0} S_2^n \ker S_2^*.$$

Also, by double commutativity, $\bigcap_{m\geq 0} S_1^m\mathcal{H} \cap \bigcap_{n\geq 0} S_2^n\mathcal{H} = \bigcap_{i\geq 0}(S_1S_2)^i\mathcal{H}$. Hence $\bigcap_{m\geq 0} S_1^m\mathcal{H} = \bigcap_{n\geq 0} S_2^n\mathcal{H} = \bigcap_{i\geq 0}(S_1S_2)^i\mathcal{H} = \{0\}$. Consequently, $\mathcal{H}_{ss} = \mathcal{H}$ (by Słociński's theorem) and S is a bi-shift. □

More generally, in the non-double commuting case we need to find a good replacement for the bi-shift operator as part of such a generalized Wold-Słociński-type decomposition.

Observe that $\bigcap_{i\geq 0} \ker S_2^* S_1^i$ ($= \ker S_2^*$ if S is double commuting) is invariant under S_1, as well as $\bigcap_{j\geq 0} \ker S_1^* S_2^j$ ($= \ker S_1^*$ if S is double commuting) under S_2. By the previous remark it is natural to introduce our bi-shift replacement by

Definition 2.5. S is said to be a *weak bi-shift* if

$$S_1|_{\bigcap_{i\geq 0}\ker S_2^* S_1^i},\ S_2|_{\bigcap_{j\geq 0}\ker S_1^* S_2^j} \text{ and } S_1S_2 \text{ are shifts.}$$

Recall that, for any contraction T acting on a Hilbert space $\mathcal{H}$, there is a maximal subspace $\mathcal{H}_u$ of $\mathcal{H}$ reducing T to a unitary operator ([La], [NF2]). More exactly,

$$\mathcal{H}_u = \{x \in \mathcal{H} \mid \|T^n x\| = \|x\| = \|T^{*n}x\|,\ n \geq 0\}. \tag{4}$$

T is said to be *completely non-unitary* (*cnu*) if its corresponding space $\mathcal{H}_u = \{0\}$.

The definition above can be reformulated in order to better illustrate the weak bi-shift structure. We need the following:

Lemma 2.6. *Let $S = (S_1, S_2)$ be a bi-isometry on $\mathcal{H}$. The following assertions are equivalent:*

(*i*) $S_1|_{\bigcap_{i\geq 0}\ker S_2^* S_1^i}$ *is a shift;*
(*ii*) $S_1(I - S_2S_2^*)$ *is cnu;*
(*iii*) $\bigcap_{m\geq 0} S_1^m\mathcal{H} \cap \bigcap_{i\geq 0}\ker S_2^* S_1^i = \{0\}$.

By symmetry, the same kind of results are also true for $S_2|_{\bigcap_{j\geq 0}\ker S_1^* S_2^j}$.

Proof. We prove first that $S_1 \bigcap_{i\geq 0}\ker S_2^* S_1^i = S_1\mathcal{H} \cap \bigcap_{i\geq 0}\ker S_2^* S_1^i$. The direct inclusion is obvious. For the converse, consider $x \in S_1\mathcal{H} \cap \bigcap_{i\geq 0}\ker S_2^* S_1^i$. Then $x = S_1S_1^*x$ and $S_2^*S_1^i x = 0$, $i \geq 0$. Consequently, $S_2^*S_1^i(S_1^*x)$ equals $S_2^*S_1^*x = S_1^*S_2^*x = 0$ for $i = 0$, and $S_2^*S_1^{i-1}x = 0$ for $i \geq 1$. Obtain that $x \in S_1 \bigcap_{i\geq 0}\ker S_2^* S_1^i$. Then the maximal subspace of $\bigcap_{i\geq 0}\ker S_2^* S_1^i$ on which $S_1|_{\bigcap_{i\geq 0}\ker S_2^* S_1^i}$ is unitary, namely $\bigcap_{m\geq 0} S_1^m \bigcap_{i\geq 0}\ker S_2^* S_1^i$ (by the theorem of Wold), coincides with $\bigcap_{m\geq 0} S_1^m\mathcal{H} \cap \bigcap_{i\geq 0}\ker S_2^* S_1^i$. The equivalence (*i*) ⇔ (*iii*) is proved.

The next step is to compute the maximal subspace reducing $S_1(I - S_2S_2^*)$ to a unitary operator (according to (4)). Proceed inductively by proving that $x \in \mathcal{H}$ satisfies

$$\|[S_1(I - S_2S_2^*)]^k x\| = \|x\| = \|[S_1(I - S_2S_2^*)]^{*k} x\|, \quad 0 \leq k \leq n, \tag{5}$$

if and only if $x \in \bigcap_{k=0}^{n-1} \ker S_2^* S_1^k \cap \bigcap_{k=0}^{n} S_1^k \mathcal{H}$. It is enough to apply successively the following classical result: if P is an orthogonal projection on $\mathcal{H}$, then $x \in P\mathcal{H}$ if and only if $\|Px\| = \|x\|$. For $n = 1$, $\|S_1(I - S_2S_2^*)x\| = \|x\|$ if and only if $x \in \ker S_2^*$ (take $P = I - S_2S_2^*$), while $\|(I - S_2S_2^*)S_1^* x\| = \|x\|$ if and only if $x \in \ker(S_1S_2)^* \cap S_1\mathcal{H}$ (take $P = S_1(I - S_2S_2^*)S_1^*$ and observe that $P\mathcal{H} = S_1 \ker S_2^* = \ker(S_1S_2)^* \cap S_1\mathcal{H}$). Suppose now that the property is true for a given $n \geq 1$. Then (5) holds for $0 \leq k \leq n+1$ if and only if

$$x \in \bigcap_{k=0}^{n-1} \ker S_2^* S_1^k \cap \bigcap_{k=0}^{n} S_1^k \mathcal{H},$$

$$\|S_1(I - S_2S_2^*)[S_1(I - S_2S_2^*)]^n x\| = \|[S_1(I - S_2S_2^*)]^n x\|$$

(i.e., $[S_1(I - S_2S_2^*)]^n x \in \ker S_2^*$) and

$$\|(I - S_2S_2^*)S_1^*[(I - S_2S_2^*)S_1^*]^n x\| = \|[(I - S_2S_2^*)S_1^*]^n x\|$$

(i.e., $[(I - S_2S_2^*)S_1^*]^n x \in \ker(S_1S_2)^* \cap S_1\mathcal{H}$). The induction is completed by the remark that, for $x \in \bigcap_{k=0}^{n-1} \ker S_2^* S_1^k \cap \bigcap_{k=0}^{n} S_1^k \mathcal{H}$ we have $[S_1(I - S_2S_2^*)]^n x = S_1^n x$ (so $S_1^n x \in \ker S_2^*$, that is, $x \in \ker S_2^* S_1^n$) and $[(I - S_2S_2^*)S_1^*]^n x = S_1^{*n} x$ (so $S_1^{*n} x \in \ker(S_1S_2)^* \cap S_1\mathcal{H}$, that is, $x = S_1^n S_1^{*n} x = S_1^n(S_1S_1^*S_1^{*n}x) = S_1^{n+1}S_1^{*n+1}x \in S_1^{n+1}\mathcal{H}$). Deduce that, by (4), the maximal subspace reducing $S_1(I - S_2S_2^*)$ to a unitary operator is $\bigcap_{m\geq 0} S_1^m \mathcal{H} \cap \bigcap_{i\geq 0} \ker S_2^* S_1^i$, the same as the maximal subspace reducing $S_1|_{\bigcap_{i\geq 0} \ker S_2^* S_1^i}$ to a unitary operator. The equivalence $(i) \Leftrightarrow (ii)$ is proved. □

Consequently:

Proposition 2.7. *The following assertions are equivalent:*

- (i) *S is a weak bi-shift;*
- (ii) *$S_1(I - S_2S_2^*)$, $S_2(I - S_1S_1^*)$ and S_1S_2 are cnu;*
- (iii)

$$\bigcap_{m\geq 0} S_1^m \mathcal{H} \cap \bigcap_{i\geq 0} \ker S_2^* S_1^i = \bigcap_{n\geq 0} S_2^n \mathcal{H} \cap \bigcap_{j\geq 0} \ker S_1^* S_2^j = \bigcap_{n\geq 0} (S_1S_2)^n \mathcal{H} = \{0\}.$$

The main result of this section is

Theorem 2.8. *Let $V = (V_1, V_2)$ be a bi-isometry on $\mathcal{H}$. There is a unique orthogonal decomposition of the form*

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{ws} \tag{6}$$

into reducing subspaces for V such that $V|_{\mathcal{H}_{uu}}$ is bi-unitary, $V|_{\mathcal{H}_{us}}$ is unitary-shift, $V|_{\mathcal{H}_{su}}$ is shift-unitary and $V|_{\mathcal{H}_{ws}}$ is a weak bi-shift.

In addition, $\mathcal{H}_{uu}$, $\mathcal{H}_{us}$ and $\mathcal{H}_{su}$ have the structure given by (1), (2) *and respectively* (3).

Proof. Define $\mathcal{H}_{uu}$ by (1), $\mathcal{H}_{us}$ by (2) and $\mathcal{H}_{su}$ by (3). If $\mathcal{H} = \mathcal{H}_u^k \oplus \mathcal{H}_s^k$ is the Wold decomposition associated to the isometry V_k, $k = 1, 2$, then $\mathcal{H}_{uu} \subset \mathcal{H}_u^1 \cap \mathcal{H}_u^2$, $\mathcal{H}_{us} \subset \mathcal{H}_u^1 \cap \mathcal{H}_s^2$ and $\mathcal{H}_{su} \subset \mathcal{H}_s^1 \cap \mathcal{H}_u^2$. The orthogonal sum $\mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su}$ is then well-defined, and its orthogonal complement $\mathcal{H}_{ws}$ is also reducing for V. Since $V|_{\mathcal{H}_{ws}}$ does not have reducing bi-unitary, unitary-shift or shift-unitary parts, we obtain

$$\Big(\bigcap_{n\geq 0}(V_1V_2)^n\mathcal{H}\Big)\cap\mathcal{H}_{ws}=\{0\},$$

$$\Big(\bigcap_{m\geq 0}V_1^m\big(\bigcap_{i\geq 0}\ker V_2^*V_1^i\big)\Big)\cap\mathcal{H}_{ws}=\{0\}$$

and

$$\Big(\bigcap_{n\geq 0}V_2^n\big(\bigcap_{j\geq 0}\ker V_1^*V_2^j\big)\Big)\cap\mathcal{H}_{ws}=\{0\}.$$

According to Definition 2.5, $V|_{\mathcal{H}_{ws}}$ is a weak bi-shift.

If $\mathcal{H}=\mathcal{H}'_{uu}\oplus\mathcal{H}'_{us}\oplus\mathcal{H}'_{su}\oplus\mathcal{H}'_{ws}$ is another decomposition into reducing summands such that $V|_{\mathcal{H}'_{uu}}$ is bi-unitary, $V|_{\mathcal{H}'_{us}}$ is unitary-shift, $V|_{\mathcal{H}'_{su}}$ is shift-unitary and $V|_{\mathcal{H}'_{ws}}$ is a weak bi-shift, then $\mathcal{H}'_{uu}\subset\mathcal{H}_{uu}$, $\mathcal{H}'_{us}\subset\mathcal{H}_{us}$ and $\mathcal{H}'_{su}\subset\mathcal{H}_{su}$ by maximality (Propositions 2.1–2.3). Then $\mathcal{H}'_{ws}\supset(\mathcal{H}_{uu}\ominus\mathcal{H}'_{uu})\oplus(\mathcal{H}_{us}\ominus\mathcal{H}'_{us})\oplus(\mathcal{H}_{su}\ominus\mathcal{H}'_{su})$. Since a weak bi-shift has null bi-unitary, unitary-shift and respectively shift-unitary maximal parts, we obtain $\mathcal{H}_{uu}\ominus\mathcal{H}'_{uu}=\{0\}$, $\mathcal{H}_{us}\ominus\mathcal{H}'_{us}=\{0\}$ and $\mathcal{H}_{su}\ominus\mathcal{H}'_{su}=\{0\}$. The decomposition is unique. □

There is a maximal bi-shift part in any bi-isometry (see [Su], [Sł], [GS1]). We give here a more precise structure:

Proposition 2.9. *There exists a maximal subspace $\mathcal{H}_s\subset\mathcal{H}_{ws}$ of $\mathcal{H}$ reducing V to a bi-shift. More precisely,*

$$\mathcal{H}_s=\bigoplus_{p\in\mathbb{Z}_+^2}V^p\Big(\bigcap_{i\geq 0}\ker V_2^*V_1^i\cap\bigcap_{j\geq 0}\ker V_1^*V_2^j\Big). \tag{7}$$

Proof. An easy computation shows that $\mathcal{W}=\bigcap_{i\geq 0}\ker V_2^*V_1^i\cap\bigcap_{j\geq 0}\ker V_1^*V_2^j$ is wandering for V. If we denote by $\mathcal{H}_s$ the orthogonal sum in (7), then $V_k\mathcal{H}_s=\mathcal{H}_s\ominus(\bigoplus_{n\geq 0}V_{3-k}^n\mathcal{W})$ and $V_k^*\mathcal{H}_s=\mathcal{H}_s$, $k=1,2$. Therefore, $\mathcal{H}_s$ reduces V to a bi-shift having $\mathcal{W}$ as the defect subspace.

Suppose that $\mathcal{H}_0$ is a subspace of $\mathcal{H}$ reducing V to a bi-shift. Since $V|_{\mathcal{H}_0}$ is double commuting, we can use, as before, the geometrical structure in the Słociński decomposition:

$$\mathcal{H}_0=\bigoplus_{p\in\mathbb{Z}_+^2}V^p(\ker V_1^*\cap\ker V_2^*\cap\mathcal{H}_0).$$

Observe that $V_2^*V_1^ix=V_1^iV_2^*x=0$ $(i\geq 0)$ and $V_1^*V_2^jx=V_2^jV_1^*x=0$ $(j\geq 0)$, for any $x\in\ker V_1^*\cap\ker V_2^*\cap\mathcal{H}_0$. Then $\mathcal{H}_0\subset\mathcal{H}_s$, and the maximality is proved.

With the notation in the proof of Theorem 2.8 it is easy to observe that $\mathcal{H}_s\subset\mathcal{H}_s^1\cap\mathcal{H}_s^2$. Consequently, $\mathcal{H}_s\perp\mathcal{H}_{uu}\oplus\mathcal{H}_{us}\oplus\mathcal{H}_{su}$ and, by Theorem 2.8, $\mathcal{H}_s\subset\mathcal{H}_{ws}$. □

In any bi-isometry we can identify a double commuting part. More precisely:

Proposition 2.10. *Let $V=(V_1,V_2)$ be a bi-isometry on $\mathcal{H}$. Then there exists a maximal subspace $\mathcal{H}_{dc}$ of $\mathcal{H}$ reducing V to a double commuting bi-isometry. More precisely,*

$$\mathcal{H}_{dc}=\{x\in\mathcal{H}\mid V_1^mV_1^{*i}V_2^nV_2^{*j}x=V_2^nV_2^{*j}V_1^mV_1^{*i}x,\ m,n,i,j\geq 0\}. \tag{8}$$

Proof. Define $\mathcal{H}_{dc}$ by (8). $\mathcal{H}_{dc}$ is a closed subspace, being an intersection of bounded operator kernels. For any $x \in \mathcal{H}_{dc}$ and $m, n, i, j \geq 0$, observe that

$$V_1^m V_1^{*i} V_2^n V_2^{*j} V_1 x = V_1^m V_1^{*i} V_1 V_2^n V_2^{*j} x = V_2^n V_2^{*j} V_1^m V_1^{*i} V_1 x$$

and

$$V_1^m V_1^{*i} V_2^n V_2^{*j} V_1^* x = V_1^m V_1^{*i} V_1^* V_2^n V_2^{*j} x = V_2^n V_2^{*j} V_1^m V_1^{*i} V_1^* x.$$

Hence $\mathcal{H}_{dc}$ reduces V_1 and, similarly, also V_2. It is obvious that $V|_{\mathcal{H}_{dc}}$ is double commuting and $\mathcal{H}_{dc}$ is maximal. ☐

Corollary 2.11. *The Słociński decomposition attached to* $V|_{\mathcal{H}_{dc}}$ *is*

$$\mathcal{H}_{dc} = \mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_s,$$

the subspaces $\mathcal{H}_{uu}$, $\mathcal{H}_{us}$, $\mathcal{H}_{su}$ *and* $\mathcal{H}_s$ *being given by* (1)–(3) *and* (7).

We can now give another definition for the weak bi-shift:

Corollary 2.12. *A bi-isometry* V *on* $\mathcal{H}$ *is a weak bi-shift if and only if* $V|_{\mathcal{H}_{dc}}$ *is a bi-shift.*

3. The associated dual bi-isometry

Let $V = (V_1, V_2)$ be a bi-isometry on a Hilbert space $\mathcal{H}$. A pair $U = (U_1, U_2)$ of commuting unitary operators on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ with $U_k|_{\mathcal{H}} = V_k,\ k = 1, 2$ (extension), and $\mathcal{K} = \bigvee_{p \in \mathbb{Z}^2} U^p \mathcal{H}$ (minimality) is said to be the *minimal unitary extension* of V. Its existence is given by a result of Ito ([It]), and the minimality condition assures its uniqueness up to an isomorphism.

Observe that $\tilde{\mathcal{H}} = \mathcal{K} \ominus \mathcal{H}$ is invariant under U^*.

Definition 3.1. The bi-isometry $\tilde{V} := U^*|_{\tilde{\mathcal{H}}}$ is said to be the *dual bi-isometry* associated to V (see [GS2]).

We can introduce some particular classes of bi-isometries by certain properties of their associated dual bi-isometries:

Definition 3.2. V is said to be:

(1) *dual double commuting* if $\tilde{V}$ is double commuting (see [GG]);

(2) *modified bi-shift* if V is completely non-unitary and $\tilde{V}$ is bi-shift (see [GS2]);

(3) *modified weak bi-shift* if V is completely non-unitary and $\tilde{V}$ is weak bi-shift.

To prove some properties of the dual bi-isometry we need the following:

Lemma 3.3. (*a*) $V_2^n \ker V_1^* V_2^n = U_1 \ker \tilde{V}_1^* \tilde{V}_2^n,\ n \geq 0$*;*

(*b*) $V_2^{n+1} \ker V_1^* V_2^n = V_2^{n+1} \mathcal{H} \cap \ker(V_1 V_2)^*,\ n \geq 0.$

By symmetry, the same kind of results are also true if we switch the roles of V_1 *and* V_2*.*

Proof. (*a*) If $y \in \ker \tilde{V}_1^* \tilde{V}_2^n$, then $P_{\tilde{\mathcal{H}}} U_1 U_2^{*n} y = \tilde{V}_1^* \tilde{V}_2^n y = 0$; that is, $U_1 U_2^{*n} y \in \mathcal{H}$. Moreover, $V_1^* V_2^n U_1 U_2^{*n} y = V_1^* U_1 y = P_{\mathcal{H}} y = 0$.

Conversely, for $x \in \ker V_1^* V_2^n$, we obtain that $P_{\mathcal{H}} U_1^* U_2^n x = V_1^* V_2^n x = 0$; that is, $U_1^* U_2^n x \in \tilde{\mathcal{H}}$. Moreover, $\tilde{V}_1^* \tilde{V}_2^n U_1^* U_2^n x = P_{\tilde{\mathcal{H}}} x = 0$.

(*b*) Observe that, for $x \in V_2^{n+1} \mathcal{H}$ (or equivalently $x = V_2^{n+1} V_2^{*n+1} x$), $x \in \ker(V_1 V_2)^*$ if and only if $(V_1 V_2)^* V_2^{n+1} V_2^{*n+1} x = 0$, that is, $V_2^{*n+1} x \in \ker V_1^* V_2^n$. ☐

Since $\ker V_2^*$ is invariant under V_1^*, we can consider $V_1^*|_{\ker V_2^*}$ as a contraction on $\ker V_2^*$ and, similarly, $V_2^*|_{\ker V_1^*}$ on $\ker V_1^*$.

Proposition 3.4. (i) *$\tilde{V}$ is always completely non-unitary (that is, $\tilde{V}_1\tilde{V}_2$ is a shift).*

- (ii) *V is completely non-unitary if and only if $\tilde{\tilde{V}} = V$.*
- (iii) *The following assertions are equivalent:*
  - (a) *V is dual double commuting;*
  - (b) $\ker V_1^* \perp \ker V_2^*$*;*
  - (c) $V_1 \ker V_2^* \supset \ker V_2^*$ *(or $V_2 \ker V_1^* \supset \ker V_1^*$);*
  - (d) $V_1^*|_{\ker V_2^*}$ *(or $V_2^*|_{\ker V_1^*}$) is isometric.*

Proof. (i) and (ii) have been proved in [Po, Remark 3.5].

(iii) By Lemma 3.3(a) for $n = 0$, $\ker V_1^* \perp \ker V_2^*$ if and only if $\tilde{V}_2 \ker \tilde{V}_1{}^* \perp \tilde{V}_1 \ker \tilde{V}_2{}^*$ or, equivalently, $[\tilde{V}_1(I - \tilde{V}_2\tilde{V}_2{}^*)]^*[\tilde{V}_2(I - \tilde{V}_1\tilde{V}_1{}^*)] = 0$ (in operator form). By an easy computation we obtain $\tilde{V}_1^*\tilde{V}_2 = \tilde{V}_2\tilde{V}_1^*$. For the second part, by Lemma 3.3(b), $V_1 \ker V_2^* \supset \ker V_2^*$ if and only if $V_1\mathcal{H} \supset \ker V_2^*$ or, equivalently, $\ker V_1^* \perp \ker V_2^*$. Finally, $V_1^*|_{\ker V_2^*}$ is isometric if and only if

$$0 = \|(I - V_2V_2^*)x\|^2 - \|V_1^*(I - V_2V_2^*)x\|^2 = \|(I - V_1V_1^*)(I - V_2V_2^*)x\|^2, \quad x \in \mathcal{H},$$

that is, if and only if $\ker V_1^* \perp \ker V_2^*$. □

Using Proposition 3.4(i), write the Wold-type decomposition in Theorem 2.8 for $\tilde{V}$ instead of V:

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{us} \oplus \tilde{\mathcal{H}}_{su} \oplus \tilde{\mathcal{H}}_{ws}. \tag{9}$$

Then $\tilde{\mathcal{H}}_{us} = \bigoplus_{n\geq 0} \tilde{V}_2{}^n \tilde{\mathcal{W}}_2 = \bigoplus_{n\leq 0} U_2^n \tilde{\mathcal{W}}_2$ with $\tilde{\mathcal{W}}_2 = \bigcap_{m\geq 0} \tilde{V}_1^m \big(\bigcap_{i\geq 0} \ker \tilde{V}_2^* \tilde{V}_1^i\big)$ and, by passing to the dual, $\widetilde{\tilde{\mathcal{H}}_{us}} = \bigoplus_{n\in\mathbb{Z}} U_2^n \tilde{\mathcal{W}}_2 \ominus \tilde{\mathcal{H}}_{us} = \bigoplus_{n\geq 0} U_2^n (U_2\tilde{\mathcal{W}}_2)$. Observe that $U_1^* \bigcap_{i\geq 0} \ker V_2^* V_1^i \supset \bigcap_{i\geq 0} \ker V_2^* V_1^i$ and, by duality, $U_1 \bigcap_{i\geq 0} \ker \tilde{V}_2{}^* \tilde{V}_1{}^i \supset \bigcap_{i\geq 0} \ker \tilde{V}_2{}^* \tilde{V}_1{}^i$. Use this observation and Lemma 3.3(a) (more precisely, $\ker \tilde{V}_2{}^* \tilde{V}_1{}^i = U_2^* U_1^i \ker V_2^* V_1^i$) to prove that

$$\begin{aligned} U_2\tilde{\mathcal{W}}_2 &= U_2 \bigcap_{m\in\mathbb{Z}} U_1^m \big(\bigcap_{i\geq 0} \ker \tilde{V}_2{}^* \tilde{V}_1{}^i\big) \\ &\overset{m+i=p}{=} \bigcap_{p\in\mathbb{Z}, i\geq 0} U_1^p \ker V_2^* V_1^i \\ &= \bigcap_{p\geq 0} V_1^p \bigcap_{i\geq 0} \ker V_2^* V_1^i. \end{aligned}$$

Deduce $\widetilde{\tilde{\mathcal{H}}_{us}} = \mathcal{H}_{us}$ and, by symmetry, $\widetilde{\tilde{\mathcal{H}}_{su}} = \mathcal{H}_{su}$. Therefore, by duality, relation (9) becomes

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \widetilde{\tilde{\mathcal{H}}_{ws}}, \tag{10}$$

that is, $\widetilde{\tilde{\mathcal{H}}_{ws}} = \mathcal{H}_{ws}$. In other words:

Corollary 3.5. *V is a weak bi-shift if and only if V is a modified weak bi-shift.*

As observed earlier (Proposition 2.9; cf. [Su], [Sł], [GS1]), the weak bi-shift part contains a maximal bi-shift. Then $\mathcal{H}_m := \widetilde{\tilde{\mathcal{H}}_s}$ is the maximal modified bi-shift subspace and is contained in $\widetilde{\tilde{\mathcal{H}}_{ws}} = \mathcal{H}_{ws}$. $\mathcal{H}_m$ was already considered in [GS2]. We give here a more precise structure:

Corollary 3.6. *There exists a maximal subspace $\mathcal{H}_m \subset \mathcal{H}_{ws}$ of $\mathcal{H}$ reducing V to a modified bi-shift. More precisely,*

$$\mathcal{H}_m = \bigoplus_{p\in\mathbb{Z}^2\setminus(-\infty,-1]^2} V^p\Big(\bigcap_{m\geq 0} V_1^m\mathcal{H} \cap \bigcap_{n\geq 0} V_2^n\mathcal{H} \cap \ker(V_1V_2)^*\Big) \tag{11}$$

(*according to the notation for* V^p *in Section* 1).

Proof. $\tilde{\mathcal{H}}_s = \bigoplus_{p\in\mathbb{Z}_+^2} \tilde{V}^p\Big(\bigcap_{i\geq 0}\ker\tilde{V}_2^*\tilde{V}_1^i \cap \bigcap_{j\geq 0}\ker\tilde{V}_1^*\tilde{V}_2^j\Big)$ by Proposition 2.9. Then $\mathcal{H}_m := \widetilde{\tilde{\mathcal{H}}_s} = \bigoplus_{p\in\mathbb{Z}^2\setminus\{-\mathbb{Z}_+^2\}} U^p\Big(\bigcap_{i\geq 0}\ker\tilde{V}_2^*\tilde{V}_1^i \cap \bigcap_{j\geq 0}\ker\tilde{V}_1^*\tilde{V}_2^j\Big)$. Use again Lemma 3.3 ($a$) and ($b$) to see that

$$\begin{aligned}\mathcal{H}_m &= \bigoplus_{p\in\mathbb{Z}^2\setminus\{-\mathbb{Z}_+^2\}} U^{p-(1,1)}\Big(\bigcap_{i\geq 0} V_1^{i+1}\ker V_2^*V_1^i \cap \bigcap_{j\geq 0} V_2^{j+1}\ker V_1^*V_2^j\Big)\\ &= \bigoplus_{p\in\mathbb{Z}^2\setminus\{-\mathbb{Z}_+^2\}} U^{p-(1,1)}\Big(\bigcap_{m\geq 0} V_1^m\mathcal{H} \cap \bigcap_{n\geq 0} V_2^n\mathcal{H} \cap \ker(V_1V_2)^*\Big).\end{aligned}$$

Moreover, $\mathcal{H}_m \subset \mathcal{H}$ implies $U^{p-(1,1)}x = P_{\mathcal{H}}U^{p-(1,1)}x = V^{p-(1,1)}x$, for any $x \in \bigcap_{m\geq 0} V_1^m\mathcal{H} \cap \bigcap_{n\geq 0} V_2^n\mathcal{H} \cap \ker(V_1V_2)^*$ and $p \notin (-\infty,0]^2$. (11) is proved. ☐

Remark 3.7. (a) Since $\widetilde{\tilde{\mathcal{H}}_m} = \mathcal{H}_s$, the method above would not provide another reducing subspace of $\mathcal{H}_{ws}$.

(b) If V_1 (resp. V_2) is a shift, then $\mathcal{H}_{uu} = \mathcal{H}_{us}$ (resp. $\mathcal{H}_{su}$) $= \mathcal{H}_m = \{0\}$; if V_1 (resp. V_2) is unitary, then $\mathcal{H}_{su}$ (resp. $\mathcal{H}_{us}$) $= \mathcal{H}_{ws} = \{0\}$.

(c) If V is dual double commuting ($\ker V_1^* \perp \ker V_2^*$ according to Proposition 3.4), then the decomposition (10) becomes the one given in [GG]:

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_m.$$

We can give here a more precise structure than in [Po]. Observe that, in this case, by Lemma 3.3(a) applied for $n = 0$, $U_2\bigcap_{m\geq 0}\tilde{V}_1^m\ker\tilde{V}_2^* = \bigcap_{m\leq 0} U_1^m\ker V_2^* = \bigcap_{m\geq 0}\ker V_2^*V_1^m$, and this implies that

$$\mathcal{H}_{us} = \bigoplus_{n\geq 0} V_2^n\Big(\bigcap_{i\geq 0}\ker V_2^*V_1^i\Big).$$

Similarly,

$$\mathcal{H}_{su} = \bigoplus_{m\geq 0} V_1^m\Big(\bigcap_{j\geq 0}\ker V_1^*V_2^j\Big).$$

Moreover, $\ker V_2^*V_1^i = V_1\ker V_2^*V_1^{i+1}$ $(i \geq 0)$ since, on the one side, $x \in V_1\ker V_2^*V_1^{i+1}$ $(x = V_1V_1^*x$ and $V_2^*V_1^{i+1}V_1^*x = 0)$ implies $x \in \ker V_2^*V_1^i$, and, on the other side, $x \in \ker V_2^*V_1^i$ implies $x \in V_1^{*i}\ker V_2^* \subset V_1^{*i}V_1^{i+1}\ker V_2^* = V_1\ker V_2^*$ (since by Proposition 3.4, $\ker V_2^* \subset V_1\ker V_2^*$), that is, $x = V_1V_1^*x \in V_1\ker V_2^*V_1^{i+1}$. Then $V_1^{i+1}\ker V_2^*V_1^i = V_1\ker V_2^*$ $(i \geq 0)$, $V_2^{j+1}\ker V_1^*V_2^j = V_2\ker V_1^*$ $(j \geq 0)$, and so, by Corollary 3.6,

$$\mathcal{H}_m = \bigoplus_{p\in\mathbb{Z}^2\setminus(-\infty,-1]^2} V^p\Big(V_1\ker V_2^* \cap V_2\ker V_1^*\Big).$$

(d) If V is simultaneously double and dual double commuting ($V_1\ker V_2^* = \ker V_2^*$ or, equivalently, $V_2\ker V_1^* = \ker V_1^*$), then, using again the notation in the proof of

Theorem 2.8, $\mathcal{H}_{us} = \mathcal{H}_s^2$, $\mathcal{H}_{su} = \mathcal{H}_s^1$ and $\mathcal{H}_{ws} = \{0\}$. More precisely, the decomposition (6) becomes

$$\mathcal{H} = (\mathcal{H}_u^1 \cap \mathcal{H}_u^2) \oplus \mathcal{H}_s^2 \oplus \mathcal{H}_s^1 \oplus \{0\}.$$

Modified bi-shifts can be characterized as follows:

Proposition 3.8. *The following assertions are equivalent:*

(i) *V is a modified bi-shift.*

(ii) *V is dual double commuting and*

$$\mathcal{H} = \bigoplus_{p\in\mathbb{Z}^2\setminus(-\infty,-1]^2} V^p \Big(V_1 \ker V_2^* \cap V_2 \ker V_1^* \Big).$$

(iii) *$V_1^*|_{\ker V_2^*}$, $V_2^*|_{\ker V_1^*}$ and V_1V_2 are shifts.*

Proof. (i) and (ii) are equivalent by Remark 3.7(c). In addition, by Remark 2.4, if V is completely non-unitary, then V is a modified bi-shift if and only if $\tilde{V}_1|_{\ker \tilde{V}_2^*}$, $\tilde{V}_2|_{\ker \tilde{V}_1^*}$ and $\tilde{V}_1\tilde{V}_2$ are shifts. The proof is completed by the observation that $\tilde{V}_1|_{\ker \tilde{V}_2^*}$ (respectively $\tilde{V}_2|_{\ker \tilde{V}_1^*}$) is a shift if and only if $V_1^*|_{\ker V_2^*}$ (respectively $V_2^*|_{\ker V_1^*}$) is a shift. We just have to see that, by Lemma 3.3(a), the relation

$$\ker \tilde{V}_2^* = \bigoplus_{m\geq 0} \tilde{V}_1^m (\ker \tilde{V}_2^* \ominus \tilde{V}_1^* \ker \tilde{V}_2^*)$$

can be rewritten as

$$\ker V_2^* = \bigoplus_{m\geq 0} U_1^{*m} (\ker V_2^* \ominus U_1^* \ker V_2^*).$$

Moreover, $U_1^* \ker V_2^* \subset \mathcal{H}$ (and consequently $U_1^* \ker V_2^* = V_1^* \ker V_2^*$) if V is dual double commuting. □

Example 3.9. We shall consider the example given in [Sł]. Let $\mathcal{H}$ be a Hilbert space having an orthonormal basis of the form $\{e_{i,j} \mid i \in \mathbb{Z}_+ \text{ or } j \in \mathbb{Z}_+\}$. The pair $V = (V_1, V_2)$ defined on $\mathcal{H}$ by

$$V_1 e_{i,j} = e_{i+1,j} \text{ and } V_2 e_{i,j} = e_{i,j+1} \ (i \geq 0 \text{ or } j \geq 0)$$

is clearly a cnu bi-isometry. It is easy to see that

$$\ker V_1^* = \bigvee \{e_{0,j} \mid j \leq -1\} \perp \bigvee \{e_{i,0} \mid i \leq -1\} = \ker V_2^*,$$

that is, $V_1^*|_{\ker V_2^*}$ and $V_2^*|_{\ker V_1^*}$ are isometries. They are shifts, since $V_1^{*m} \ker V_2^* = \bigvee \{e_{i,0} \mid i \leq -m-1\}$ implies

$$\bigcap_{m\geq 0} V_1^{*m} \ker V_2^* = \{0\}$$

and, similarly,

$$\bigcap_{n\geq 0} V_2^{*n} \ker V_1^* = \{0\}.$$

It follows by Proposition 3.8 that V is a modified bi-shift.

As proved in [GS1] (or deduced directly following (7) and (11)), $\mathcal{H}_s \perp \mathcal{H}_m$. Then, by Propositions 2.1–2.3, 2.9, Theorem 2.8 and Corollary 3.6, we can conclude the following theorem.

Theorem 3.10. *Let V be a bi-isometry on $\mathcal{H}$. There is a unique orthogonal decomposition of the form*

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{s} \oplus \mathcal{H}_{m} \oplus \mathcal{H}_{e}$$

into subspaces of $\mathcal{H}$ reducing V and maximal on which V is, respectively, bi-unitary, unitary-shift, shift-unitary, bi-shift or modified bi-shift.

Since $\mathcal{H}_e$ vanishes in some important cases (for example, if V is double or dual double commuting), we call $V|_{\mathcal{H}_e}$ the *evanescent* part of V.

We give some necessary and sufficient conditions on V in order to have no evanescent part:

Proposition 3.11. *Let V be a bi-isometry on $\mathcal{H}$. Then*

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{s} \oplus \mathcal{H}_{m} \tag{12}$$

if and only if

$$(I - V_1V_1^*)(I - V_2V_2^*) \bigcap_{m\geq 0} V_1^m\mathcal{H} = \{0\},$$

$$V_2^*V_1^i(I - V_1V_1^*)V_1^{*n}(I - V_2V_2^*)(I - V_1V_1^*) = 0,$$

and

$$V_1^*V_2^j(I - V_1V_1^*)V_1^{*n}(I - V_2V_2^*)(I - V_1V_1^*) = 0,$$

for all $n, i, j \geq 0$.

Proof. Just observe that there exists an orthogonal decomposition of the form (12) if and only if $V|_{\mathcal{H}\ominus\mathcal{H}_s}$ is dual double commuting, that is,

$$\ker V_2^* \cap (\mathcal{H} \ominus \mathcal{H}_s) \subset V_1(\mathcal{H} \ominus \mathcal{H}_s)$$

by Proposition 3.4(iii). Equivalently, by passing to orthogonal complements,

$$\ker V_1^* \subset V_2\mathcal{H} \oplus \bigoplus_{m\geq 0} V_1^m\mathcal{W}, \text{ with } \mathcal{W} = \bigcap_{i\geq 0} \ker V_2^*V_1^i \cap \bigcap_{j\geq 0} \ker V_1^*V_2^j. \tag{13}$$

(13) can be rewritten as $(I - V_2V_2^*)(I - V_1V_1^*)\mathcal{H} \subset \bigoplus_{m\geq 0} V_1^m\mathcal{W}$ or, equivalently, $(I-V_1V_1^*)(I-V_2V_2^*)\bigcap_{m\geq 0} V_1^m\mathcal{H} = \{0\}$ and $(I-V_1V_1^*)V_1^{*n}(I-V_2V_2^*)(I-V_1V_1^*)\mathcal{H} \subset \mathcal{W}$ $(n \geq 0)$. The conclusion follows by the definition of $\mathcal{W}$. ☐

In some other cases V can have only evanescent part:

Example 3.12. Let S be a shift on $\mathcal{H}$ and $V = (S, S)$. Then V has no double commuting part. Moreover, by Remark 3.7 (b), the modified bi-shift part is also null. Then $\mathcal{H} = \mathcal{H}_e$, as required.

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Department of Mathematics, University of the West Timişoara, RO-300223 Timişoara, Bd. Vasile Pârvan nr. 4, Romania

E-mail address: `popovici@math.uvt.ro`