

PERIODIC SOLUTIONS OF NONLINEAR IMPULSIVE DIFFERENTIAL INCLUSIONS WITH CONSTRAINTS

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ABSTRACT. In this paper we obtain the existence of periodic solutions for nonlinear “invariance” problems monitored by impulsive differential inclusions subject to impulse effects.

1. INTRODUCTION

The impulsive differential equations appear to represent a natural framework for mathematical modelings of several real world phenomena. For instance, systems with impulse effects have applications in physics, in biotechnology, in industrial robotics, in radiotechnology, in pharmacokinetics, in population dynamics, in ecology, in optimal control, in the study of microorganism reproduction, in economics, in production theory, and so on. The qualitative investigation of impulsive differential equations began in 1960 with the work of Mil’man and Myshkis (see [8]). The possibility of wide practical applications of impulsive differential equations explains the still growing interest of many authors in the investigation of these equations in recent years and the publication of monographs about this subject by Samoilenko and Perestyuk (see [11]), Lakshmikantham, Bainov and Simeonov (see [7]), and Bainov and Simeonov (see [1]). In the recent past, attention has been given to impulsive differential inclusions, and interesting results concerning the existence of periodic solutions for first-order impulsive differential inclusions have appeared: see, for instance, the papers by Watson (see [12]) and by Benchohra, Henderson, and Ntouyas (see [2]). But, unlike Benchohra, Henderson and Ntouyas, in [12] Watson studies an impulsive “invariance” problem. The fundamental tools to prove existence results for noninvariance problems are essentially fixed point arguments, nonlinear alternative, degree theory, or iterative methods combined with upper and lower solutions.

In this note we study the existence of periodic solutions for the following impulsive “invariance” problem:

$$(\mathcal{P}) \quad \begin{cases} x'(t) \in F(t, x(t)) & \text{a.e. } t \in [0, T] \setminus \{t_1, \dots, t_p\}, \\ x(t_k^+) = x(t_k) + I_k(x(t_k)) & \text{for any } k \in \{1, \dots, p\}, \\ x(0) = x(T), \end{cases}$$

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where $\bar{\Omega} \subset \mathbb{R}^N, N \geq 1$, is a canonical domain, $F : [0, T] \times \bar{\Omega} \rightarrow \mathcal{P}(\mathbb{R}^N)$ is a set-valued map, $0 = t_0 < t_1 < \dots < t_{p+1} = T, I_k : \bar{\Omega} \rightarrow \mathbb{R}^N$ is an impulse function for any $k \in \{1, \dots, p\}$, and $x(t^+) = \lim_{s \rightarrow t^+} x(s)$.

This problem is important in the case that one needs to study systems for which the state is required to stay in a set that may be constant. The pioneering work of Nagumo (see [9]) illustrates that the resolution of an “invariance” problem needs a tangential condition involving some kind of tangential approximation to the constraint set. More precisely, Nagumo proved that, for a continuous function $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$, a necessary and sufficient condition for the differential equation $x' = f(x)$ a.e. on \mathbb{R}^N to leave K invariant is that $f(x) \in T_K(x)$ for all $x \in K$, where $T_K(x)$ denotes the Bouligand contingent cone to K at $x \in K$.

In this paper we obtain existence results for the impulsive “invariance” problem (P) (see Theorems 3.1 and 3.2). As in [12], in our proof we use the result due to Hristova and Bainov concerning the existence of a periodic solution for impulsive differential equations (see [3]), together with an approximation argument. Our Theorems 3.1 and 3.2 extend the existence result proved by Watson (see Remarks 3.1 and 3.2) and, moreover, improve the Hristova - Bainov existence theorem in the case of “invariance” problems involving impulsive differential equations.

2. PRELIMINARIES

Let $\Omega \subset \mathbb{R}^N, N \geq 1$, be a domain with boundary $\partial\Omega$, and let us denote by $\bar{\Omega} = \Omega \cup \partial\Omega$ its closure and by $\text{int}\bar{\Omega}$ the interior of $\bar{\Omega}$. We say that $\bar{\Omega}$ is a canonical domain (see [3]) if Ω is bounded, convex and there exists a finite family of real-valued continuously differentiable maps $\{\Phi_i\}_{i \in \{1, \dots, q\}}$ such that the following conditions hold:

$$(\Omega 1) \quad \bar{\Omega} = \bigcap_{i=1}^q \left\{ x \in \mathbb{R}^N : \Phi_i(x) \leq 0 \right\};$$

if there exist $x_0 \in \partial\Omega$ and $i \in \{1, \dots, q\}$ such that $\Phi_i(x_0) = 0$,

$$(\Omega 2) \quad \text{then } \nabla\Phi_i(x_0) \neq 0.$$

Let us note that a compact interval of \mathbb{R} is a canonical domain.

Let $\bar{\Omega}$ be a canonical domain defined by the family $\{\Phi_i\}_{i \in \{1, \dots, q\}}$, and let us define $\alpha(x) = \{ i \in \{1, \dots, q\} : \Phi_i(x) = 0 \}$, for any $x \in \partial\Omega$, and

$$P_{\bar{\Omega}}(x) = \begin{cases} \{ y \in \mathbb{R}^N : \langle \nabla\Phi_i(x), y \rangle \leq 0, \forall i \in \alpha(x) \} & \text{if } x \in \partial\Omega, \\ \mathbb{R}^N & \text{if } x \in \Omega, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^N .

We recall the notion of Bouligand contingent cone to $\bar{\Omega}$ at $x \in \bar{\Omega}$:

$$T_{\bar{\Omega}}(x) = \left\{ y \in \mathbb{R}^N : \liminf_{\lambda \rightarrow 0^+} \frac{\rho(x + \lambda y, \bar{\Omega})}{\lambda} = 0 \right\},$$

where $\rho(z, \bar{\Omega}) = \inf_{v \in \bar{\Omega}} \|z - v\|$. We note that $P_{\bar{\Omega}}(x) = T_{\bar{\Omega}}(x)$, for any $x \in \bar{\Omega}$ (see [12], Remark 1.3).

Moreover, we recall the definition of normal cone to $\bar{\Omega}$ at $x \in \bar{\Omega}$:

$$N_{\bar{\Omega}}(x) = \left\{ x^* \in \mathbb{R}^N : \langle x^*, x \rangle = \sup_{z \in \bar{\Omega}} \langle x^*, z \rangle \right\}.$$

The following property holds:

$$(1) \quad y \in T_{\overline{\Omega}}(x) \iff \langle x^*, y \rangle \leq 0, \forall x^* \in N_{\overline{\Omega}}(x).$$

Let $F : [0, T] \times \overline{\Omega} \rightarrow \mathcal{P}(\mathbb{R}^N)$ be a set-valued map. F is said to be lower semicontinuous (l.s.c.) in $(\bar{t}, \bar{x}) \in [0, T] \times \overline{\Omega}$ if, for any open set $A \subseteq \mathbb{R}^N$ such that $A \cap F(\bar{t}, \bar{x}) \neq \emptyset$, there exists a neighborhood U of (\bar{t}, \bar{x}) such that $A \cap F(t, x) \neq \emptyset$, for any $(t, x) \in U \cap ([0, T] \times \overline{\Omega})$. F is said to be upper semicontinuous (u.s.c.) in $(\bar{t}, \bar{x}) \in [0, T] \times \overline{\Omega}$ if, for any open set $A \subseteq \mathbb{R}^N$ such that $F(\bar{t}, \bar{x}) \subseteq A$, there exists a neighborhood U of (\bar{t}, \bar{x}) such that $F(t, x) \subseteq A$, for any $(t, x) \in U \cap ([0, T] \times \overline{\Omega})$.

In this work we consider the following impulsive periodic “invariance” problem:

$$(P) \quad \begin{cases} x'(t) \in F(t, x(t)) & \text{a.e. } t \in [0, T] \setminus \{t_1, \dots, t_p\}, \\ x(t_k^+) = x(t_k) + I_k(x(t_k)) & \text{for any } k \in \{1, \dots, p\}, \\ x(0) = x(T), \end{cases}$$

where $\overline{\Omega} \subset \mathbb{R}^N$, $N \geq 1$, is a canonical domain, $F : [0, T] \times \overline{\Omega} \rightarrow \mathcal{P}(\mathbb{R}^N)$ is a set-valued map, $0 = t_0 < t_1 < \dots < t_{p+1} = T$, $I_k : \overline{\Omega} \rightarrow \mathbb{R}^N$ is an impulse function for any $k \in \{1, \dots, p\}$, and $x(t^+) = \lim_{s \rightarrow t^+} x(s)$.

Definition 2.1. A solution of problem (P) is a function $x : [0, T] \rightarrow \overline{\Omega}$, absolutely continuous in the closed interval $[0, t_1]$ and in the interval $]t_k, t_{k+1}]$ for any $k \in \{1, \dots, p\}$, such that

$$\begin{aligned} x'(t) &\in F(t, x(t)) && \text{a.e. } t \in [0, T] \setminus \{t_1, \dots, t_p\}; \\ x(t_k^+) &= x(t_k) + I_k(x(t_k)) && \text{for any } k \in \{1, \dots, p\}; \end{aligned}$$

and

$$x(0) = x(T).$$

To prove our existence results for problem (P) we need the following theorem due to Hristova and Bainov ([3], Theorem 2) in the version proposed by Watson ([12], Remarks 1.1 and 1.2, p. 397):

Theorem 2.1. Let $\overline{\Omega} \subset \mathbb{R}^N$, $N \geq 1$, be a canonical domain defined by the family $\{\Phi_i\}_{i \in \{1, \dots, q\}}$, and let $f : \mathbb{R} \times \overline{\Omega} \rightarrow \mathbb{R}^N$ and $I_k : \overline{\Omega} \rightarrow \mathbb{R}^N$, $k \in \mathbb{Z}$, be functions such that

- (B1) f is continuous in $\mathbb{R} \times \overline{\Omega}$;
- (B2) $\exists T > 0$ such that $f(t + T, x) = f(t, x)$, $\forall (t, x) \in \mathbb{R} \times \overline{\Omega}$;
- (B3) $\langle \nabla \Phi_i(x), f(t, x) \rangle \leq 0$, $\forall (t, x) \in \mathbb{R} \times \partial \Omega$, $\forall i \in \alpha(x)$;
- (B4) I_k is continuous in $\overline{\Omega}$;
- (B5) $\exists L > 0 : \|f(t, x) - f(t, y)\| \leq L \|x - y\|$, $\forall t \in \mathbb{R}$, $\forall x, y \in \overline{\Omega}$;
- (B6) $\exists h \in \mathbb{N} : t_{k+h} = t_k + T$, $I_{k+h}(x) = I_k(x)$, $\forall x \in \overline{\Omega}$;
- (B7) $x + I_k(x) \in \overline{\Omega}$, $\forall x \in \overline{\Omega}$.

Then, the system of impulsive differential equations

$$(HB) \quad \begin{cases} x'(t) = f(t, x(t)) & \text{a.e. } t \in \mathbb{R} \setminus \{t_k : k \in \mathbb{Z}\}, \\ x(t_k^+) = x(t_k) + I_k(x(t_k)) & \text{for any } k \in \mathbb{Z}, \end{cases}$$

has at least one T -periodic solution lying in $\overline{\Omega}$.

In this paper we also need the following theorem by Papageorgiou (see [10], Theorem 3.1).

Theorem 2.2. *Let (M, Σ, μ) be a complete, σ -finite measure space, X a separable Banach space and $r \in [1, \infty[$. If $f_n, f \in L^r_X(M)$, for any $n \in \mathbb{N}$, $f_n(\cdot) \xrightarrow{w} f(\cdot)$ in $L^r_X(M)$ and, for μ - a.e. $t \in M$, there exists a nonempty, weakly-compact subset $G(t)$ of X such that $f_n(t) \in G(t)$, then*

$$f(t) \in \overline{\text{co}} \, w\text{-}\overline{\lim}_{n \rightarrow \infty} \{f_n(t)\} \quad \text{for } \mu\text{-a.e. } t \in M,$$

where $w\text{-}\overline{\lim}_{n \rightarrow \infty} \{f_n(t)\} = \left\{ x \in X : x = w\text{-}\lim_{k \rightarrow \infty} f_{n_k}(t), (f_{n_k}(t))_k \subseteq (f_n(t))_n \right\}$.

3. MAIN RESULTS

In this section we state and prove our existence results on periodic solutions for the impulsive problem (\mathcal{P}) .

Theorem 3.1. *Let $\overline{\Omega} \subset \mathbb{R}^N$, $N \geq 1$, be a canonical domain such that $\text{int } \overline{\Omega} \neq \emptyset$, and let $F : [0, T] \times \overline{\Omega} \rightarrow \mathcal{P}(\mathbb{R}^N)$ be a set-valued map such that the following conditions hold:*

- (H1) $F(t, x)$ is nonempty, convex, and closed, $\forall (t, x) \in [0, T] \times \overline{\Omega}$;
- (H2) $F(t, \cdot)$ is u.s.c. in $\overline{\Omega}$, a.e. on $[0, T]$;
- (H3) F is l.s.c. in $[0, T] \times \overline{\Omega}$;
- (H4) $F(t, x) \cap T_{\overline{\Omega}}(x) \neq \emptyset$, $\forall (t, x) \in [0, T] \times \overline{\Omega}$;
- (H5) $\exists \gamma \in L^\infty([0, T])$ such that $\| F(t, x) \| \leq \gamma(t)$ a.e. on $[0, T]$, $\forall x \in \overline{\Omega}$;

and let $I_k : \overline{\Omega} \rightarrow \mathbb{R}^N$, $k \in \{1, \dots, p\}$, be an impulse function such that

- (B4) I_k is continuous in $\overline{\Omega}$;
- (B7) $x + I_k(x) \in \overline{\Omega}$, $\forall x \in \overline{\Omega}$.

Then, there exists a solution for the impulsive differential problem (\mathcal{P}) .

Proof. Let $(\delta_n)_{n \in \mathbb{N}}$, $0 < \delta_n \leq 1$ for any $n \in \mathbb{N}$, be a sequence such that $\lim_{n \rightarrow \infty} \delta_n = 0$ and, for any $n \in \mathbb{N}$, let $\hat{F}_n : [0, T] \times \overline{\Omega} \rightarrow \mathcal{P}(\mathbb{R}^N)$ be a set-valued map defined as follows:

$$(2) \quad \hat{F}_n(t, x) = F(t, x) + \delta_n B(0, 1), \quad \forall (t, x) \in [0, T] \times \overline{\Omega},$$

where $B(0, 1)$ is the ball of center 0 and radius 1.

First of all, for a fixed $n \in \mathbb{N}$, we want to show that the set-valued map $\hat{F}_n(\cdot, \cdot) \cap T_{\overline{\Omega}}(\cdot)$ is l.s.c. in $[0, T] \times \overline{\Omega}$. Since $\text{int } \overline{\Omega} \neq \emptyset$ yields $\text{int } T_{\overline{\Omega}}(x) \neq \emptyset$, $\forall x \in \overline{\Omega}$ (see [5], p. 281), by using (H4), we have that $\hat{F}_n(t, x) \cap \text{int } T_{\overline{\Omega}}(x) \neq \emptyset$, $\forall (t, x) \in [0, T] \times \overline{\Omega}$. Moreover, (H3) and Proposition 2.59 of [4] imply that \hat{F}_n is l.s.c. in $[0, T] \times \overline{\Omega}$. By Proposition 5.35 of [4] the set-valued map $(t, x) \mapsto T_{\overline{\Omega}}(x)$, $(t, x) \in [0, T] \times \overline{\Omega}$, is l.s.c. in $[0, T] \times \overline{\Omega}$ and has convex values. So, by Proposition 2.54 of [4], we can deduce that $\hat{F}_n(\cdot, \cdot) \cap T_{\overline{\Omega}}(\cdot)$ is l.s.c. in $[0, T] \times \overline{\Omega}$. Taking into account that the set-valued map $\hat{F}_n(\cdot, \cdot) \cap T_{\overline{\Omega}}(\cdot)$ has nonempty, convex and closed values, from Michael's selection theorem we can say that there exists a continuous selection $g_n : [0, T] \times \overline{\Omega} \rightarrow \mathbb{R}^N$ for the set-valued map $\hat{F}_n(\cdot, \cdot) \cap T_{\overline{\Omega}}(\cdot)$ such that

$$(3) \quad g_n(t, x) \in \hat{F}_n(t, x) \cap T_{\overline{\Omega}}(x), \quad \forall (t, x) \in [0, T] \times \overline{\Omega}.$$

Now, by Lemma 1 of Lasota and Yorke (see [6]), we have that there exists a locally Lipschitz function $\hat{g}_n : [0, T] \times \overline{\Omega} \rightarrow \mathbb{R}^N$ such that

$$(4) \quad \| g_n(t, x) - \hat{g}_n(t, x) \| \leq \delta_n \xi, \quad \forall (t, x) \in [0, T] \times \overline{\Omega},$$

where ξ is a positive constant such that

$$(5) \quad \xi \leq 1 \quad \text{and} \quad \xi \| x^* \| \leq \langle x^*, x \rangle, \quad \forall x^* \in N_{\overline{\Omega}}(x).$$

At this point, let us define $f_n : [0, T] \times \overline{\Omega} \rightarrow \mathbb{R}^N$ as follows:

$$(6) \quad f_n(t, x) = \hat{g}_n(t, x) - \delta_n x, \quad \forall (t, x) \in [0, T] \times \overline{\Omega}.$$

Now we prove that

$$(7) \quad f_n(t, x) \in T_{\overline{\Omega}}(x), \quad \forall (t, x) \in [0, T] \times \overline{\Omega}.$$

In order to obtain (7), let us fix $(t, x) \in [0, T] \times \overline{\Omega}$ and $x^* \in N_{\overline{\Omega}}(x)$. Taking into account (6), (3), (5), (4) and (1), we have that

$$\begin{aligned} \langle x^*, f_n(t, x) \rangle &= \langle x^*, \hat{g}_n(t, x) - \delta_n x \rangle \\ &= \langle x^*, g_n(t, x) \rangle + \langle x^*, \hat{g}_n(t, x) - g_n(t, x) \rangle - \delta_n \langle x^*, x \rangle \\ &\leq \|x^*\| \|\hat{g}_n(t, x) - g_n(t, x)\| - \delta_n \xi \|x^*\| \\ &\leq \delta_n \xi \|x^*\| - \delta_n \xi \|x^*\| = 0, \end{aligned}$$

and so (7) is proved.

Recalling that f_n is Lipschitz in $[0, T] \times \overline{\Omega}$ (being locally Lipschitz in the compact, convex set $[0, T] \times \overline{\Omega}$), Hristova and Bainov's theorem (see Theorem 2.1) implies that the impulsive periodic problem

$$(P_n) \quad \begin{cases} x'(t) = f_n(t, x(t)) & \text{a.e. } t \in [0, T] \setminus \{t_1, \dots, t_p\}, \\ x(t_k^+) = x(t_k) + I_k(x(t_k)) & \text{for any } k \in \{1, \dots, p\}, \\ x(0) = x(T) \end{cases}$$

has a solution $x_n : [0, T] \rightarrow \overline{\Omega}$, absolutely continuous in the closed interval $[0, t_1]$ and in the interval $]t_k, t_{k+1}]$ for any $k \in \{1, \dots, p\}$.

Let us define $y_n : [0, T] \rightarrow \mathbb{R}^N$ as follows:

$$(8) \quad y_n(t) = \begin{cases} x_n(t) & \text{if } t \in [0, t_1], \\ x_n(t) - \sum_{j=1}^k I_j(x_n(t_j)) & \text{if } t \in]t_k, t_{k+1}], k \in \{1, \dots, p\}, \end{cases}$$

for any $n \in \mathbb{N}$. Now, we prove that the sequence $(y_n)_{n \in \mathbb{N}}$ is equibounded in $[0, T]$. First of all, we observe that, for any $n \in \mathbb{N}$,

$$x_n(t) = \begin{cases} x_n(0) + \int_0^t f_n(s, x_n(s)) ds & \text{if } t \in [0, t_1], \\ x_n(0) + \sum_{j=1}^k I_j(x_n(t_j)) + \int_0^t f_n(s, x_n(s)) ds & \text{if } t \in]t_k, t_{k+1}], \\ & k \in \{1, \dots, p\}. \end{cases}$$

Then it follows that

$$(9) \quad y_n(t) = x_n(0) + \int_0^t f_n(s, x_n(s)) ds,$$

for all $t \in [0, T]$ and for any $n \in \mathbb{N}$. So, the fact that $\overline{\Omega}$ is bounded (this implies that there exists $R > 0$ such that $\overline{\Omega} \subseteq B(0, R)$), (9), (6), (4), (3), (2), (H5) and the choice of $(\delta_n)_{n \in \mathbb{N}}$ yield

$$\|y_n(t)\| \leq \|x_n(0)\| + \int_0^t \|f_n(s, x_n(s))\| ds \leq R + KT,$$

for all $t \in [0, T]$ and for any $n \in \mathbb{N}$, where $K = 1 + \xi + R + \|\gamma\|_{L^\infty([0, T])}$.

Now let us prove that the sequence $(y_n)_{n \in \mathbb{N}}$ is equicontinuous in $[0, T]$. Let us fix $t, s \in [0, T]$. By (9), (6), (4), (3), (2), (H5), the choice of $(\delta_n)_{n \in \mathbb{N}}$ and the

boundedness of $\overline{\Omega}$ we have that

$$(10) \quad \| y_n(t) - y_n(s) \| = \left\| \int_s^t f_n(z, x_n(z)) dz \right\| \leq K | t - s |,$$

for any $n \in \mathbb{N}$; so $(y_n)_{n \in \mathbb{N}}$ is equicontinuous in $[0, T]$.

Then, by the Ascoli - Arzelà theorem, there exists a subsequence of $(y_n)_{n \in \mathbb{N}}$, which we again denote by $(y_n)_{n \in \mathbb{N}}$, such that

$$(11) \quad y_n \longrightarrow y_0 \text{ uniformly on } [0, T].$$

By (8), (11) and (B4) it follows that there exists a subsequence of $(x_n)_{n \in \mathbb{N}}$, denoted again by $(x_n)_{n \in \mathbb{N}}$, such that

$$(12) \quad x_n \longrightarrow x_0 \text{ uniformly on } [0, T].$$

We claim that x_0 is a solution of the impulsive differential inclusion (\mathcal{P}) . The fact that x_n solves problem (\mathcal{P}_n) , for any $n \in \mathbb{N}$, (12) and (B4) yield

$$x_0(t) \in \overline{\Omega}, \forall t \in [0, T], \quad x_0(0) = x_0(T),$$

and

$$\begin{aligned} x_0(t_k^+) &= \lim_{t \rightarrow t_k^+} x_0(t) = \lim_{t \rightarrow t_k^+} \lim_{n \rightarrow \infty} x_n(t) = \lim_{n \rightarrow \infty} x_n(t_k^+) \\ &= \lim_{n \rightarrow \infty} [x_n(t_k) + I_k(x_n(t_k))] = x_0(t_k) + I_k(x_0(t_k)), \end{aligned}$$

for any $k \in \{1, \dots, p\}$. Now we prove that x_0 is absolutely continuous in the closed interval $[0, t_1]$ and in the interval $]t_k, t_{k+1}]$ for any $k \in \{1, \dots, p\}$. Actually, by (10), the sequence $(y_n)_{n \in \mathbb{N}}$ is equi-Lipschitz in $[0, T]$. So, by using (8), we can say that $(x_n)_{n \in \mathbb{N}}$ is equi-Lipschitz in $[0, t_1]$ and in $]t_k, t_{k+1}]$, for any $k \in \{1, \dots, p\}$. By passing to the limit as n goes to infinity, we obtain that x_0 is Lipschitz in $[0, t_1]$ and in $]t_k, t_{k+1}]$, for any $k \in \{1, \dots, p\}$. It remains to show that $x_0'(t) \in F(t, x_0(t))$ a.e. $t \in [0, T] \setminus \{t_1, \dots, t_p\}$. Without loss of generality, given a fixed $k \in \{1, \dots, p\}$, we can suppose that x_n and x_0 are continuous in $[t_k, t_{k+1}]$ by defining them in t_k in this way:

$$x_n(t_k) = x_n(t_k^+) \quad \text{and} \quad x_0(t_k) = x_0(t_k^+),$$

for any $n \in \mathbb{N}$. By using the fact that x_n is a solution of problem (\mathcal{P}_n) , for any $n \in \mathbb{N}$, (6), (4), (3), (2), (H5), the choice of $(\delta_n)_{n \in \mathbb{N}}$ and the boundedness of $\overline{\Omega}$ yield that the sequence $(x_n')_{n \in \mathbb{N}}$ is bounded in $L^2([t_k, t_{k+1}])$. So, there exist $v_k \in L^2([t_k, t_{k+1}])$ and a subsequence of $(x_n')_{n \in \mathbb{N}}$, denoted again by $(x_n')_{n \in \mathbb{N}}$, such that

$$(13) \quad x_n' \xrightarrow{w} v_k \text{ in } L^2([t_k, t_{k+1}]).$$

Since x_n is absolutely continuous in $[t_k, t_{k+1}]$, for any $n \in \mathbb{N}$ we have that

$$(14) \quad x_n(t) - x_n(t_k) = \int_{t_k}^t x_n'(s) ds,$$

for all $t \in [t_k, t_{k+1}]$. By (12), (13), (14) and the absolute continuity of x_0 in $[t_k, t_{k+1}]$ it follows that $x_0' = v_k$ a.e. on $[t_k, t_{k+1}]$. The same argument can be used to prove that there exists $v_0 \in L^2([0, t_1])$ such that $x_0' = v_0$ a.e. on $[0, t_1]$. So

$$(15) \quad x_n' \xrightarrow{w} x_0' \text{ in } L^2([0, T]).$$

Let us denote by $N \subset [0, T]$ the set such that $\mu N = 0$ and hypotheses (H2) and (H5) are satisfied in $([0, T] \setminus N) \times \bar{\Omega}$. Now let us fix $t \in [0, T] \setminus N$ and $\epsilon > 0$. By the choice of $(\delta_n)_{n \in \mathbb{N}}$, it follows that

$$(16) \quad \exists n_1 = n_1(\epsilon) \in \mathbb{N} : \forall n \geq n_1 \delta_n < \epsilon.$$

By using (H2), we can say that

$$(17) \quad \exists \rho = \rho(\epsilon, x_0(t)) > 0 : \forall x \in \bar{\Omega}, \|x - x_0(t)\| < \rho, F(t, x) \subseteq F(t, x_0(t)) + B(0, \epsilon).$$

By (12), we have that

$$(18) \quad \exists n_2(\epsilon, t) = n_2(\rho(\epsilon, x_0(t))) : \forall n \geq n_2(\epsilon, t) \|x_n(s) - x_0(s)\| < \rho, \forall s \in [0, T].$$

Then, by (17) and (18), we obtain that

$$(19) \quad F(t, x_n(t)) \subseteq F(t, x_0(t)) + B(0, \epsilon), \forall n \geq n_2(\epsilon, t).$$

Then, setting $n(\epsilon, t) = \max\{n_1(\epsilon), n_2(\epsilon, t)\}$, by (2), (16) and (19) we have that

$$(20) \quad \hat{F}_n(t, x_n(t)) \subseteq F(t, x_0(t)) + B(0, 2\epsilon), \forall n \geq n(\epsilon, t).$$

Since x_n is a solution for problem (\mathcal{P}_n) , for any $n \in \mathbb{N}$, (6), (3), (4) and the boundedness of $\bar{\Omega}$ imply

$$(21) \quad \begin{aligned} x'_n(t) &= f_n(t, x_n(t)) = \hat{g}_n(t, x_n(t)) - \delta_n x_n(t) \\ &= g_n(t, x_n(t)) + \hat{g}_n(t, x_n(t)) - g_n(t, x_n(t)) - \delta_n x_n(t) \\ &\in \hat{F}_n(t, x_n(t)) + (\xi + R)\delta_n B(0, 1) \end{aligned}$$

a.e. $t \in [0, T] \setminus \{t_1, \dots, t_p\}$ and for any $n \in \mathbb{N}$. By using (21), (2), (H5) and the choice of $(\delta_n)_{n \in \mathbb{N}}$ and ξ , we have that

$$(22) \quad \begin{aligned} x'_n(t) &\in F(t, x_n(t)) + \delta_n B(0, 1) + (\xi + R)\delta_n B(0, 1) \\ &\subseteq (\gamma(t) + 2 + R)B(0, 1) \equiv G(t) \end{aligned}$$

a.e. $t \in [0, T] \setminus \{t_1, \dots, t_p\}$ and for any $n \in \mathbb{N}$. Taking into account that $G(t)$ is a convex and weakly compact subset of \mathbb{R}^N , (15) and (22) allow us to deduce that (see Theorem 2.2)

$$(23) \quad x'_0(t) \in \overline{\text{co}} \text{ w-}\lim_{n \rightarrow \infty} \{x'_n(t)\} \text{ a.e. } t \in [0, T].$$

Now, by (16) and (21) it follows that

$$(24) \quad x'_n(t) \in \hat{F}_n(t, x_n(t)) + (\xi + R)B(0, \epsilon)$$

a.e. $t \in [0, T] \setminus \{t_1, \dots, t_p\}$, for any $n \geq n(\epsilon, t)$. Then (23) and (24) yield

$$(25) \quad x'_0(t) \in \overline{\text{co}} \text{ w-}\lim_{n \rightarrow \infty} \left(\hat{F}_n(t, x_n(t)) + (\xi + R)B(0, \epsilon) \right)$$

a.e. $t \in [0, T] \setminus \{t_1, \dots, t_p\}$. So, by (25), (20) and (H1), we obtain that

$$(26) \quad x'_0(t) \in \overline{\text{co}} \left(F(t, x_0(t)) + (2 + \xi + R)B(0, \epsilon) \right) = F(t, x_0(t)) + (2 + \xi + R)B(0, \epsilon)$$

a.e. $t \in [0, T] \setminus \{t_1, \dots, t_p\}$. So, the arbitrariness of ϵ , (26) and (H1) yield

$$x'_0(t) \in F(t, x_0(t)) \text{ a.e. } t \in [0, T] \setminus \{t_1, \dots, t_p\}. \quad \square$$

Remark 3.1. We note that there exist set-valued maps satisfying the hypotheses of Theorem 3.1, but not the conditions of Theorem 2.1 due to Watson (see [12]). For example, we can consider the set-valued map $F : [0, 1] \times [0, 1] \rightarrow \mathcal{P}(\mathbb{R})$ defined as follows:

$$F(t, x) = \begin{cases} \{0\} & \text{if } t = 0, x \in [0, 1], \\ [0, 1] & \text{if } t \in]0, 1], x \in [0, 1]. \end{cases}$$

Now, by using Theorem 3.1, we are able to prove the following result.

Theorem 3.2. *Let $\overline{\Omega} \subset \mathbb{R}^N$, $N \geq 1$, be a canonical domain such that $\text{int } \overline{\Omega} \neq \emptyset$, and let $F : [0, T] \times \overline{\Omega} \rightarrow \mathcal{P}(\mathbb{R}^N)$ be a set-valued map such that the following conditions hold:*

- (C1) $F(t, x)$ is nonempty, convex, and closed a.e. on $[0, T]$, $\forall x \in \overline{\Omega}$;
- (H2) $F(t, \cdot)$ is u.s.c. in $\overline{\Omega}$, a.e. on $[0, T]$;
- (C2) $\exists (F_n)_{n \in \mathbb{N}}$, $F_n : [0, T] \times \overline{\Omega} \rightarrow \mathcal{P}(\mathbb{R}^N)$ such that
 - (C2.1) $F_n(t, x)$ is nonempty, convex, and closed, $\forall (t, x) \in [0, T] \times \overline{\Omega}$, $\forall n \in \mathbb{N}$;
 - (C2.2) F_n is l.s.c. in $[0, T] \times \overline{\Omega}$, $\forall n \in \mathbb{N}$;
 - (C2.3) $F_n(t, x) \cap T_{\overline{\Omega}}(x) \neq \emptyset$, $\forall (t, x) \in [0, T] \times \overline{\Omega}$, $\forall n \in \mathbb{N}$;
 - (C2.4) $\exists \gamma \in L^\infty([0, T])$ such that $\|F_n(t, x)\| \leq \gamma(t)$ a.e. on $[0, T]$, $\forall x \in \overline{\Omega}$, $\forall n \in \mathbb{N}$;
 - (C2.5) a.e. $t \in [0, T]$, $\forall \epsilon > 0 \exists \bar{n} = \bar{n}(\epsilon, t) \in \mathbb{N}$ such that $F_n(t, x) \subseteq F(t, x) + B(0, \epsilon)$, $\forall n \geq \bar{n}$, $\forall x \in \overline{\Omega}$;

and let $I_k : \overline{\Omega} \rightarrow \mathbb{R}^N$, $k \in \{1, \dots, p\}$, be an impulse function such that

- (B4) I_k is continuous in $\overline{\Omega}$;
- (B7) $x + I_k(x) \in \overline{\Omega}$, $\forall x \in \overline{\Omega}$.

Then, there exists a solution for the impulsive differential problem (P).

Proof. Let $(\delta_n)_{n \in \mathbb{N}}$, $0 < \delta_n \leq 1$ for any $n \in \mathbb{N}$, be a sequence such that $\lim_{n \rightarrow \infty} \delta_n = 0$ and, for any $n \in \mathbb{N}$, let $\hat{F}_n : [0, T] \times \overline{\Omega} \rightarrow \mathcal{P}(\mathbb{R}^N)$ be a set-valued map defined as follows:

$$(27) \quad \hat{F}_n(t, x) = F_n(t, x) + \delta_n B(0, 1), \quad \forall (t, x) \in [0, T] \times \overline{\Omega}.$$

In order to prove the existence of a continuous selection $g_n : [0, T] \times \overline{\Omega} \rightarrow \mathbb{R}^N$ for the set-valued map $\hat{F}_n(\cdot, \cdot) \cap T_{\overline{\Omega}}(\cdot)$, we can apply the same arguments used in the first part of the proof of Theorem 3.1. Now let us consider the impulsive periodic problem

$$(P_n) \quad \begin{cases} x'(t) = g_n(t, x(t)) & \text{a.e. } t \in [0, T] \setminus \{t_1, \dots, t_p\}, \\ x(t_k^+) = x(t_k) + I_k(x(t_k)) & \text{for any } k \in \{1, \dots, p\}, \\ x(0) = x(T). \end{cases}$$

By Theorem 3.1 there exists a solution $x_n : [0, T] \rightarrow \overline{\Omega}$ for problem (P_n) , for any $n \in \mathbb{N}$. Arguing as in the proof of Theorem 3.1, we obtain that

$$(28) \quad x_n \rightarrow x_0 \text{ uniformly on } [0, T],$$

where x_0 is a function, absolutely continuous in the closed interval $[0, t_1]$ and in the interval $]t_k, t_{k+1}[$ for any $k \in \{1, \dots, p\}$, such that

$$\begin{aligned} x_0(t) &\in \overline{\Omega}, \quad \forall t \in [0, T], \quad x_0(0) = x_0(T), \\ x_0(t_k^+) &= x_0(t_k) + I_k(x_0(t_k)), \quad k \in \{1, \dots, p\}, \end{aligned}$$

and

$$(29) \quad x'_n \xrightarrow{w} x'_0 \text{ in } L^2([0, T]).$$

To prove that x_0 is a solution for the impulsive differential problem (\mathcal{P}) , it remains to show that $x'_0(t) \in F(t, x_0(t))$ a.e. $t \in [0, T] \setminus \{t_1, \dots, t_p\}$.

Let us denote by $N^* \subset [0, T]$ the set such that $\mu N^* = 0$ and $(C1), (H2), (C2.4)$ and $(C2.5)$ are satisfied in $([0, T] \setminus N^*) \times \overline{\Omega}$. Now, let us fix $t \in [0, T] \setminus N^*$ and $\epsilon > 0$. Hypothesis $(C2.5)$ implies that

$$(30) \quad \exists \bar{n} = \bar{n}(\epsilon, t) \in \mathbb{N} : \forall n \geq \bar{n}(\epsilon, t) \quad F_n(t, x_n(t)) \subseteq F(t, x_n(t)) + B(0, \epsilon).$$

As in Theorem 3.1, by the choice of $(\delta_n)_{n \in \mathbb{N}}$, $(H2)$ and (28) , we can say that there exists $\tilde{n} = \tilde{n}(\epsilon, t) \in \mathbb{N}$ such that

$$(31) \quad \delta_n < \epsilon \quad \text{and} \quad F(t, x_n(t)) \subseteq F(t, x_0(t)) + B(0, \epsilon), \quad \forall n \geq \tilde{n}(\epsilon, t).$$

Then, setting $n^*(\epsilon, t) = \max \{ \bar{n}(\epsilon, t), \tilde{n}(\epsilon, t) \}$, by (30) and (31) we obtain that

$$(32) \quad \hat{F}_n(t, x_n(t)) \subseteq F(t, x_0(t)) + B(0, 3\epsilon), \quad \forall n \geq n^*(\epsilon, t).$$

Since x_n is a solution for problem (\mathcal{P}_n) and g_n is a selection for $\hat{F}_n(\cdot, \cdot) \cap T_{\overline{\Omega}}(\cdot)$, for any $n \in \mathbb{N}$, (27) , $(C2.4)$ and the choice of $(\delta_n)_{n \in \mathbb{N}}$ imply

$$(33) \quad x'_n(t) \in \hat{F}_n(t, x_n(t)) = F_n(t, x_n(t)) + \delta_n B(0, 1) \subseteq B(0, \gamma(t) + 1) \equiv G(t)$$

a.e. $t \in [0, T] \setminus \{t_1, \dots, t_p\}$ and for any $n \in \mathbb{N}$. Taking into account that $G(t)$ is a convex and weakly compact subset of \mathbb{R}^N , (29) and (33) allow us to deduce that (see Theorem 2.2)

$$(34) \quad x'_0(t) \in \overline{\text{co}} \text{ w-}\lim_{n \rightarrow \infty} \{x'_n(t)\} \text{ a.e. } t \in [0, T].$$

Since $\{x'_n(t)\} \subseteq \hat{F}_n(t, x_n(t))$ a.e. $t \in [0, T] \setminus \{t_1, \dots, t_p\}$, for any $n \in \mathbb{N}$, by (34) it follows that

$$(35) \quad x'_0(t) \in \overline{\text{co}} \text{ w-}\lim_{n \rightarrow \infty} \hat{F}_n(t, x_n(t))$$

a.e. $t \in [0, T] \setminus \{t_1, \dots, t_p\}$. Then, by (35) , (32) and $(C1)$, we obtain that

$$(36) \quad x'_0(t) \in \overline{\text{co}} \left(F(t, x_0(t)) + B(0, 3\epsilon) \right) = F(t, x_0(t)) + B(0, 3\epsilon)$$

a.e. $t \in [0, T] \setminus \{t_1, \dots, t_p\}$. So, by the arbitrariness of ϵ , (36) and $(C1)$, we can conclude that x_0 is a solution of problem (\mathcal{P}) . □

Remark 3.2. Let us remark that Theorem 3.2 improves Theorem 3.1 (and so it extends the result due to Watson (see Remark 3.1)). It is obvious that, if F satisfies all the hypotheses of Theorem 3.1, we can take $F_n = F$, for any $n \in \mathbb{N}$. Moreover, there exist set-valued maps F satisfying the hypotheses of Theorem 3.2 that are not l.s.c. For example, we can consider the set-valued map $F : [0, 2] \times [0, 1] \rightarrow \mathcal{P}(\mathbb{R})$ defined as follows:

$$(37) \quad F(t, x) = \begin{cases} [-1, 0] & \text{if } t \in [0, 1[, x \in [0, 1], \\ [-1, 1] & \text{if } t = 1, x \in [0, 1], \\ [-1, 2] & \text{if } t \in]1, 2], x \in [0, 1]. \end{cases}$$

F is not l.s.c. in $[0, 2] \times [0, 1]$, but it satisfies hypotheses (C1) and (H2) of Theorem 3.2. Moreover, we can construct the sequence $(F_n)_{n \in \mathbb{N}}$, $F_n : [0, 2] \times [0, 1] \longrightarrow \mathcal{P}(\mathbb{R})$, in this way:

$$F_n(t, x) = \begin{cases} [-1, t^n] & \text{if } t \in [0, 1], x \in [0, 1], \\ [-1, 2] & \text{if } t \in]1, 2], x \in [0, 1], \end{cases}$$

for any $n \in \mathbb{N}$.

This sequence satisfies conditions (C2.1), (C2.2), (C2.3), (C2.4) and, since it converges to F , (C2.5).

We also want to observe that in Theorem 3.2 we do not need that the sequence $(F_n)_{n \in \mathbb{N}}$ converges to F . Indeed, the sequence $(\tilde{F}_n)_{n \in \mathbb{N}}$, $\tilde{F}_n : [0, 2] \times [0, 1] \longrightarrow \mathcal{P}(\mathbb{R})$ defined by

$$\tilde{F}_n(t, x) = \begin{cases} [-1, \frac{1}{n}] & \text{if } t \in [0, 1], x \in [0, 1], \\ [-1, 2] & \text{if } t \in]0, 1], x \in [0, 1], \end{cases}$$

for any $n \in \mathbb{N}$, does not converge to the set-valued map F defined as in (37), but it satisfies all the hypotheses of Theorem 3.2.

Remark 3.3. Let us note that there exist u.s.c., but not l.s.c., set-valued maps F satisfying the hypotheses of Theorem 3.2. For example, we can consider the set-valued map $F : [0, 2] \times [0, 1] \longrightarrow \mathcal{P}(\mathbb{R})$ defined as follows:

$$F(t, x) = \begin{cases} [-1, 0] & \text{if } t \in [0, 1[, x \in [0, 1], \\ [-1, 1] & \text{if } t \in [1, 2], x \in [0, 1]. \end{cases}$$

In this case we can construct the sequence $(F_n)_{n \in \mathbb{N}}$, $F_n : [0, 2] \times [0, 1] \longrightarrow \mathcal{P}(\mathbb{R})$, in this way:

$$F_n(t, x) = \begin{cases} [-1, \frac{1}{n}] & \text{if } t \in [0, 1], x \in [0, 1], \\ [-1, 1] & \text{if } t \in]1, 2], x \in [0, 1], \end{cases}$$

for any $n \in \mathbb{N}$.

Remark 3.4. Finally, we observe that there exist set-valued maps satisfying the hypotheses of Theorem 3.2 that are neither l.s.c. nor u.s.c. For example, we can consider the set-valued map F defined as in (37).

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