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FUNCTIONAL ANALYSIS PROOFS OF ABEL'S THEOREMS

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ABSTRACT. We give alternative proofs to the classical theorems of Abel, using the concept of Berezin symbol.

1.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. The sequence $\{a_n\}_{n=0}^{\infty}$ is Abel convergent (written (A) convergent) to a if the limit

$$\lim_{t\to 1^-}(1-t)\sum_{n=0}^{\infty} a_n t^n = a$$

exists. The series $\sum\limits_{n=0}^{\infty} a_n$ is (A) convergent to L if the sequence of partial sums $\{s_n\}_{n=0}^{\infty}$ (where $s_n \overset{\text{def}}{=} \sum\limits_{k=0}^{n} a_k$) is (A) convergent to L.

The following famous results are due to Abel (e.g., see [1], [2]).

Theorem 1 (Theorem of Abel)**.** *If $\{a_n\}_{n=0}^{\infty}$ converges to a, then $\{a_n\}_{n=0}^{\infty}$ (A) converges to a.*

Theorem 2 (Theorem of Abel)**.** *If the series $\sum\limits_{n=0}^{\infty} a_n$ converges to L, then $\sum\limits_{n=0}^{\infty} a_n$ is (A) convergent to L.*

This paper presents functional analysis proofs of these results. To give our proofs, we first define what is meant by a Berezin symbol.

2.

Let H^2 denote the Hardy space of functions analytic on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For a general bounded operator A on the Hardy space, the Berezin symbol of A is (see [3]) the function $\tilde{A}$ defined by

$$\tilde{A}(\lambda) = \left(A\hat{k}_\lambda, \hat{k}_\lambda\right), \ \lambda \in \mathbb{D},$$

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where $\hat{k}_\lambda$ is the normalized reproducing kernel of H^2. The function $\hat{k}_\lambda$ is defined by

$$\hat{k}_\lambda (z) = \frac{\sqrt{1-|\lambda|^2}}{1-\bar{\lambda}z}$$

for $z \in \mathbb{D}$ and has the property that $\left(f, \hat{k}_\lambda\right) = \sqrt{1-|\lambda|^2}\, f(\lambda)$, for $f \in H^2$, and this obviously approaches 0 for $f \in H^\infty$ (the space of all bounded analytic functions on $\mathbb{D}$), and hence for all $f \in H^2$, whenever $|\lambda| \to 1^-$. Thus, the kernels $\hat{k}_\lambda$ converge weakly to zero as λ approaches the unit circle $\partial\mathbb{D}$ (i.e., the Hardy space H^2 is standard [3]). Then we have that if A is a compact operator on H^2, then $\tilde{A}(\lambda) \to 0$ as $\lambda \to \partial\mathbb{D}$. In this sense, the Berezin symbol of a compact operator on H^2 vanishes on the boundary.

3.

Proof of Theorem 1. Let us consider the diagonal operator $\mathcal{D}_{\{a_n\}}$ on H^2 defined by

$$\mathcal{D}_{\{a_n\}} z^k = a_k z^k, \quad k = 0, 1, 2, \ldots .$$

Since $\{a_n\}$ is a bounded sequence, $\mathcal{D}_{\{a_n\}}$ is a bounded operator on H^2. We now calculate the Berezin symbol of an operator $\mathcal{D}_{\{a_n\}}$. We have

$$\begin{aligned}
\tilde{\mathcal{D}}_{\{a_n\}}(\lambda) &= \left(\mathcal{D}_{\{a_n\}}, \hat{k}_\lambda, \hat{k}_\lambda\right) = \sqrt{1-|\lambda|^2}\left(\mathcal{D}_{\{a_n\}} \sum_{k=0}^{\infty} \bar{\lambda}^k z^k,\, \hat{k}_\lambda\right) \\
&= \sqrt{1-|\lambda|^2}\left(\sum_{k=0}^{\infty} \bar{\lambda}^k \mathcal{D}_{\{a_n\}} z^k,\, \hat{k}_\lambda\right) \\
&= \sqrt{1-|\lambda|^2}\left(\sum_{k=0}^{\infty} \bar{\lambda}^k a_k z^k,\, \hat{k}_\lambda\right) \\
&= \left(1-|\lambda|^2\right) \sum_{k=0}^{\infty} a_k |\lambda|^{2k}.
\end{aligned}$$

Thus,

$$\tilde{\mathcal{D}}_{\{a_n\}}(\lambda) = \left(1-|\lambda|^2\right) \sum_{k=0}^{\infty} a_k |\lambda|^{2k} \;, \quad \lambda \in \mathbb{D}$$

(i.e., $\tilde{\mathcal{D}}_{\{a_n\}}$ is a radial function, $\tilde{\mathcal{D}}_{\{a_n\}}(\lambda) = \tilde{\mathcal{D}}_{\{a_n\}}(|\lambda|)$), which yields

$$\text{(1)} \qquad \tilde{\mathcal{D}}_{\{a_n\}}\left(\sqrt{t}\right) = (1-t) \sum_{k=0}^{\infty} a_k t^k, \quad 0 < t < 1.$$

Then from (1) we have

$$(1-t)\sum_{k=0}^{\infty} a_k t^k = (1-t)\sum_{k=0}^{\infty} (a_k - a)\, t^k + a(1-t)\sum_{k=0}^{\infty} t^k = \tilde{\mathcal{D}}_{\{a_n - a\}}\left(\sqrt{t}\right) + a.$$

Since by the condition of the theorem $a_n - a \to 0$ as $n \to \infty$, $\tilde{\mathcal{D}}_{\{a_n - a\}}$ is a compact operator on H^2. Hence, its Berezin symbol vanishes on the boundary, i.e.,

$$\lim_{t \to 1^-} \tilde{\mathcal{D}}_{\{a_n - a\}}\left(\sqrt{t}\right) = 0.$$

Then the last equality yields

$$\lim_{t\to 1^-}(1-t)\sum_{k=0}^{\infty}a_k t^k = a,$$

which completes the proof. □

4.

Proof of Theorem 2. The argument that was used to prove Theorem 1 can easily be modified to prove the equality

$$\tilde{\mathcal{D}}_{\{s_n\}}\left(\sqrt{t}\right) = \sum_{k=0}^{\infty} a_k t^k\,, \quad 0<t<1. \tag{2}$$

Formula (2) implies that for each $t \in (0,1)$ the series $\sum\limits_{k=0}^{\infty} a_k t^k$ is convergent. On the other hand,

$$\mathcal{D}_{\{s_n\}} = LI + \mathcal{D}_{\{s_n - L\}}\,,$$

where the diagonal operator $\mathcal{D}_{\{s_n-L\}}$ is compact since by the hypothesis of the theorem $s_n - L \to 0$ as $n\to\infty$, and therefore from the formula (2) we have

$$\begin{aligned}\lim_{t\to 1^-}\sum_{k=0}^{\infty}a_k t^k &= \lim_{t\to 1^-}\tilde{\mathcal{D}}_{\{s_n\}}\left(\sqrt{t}\right)\\ &= \lim_{t\to 1^-}\left(L+\tilde{\mathcal{D}}_{\{s_n-L\}}\left(\sqrt{t}\right)\right)\\ &= L+\lim_{t\to 1^-}\tilde{\mathcal{D}}_{\{s_n-L\}}\left(\sqrt{t}\right)\\ &= L,\end{aligned}$$

which means that the series $\sum\limits_{k=0}^{\infty} a_k$ (A) converges to L. The theorem is proved. □

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