FUNCTIONAL ANALYSIS PROOFS OF ABEL’S THEOREMS

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Abstract. We give alternative proofs to the classical theorems of Abel, using the concept of Berezin symbol.

1.

Let \( \{a_n\}_{n=0}^{\infty} \) be a sequence of complex numbers. The sequence \( \{a_n\}_{n=0}^{\infty} \) is Abel convergent (written \((A)\) convergent) to \( a \) if the limit

\[
\lim_{t \to 1^-} (1 - t) \sum_{n=0}^{\infty} a_n t^n = a
\]

exists. The series \( \sum_{n=0}^{\infty} a_n \) is \((A)\) convergent to \( L \) if the sequence of partial sums \( \{s_n\}_{n=0}^{\infty} \) (where \( s_n \stackrel{\text{def}}{=} \sum_{k=0}^{n} a_k \)) is \((A)\) convergent to \( L \).

The following famous results are due to Abel (e.g., see \[1, 2]\).

Theorem 1 (Theorem of Abel). If \( \{a_n\}_{n=0}^{\infty} \) converges to \( L \), then \( \{a_n\}_{n=0}^{\infty} \) \((A)\) converges to \( L \).

Theorem 2 (Theorem of Abel). If the series \( \sum_{n=0}^{\infty} a_n \) converges to \( L \), then \( \sum_{n=0}^{\infty} a_n \) is \((A)\) convergent to \( L \).

This paper presents functional analysis proofs of these results. To give our proofs, we first define what is meant by a Berezin symbol.

2.

Let \( H^2 \) denote the Hardy space of functions analytic on the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). For a general bounded operator \( A \) on the Hardy space, the Berezin symbol of \( A \) is (see \[3]\) the function \( \tilde{A} \) defined by

\[
\tilde{A} (\lambda) = \left( A \hat{k}_{\lambda}, \hat{k}_{\lambda} \right) , \ \lambda \in \mathbb{D} ,
\]

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where \( \hat{k}_\lambda \) is the normalized reproducing kernel of \( H^2 \). The function \( \hat{k}_\lambda \) is defined by

\[
\hat{k}_\lambda(z) = \frac{\sqrt{1 - |\lambda|^2}}{1 - \lambda z}
\]

for \( z \in \mathbb{D} \) and has the property that \( (f, \hat{k}_\lambda) = \sqrt{1 - |\lambda|^2} f(\lambda), \) for \( f \in H^2 \), and this obviously approaches 0 for \( f \in H^\infty \) (the space of all bounded analytic functions on \( \mathbb{D} \)), and hence for all \( f \in H^2 \), whenever \( |\lambda| \to 1^- \). Thus, the kernels \( \hat{k}_\lambda \) converge weakly to zero as \( \lambda \) approaches the unit circle \( \partial \mathbb{D} \) (i.e., the Hardy space \( H^2 \) is standard \( \mathbb{B} \)). Then we have that if \( A \) is a compact operator on \( H^2 \), then \( A(\lambda) \to 0 \) as \( \lambda \to \partial \mathbb{D} \). In this sense, the Berezin symbol of a compact operator on \( H^2 \) vanishes on the boundary.

3.

**Proof of Theorem 1.** Let us consider the diagonal operator \( D_{\{a_n\}} \) on \( H^2 \) defined by

\[
D_{\{a_n\}} z^k = a_k z^k, \quad k = 0, 1, 2, \ldots.
\]

Since \( \{a_n\} \) is a bounded sequence, \( D_{\{a_n\}} \) is a bounded operator on \( H^2 \). We now calculate the Berezin symbol of an operator \( D_{\{a_n\}} \). We have

\[
\hat{D}_{\{a_n\}}(\lambda) = \left( D_{\{a_n\}}, \hat{k}, \hat{k} \right) = \sqrt{1 - |\lambda|^2} \left( D_{\{a_n\}} \sum_{k=0}^{\infty} \lambda^k z^k, \hat{k} \right)
\]

\[
= \sqrt{1 - |\lambda|^2} \left( \sum_{k=0}^{\infty} \lambda^k D_{\{a_n\}} z^k, \hat{k} \right)
\]

\[
= \sqrt{1 - |\lambda|^2} \left( \sum_{k=0}^{\infty} \lambda^k a_k z^k, \hat{k} \right)
\]

\[
= \left( 1 - |\lambda|^2 \right) \sum_{k=0}^{\infty} a_k |\lambda|^{2k}.
\]

Thus,

\[
\hat{D}_{\{a_n\}}(\lambda) = \left( 1 - |\lambda|^2 \right) \sum_{k=0}^{\infty} a_k |\lambda|^{2k}, \quad \lambda \in \mathbb{D}
\]

(i.e., \( \hat{D}_{\{a_n\}} \) is a radial function, \( \hat{D}_{\{a_n\}}(\lambda) = \hat{D}_{\{a_n\}}(|\lambda|) \)), which yields

\[
(1)
\]

\[
\hat{D}_{\{a_n\}}(\sqrt{t}) = (1 - t) \sum_{k=0}^{\infty} a_k t^k, \quad 0 < t < 1.
\]

Then from \( (1) \) we have

\[
(1 - t) \sum_{k=0}^{\infty} a_k t^k = (1 - t) \sum_{k=0}^{\infty} (a_k - a) t^k + a (1 - t) \sum_{k=0}^{\infty} t^k = \hat{D}_{\{a_n-a\}}(\sqrt{t}) + a.
\]

Since by the condition of the theorem \( a_n - a \to 0 \) as \( n \to \infty \), \( \hat{D}_{\{a_n-a\}} \) is a compact operator on \( H^2 \). Hence, its Berezin symbol vanishes on the boundary, i.e.,

\[
\lim_{t \to 1^-} \hat{D}_{\{a_n-a\}}(\sqrt{t}) = 0.
\]
Then the last equality yields
\[
\lim_{t \to 1^-} (1 - t) \sum_{k=0}^{\infty} a_k t^k = a,
\]
which completes the proof. \(\Box\)

4.

**Proof of Theorem 2.** The argument that was used to prove Theorem 1 can easily be modified to prove the equality
\[
(2) \quad \hat{D}_{\{s_n\}} \left( \sqrt{t} \right) = \sum_{k=0}^{\infty} a_k t^k, \quad 0 < t < 1.
\]

Formula (2) implies that for each \( t \in (0, 1) \) the series \( \sum_{k=0}^{\infty} a_k t^k \) is convergent. On the other hand,
\[
\hat{D}_{\{s_n\}} = LI + \hat{D}_{\{s_n-L\}},
\]
where the diagonal operator \( \hat{D}_{\{s_n-L\}} \) is compact since by the hypothesis of the theorem \( s_n - L \to 0 \) as \( n \to \infty \), and therefore from the formula (2) we have
\[
\lim_{t \to 1^-} \sum_{k=0}^{\infty} a_k t^k = \lim_{t \to 1^-} \hat{D}_{\{s_n\}} \left( \sqrt{t} \right)
\]
\[
= \lim_{t \to 1^-} \left( L + \hat{D}_{\{s_n-L\}} \left( \sqrt{t} \right) \right)
\]
\[
= L + \lim_{t \to 1^-} \hat{D}_{\{s_n-L\}} \left( \sqrt{t} \right)
\]
\[
= L,
\]
which means that the series \( \sum_{k=0}^{\infty} a_k \) converges to \( L \). The theorem is proved. \(\Box\)

**References**


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