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CARLSON TYPE INEQUALITIES AND EMBEDDINGS OF INTERPOLATION SPACES

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ABSTRACT. We discuss the close relation between Carlson type inequalities

$$\|a\|_X \leq K\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^{\theta}$$

and interpolation, and prove embedding results for real interpolation spaces, in particular into weighted L_p-spaces.

1. INTRODUCTION

In 1934, F. Carlson [3] proved that the inequality

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 < \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} n^2 a_n^2 \tag{1}$$

holds for all nonzero sequences $\{a_n\}_{n=1}^{\infty}$ of nonnegative numbers. He also noted that the corresponding integral inequality

$$\left(\int_0^{\infty} f(x)\,dx\right)^4 \leq \pi^2 \int_0^{\infty} f^2(x)\,dx \int_0^{\infty} x^2 f^2(x)\,dx \tag{2}$$

holds. Carlson's inequalities (1) and (2) have been generalized in various directions and applied in several areas (see, e.g., [1], [6] and [7], and the references given there). In general, by a *Carlson type inequality*, we shall mean an inequality of the form

$$\|a\|_X \leq K\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^{\theta}, \tag{3}$$

where X, A_0 and A_1 are normed spaces, $0 < \theta < 1$, and the constant K is independent of a. We assume, here and hereafter, that A_0 and A_1 are embedded in a Hausdorff topological vector space.

The inequality (3) is equivalent to X being of class $\mathcal{C}_J(\theta; A_0, A_1)$; that is, there is a constant K such that for all $t > 0$

$$\|a\|_X \leq K t^{-\theta} J(t, a; A_0, A_1). \tag{4}$$

Here, J is the Peetre J-functional, defined for all $a \in A_0 \cap A_1$ by

$$J(t, a; A_0, A_1) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \quad t > 0.$$

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To see this equivalence, note that (4) can be written as

$$\|a\|_X \leq K \max\{t^{-\theta}\|a\|_{A_0}, t^{1-\theta}\|a\|_{A_1}\}. \tag{5}$$

The maximum of the two numbers above is smallest when

$$t = \frac{\|a\|_{A_0}}{\|a\|_{A_1}},$$

in which case it equals

$$\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^{\theta}.$$

Thus (5) holds precisely when (3) does.

If X is complete, then X is of class $\mathcal{C}_J(\theta; A_0, A_1)$ if and only if we have the continuous embedding

$$(A_0, A_1)_{\theta,1} \subseteq X,$$

where $(A_0, A_1)_{\theta,r}$, $1 \leq r \leq \infty$, are the interpolation spaces arising from the real interpolation method (any of the equivalent J- and K-methods). For a proof of this, see [2]. In the remainder of this paper, we discuss other embeddings of interpolation spaces; in particular, we obtain known and new embedding results for interpolation spaces into weighted L_p-spaces, but it is also discussed how even more general embeddings can be proved. More precisely, we investigate the embedding $(A_0, A_1)_{\theta,r} \subseteq X$ for $r > 1$.

2. The results

We have the following embedding result.

Theorem 1. *Let (Ω, μ) be a measure space. Let the weights α, α_0 and α_1 be defined on Ω, and suppose that $\theta \in (0, 1)$ and $p, p_0, p_1 \in (0, \infty]$ are such that*

$$\frac{1}{q} := \frac{1}{p} - \frac{1-\theta}{p_0} - \frac{\theta}{p_1} \geq 0. \tag{6}$$

Suppose, moreover, that there is a constant C such that

$$\mu\left(\left\{\omega \in \Omega; 2^m \leq \frac{\alpha_0(\omega)}{\alpha_1(\omega)} < 2^{m+1}\right\}\right) \leq C, \ m \in \mathbb{Z}, \tag{7}$$

and that

$$\frac{\alpha}{\alpha_0^{1-\theta}\alpha_1^{\theta}} \in L_s(d\mu) \tag{8}$$

for some $s \in [q, \infty]$. Let $X = L_p(\alpha^p \, d\mu)$ and $A_i = L_{p_i}(\alpha_i^{p_i} \, d\mu)$, $i = 0, 1$. Then

$$(A_0, A_1)_{\theta,p} \subseteq X. \tag{9}$$

Remark 1. When $p = p_\theta$, defined as

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

the space $(A_0, A_1)_{\theta,p}$ is a weighted L_p-space. In general, however, this interpolation space can be complicated, and difficult to characterize (see, e.g., [8]).

Proof of Theorem 1. Take θ_i, $i = 0, 1$, such that $0 < \theta_0 < \theta < \theta_1 < 1$, and define

$$\eta = \frac{\theta - \theta_0}{\theta_1 - \theta_0}.$$

For $i = 0, 1$, let

$$\beta_i = \alpha \left(\frac{\alpha_0}{\alpha_1} \right)^{\theta - \theta_i}$$

and define r_i by

$$\frac{1}{r_i} = \frac{1}{q} + \frac{1 - \theta_i}{p_0} + \frac{\theta_i}{p_1}.$$

Let $X_i = L_{r_i}(\beta_i^{r_i}\, d\mu)$, $i = 0, 1$. By assumption,

$$\frac{1}{r_i} - \frac{1 - \theta_i}{p_0} - \frac{\theta_i}{p_1} = \frac{1}{q} \geq 0$$

and

$$\frac{\beta_i}{\alpha_0^{1-\theta_i} \alpha_1^{\theta_i}} = \frac{\alpha}{\alpha_0^{1-\theta} \alpha_1^{\theta}} \in L_s(d\mu),$$

and so it follows by [6, Theorem 2] that there are constants K_i such that

$$\|f\|_{X_i} \leq K_i \|f\|_{A_0}^{1-\theta_i} \|f\|_{A_1}^{\theta_i}.$$

In other words, $X_i \in \mathcal{C}_J(\theta_i; A_0, A_1)$. By (the proof of) the Reiteration Theorem [2, Theorem 3.5.3], it follows that

$$(A_0, A_1)_{\theta,p} \subseteq (X_0, X_1)_{\eta,p}.$$

Now,

$$\beta_0^{1-\eta} \beta_1^{\eta} = \alpha$$

and

$$\frac{1 - \eta}{r_0} + \frac{\eta}{r_1} = \frac{1}{p}.$$

So [2, Theorem 5.5.1] implies that $(X_0, X_1)_{\eta,p} = X$. This completes the proof. □

The scale of spaces $(A_0, A_1)_{\theta,r}$ is increasing with r. Thus (9) holds true if p is replaced by any $r \in [1, p)$. However, the following partial converse to Theorem 1 shows that we cannot, in general, go beyond p.

For each $t > 0$, we define the Peetre K-functional on $A_0 + A_1$ by

$$K(t, f; A_0, A_1) = \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1}; f = f_0 + f_1, f_i \in A_i, i = 0, 1\}.$$

Proposition 2. *Given $p, p_0, p_1 \in (0, \infty]$ and $\theta \in (0, 1)$ satisfying* (6), *for any $r \in (p, \infty]$, there are a measure space $(\Omega, d\mu)$ and weights $\alpha, \alpha_0, \alpha_1$ satisfying* (7) *and* (8) *such that*

$$(A_0, A_1)_{\theta,r} \not\subseteq X.$$

Proof. We consider $\Omega = (e, \infty)$ with the measure

$$d\mu(x) = \frac{dx}{x},$$

and define $\alpha(x) = 1$, $\alpha_0(x) = x^{\theta}$, $\alpha_1(x) = x^{-(1-\theta)}$. Then the weights satisfy the hypotheses of Theorem 1, with $C = \log 2$ and $s = \infty$.

Let

$$f(x) = \frac{1}{(\log x)^{1/p}}.$$

Then $f \notin X$. We want to show that $f \in (A_0, A_1)_{\theta,r}$. The norm on $(A_0, A_1)_{\theta,r}$ is given by

$$\|f\|_{\theta,r}^r = \int_0^\infty (t^{-\theta} K(t, f; A_0, A_1))^r \frac{dt}{t},$$

where K is the Peetre K-functional, with the usual convention when $r = \infty$. If $0 < t \le e$, define $f_0^{(t)} = 0$ and $f_1^{(t)} = f$. Then $\|f_0^{(t)}\|_{A_0} = 0$, while

$$\|f_1^{(t)}\|_{A_1} = \left(\int_e^\infty \frac{x^{-p_1(1-\theta)}}{(\log x)^{p_1/p}} \frac{dx}{x}\right)^{1/p_1} = B.$$

Thus

$$\begin{aligned} t^{-\theta} K(t, f; A_0, A_1) &\le t^{-\theta} \|f_0^{(t)}\|_{A_0} + t^{1-\theta} \|f_1^{(t)}\|_{A_1} \\ &= Bt^{1-\theta}. \end{aligned}$$

If $t > e$, let $f_0^{(t)} = f\chi_{(e,t)}$ and $f_1^{(t)} = f\chi_{[t,\infty)}$. Then $f_0^{(t)} + f_1^{(t)} = f$ for all t. We have

$$\begin{aligned} \int_\Omega (f_0^{(t)} \alpha_0)^{p_0} \, d\mu &= \int_e^t \frac{x^{p_0\theta}}{(\log x)^{p_0/p}} \frac{dx}{x} \\ &\le C_0^{p_0} \frac{t^{p_0\theta}}{(\log t)^{p_0/p}} \end{aligned}$$

and

$$\int_\Omega (f_1^{(t)} \alpha_1)^{p_1} \, d\mu \le C_1^{p_1} \frac{t^{-p_1(1-\theta)}}{(\log t)^{p_1/p}},$$

so that

$$t^{-\theta} \|f_0^{(t)}\|_{A_0} + t^{1-\theta} \|f_1^{(t)}\|_{A_1} \le \frac{C_0 + C_1}{(\log t)^{1/p}}.$$

It follows that

$$\|f\|_{\theta,r}^r \le B^r \int_0^e t^{(1-\theta)r} \frac{dt}{t} + (C_0 + C_1)^r \int_e^\infty \frac{dt}{t(\log t)^{r/p}}.$$

Since $r > p$, the last integral converges (and so does the first). It follows that $f \in (A_0, A_1)_{\theta,r}$. □

For completeness, we give the following multi-dimensional version of Theorem 1.

Theorem 3. *Let n be a positive integer, and let measure spaces $(\Omega^{(j)}, \mu^{(j)})$, $j = 1, \dots, n$, be given. Suppose that weights $\alpha^{(j)}$, $\alpha_0^{(j)}$ and $\alpha_1^{(j)}$ are defined on $\Omega^{(j)}$, and that the parameters $\theta \in (0,1)$ and $p, p_0, p_1 \in (0, \infty]$ satisfy*

$$\frac{1}{q} = \frac{1}{p} - \frac{1-\theta}{p_0} - \frac{\theta}{p_1} \ge 0.$$

Let $1 \le k \le n$, and suppose that there are constants C_j, $j = 1, \dots, k$, such that

$$\mu^{(j)}\left(\left\{\omega_j \in \Omega^{(j)}; 2^m \le \frac{\alpha_0^{(j)}(\omega_j)}{\alpha_1^{(j)}(\omega_j)} < 2^{m+1}\right\}\right) \le C_j, \ m \in \mathbb{Z},$$

and that

$$\frac{\alpha^{(j)}}{\left(\alpha_0^{(j)}\right)^{1-\theta} \left(\alpha_1^{(j)}\right)^{\theta}} \in L_{s_j}\left(d\mu^{(j)}\right), \tag{10}$$

where

$$s_j \in [q, \infty] \quad \textit{and} \quad \frac{1}{s_1} + \cdots + \frac{1}{s_k} \geq \frac{k-1}{q}.$$

Suppose also that

$$\frac{\alpha^{(j)}}{\left(\alpha_0^{(j)}\right)^{1-\theta}\left(\alpha_1^{(j)}\right)^{\theta}} \in L_q\left(d\mu^{(j)}\right), \; j = k+1, \ldots, n. \tag{11}$$

If

$$A_i = L_{p_i}\left(\prod_{j=1}^{n} \alpha_i^{(j)}\, d\mu^{(j)}\right), \; i = 0, 1,$$

and

$$X = L_p\left(\prod_{j=1}^{n} \alpha^{(j)}\, d\mu^{(j)}\right),$$

then (9) *holds.*

Sketch of Proof. As in the proof of Theorem 1, we define the spaces X_i, $i = 0, 1$, and use [6, Theorem 5] to show that $X_i \in \mathcal{C}_J(\theta_i; A_0, A_1)$. The desired embedding follows from the Reiteration Theorem and the fact that $X = (X_0, X_1)_{\eta,p}$. □

3. Concluding remarks

Remark 2. The proofs of Theorems 1 and 3 rely on the fact that we have a Carlson type inequality for the spaces involved, and that we can rewrite our space X as an interpolation space and use the Reiteration Theorem. We can state the conclusion in more general terms; namely, as soon as we can prove a Carlson type inequality for the auxiliary spaces X_0 and X_1, or equivalently, can show that X_i is of class $\mathcal{C}_J(\theta_i; A_0, A_1)$, $i = 0, 1$, we have the embedding

$$(A_0, A_1)_{\theta,p} \subseteq (X_0, X_1)_{\eta,p}.$$

The spaces X and A_i, $i = 0, 1$, need not necessarily be weighted L_p-spaces, as long as we have a Carlson type inequality and a corresponding reiteration theorem. Much more general embeddings could be achieved using the methods of this paper.

Remark 3. Carlson type inequalities have previously been used in the interpolation context, for instance by N. Ya. Kruglyak, L. Maligranda and L.-E. Persson [5], who applied the Brundyĭ–Kruglyak construction, which was also used in the solution to the classical K-divisibility problem. Their version of the inequality involved "optimal blocks", and was used to characterize the Peetre interpolation functor $\langle\,\rangle_{\varphi}$. See also [4], where the ± method was introduced.

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