

A STRONG LEBESGUE POINT PROPERTY FOR SOBOLEV FUNCTIONS

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ABSTRACT. We show that first-order Sobolev functions fulfill a Wiener integral type Lebesgue point property outside a set of Sobolev capacity zero. Our condition is stronger than the standard Lebesgue point property, but the exceptional set is slightly larger.

1. INTRODUCTION

The classical Lebesgue differentiation theorem in \mathbf{R}^n asserts that almost every point x_0 of a locally integrable function u is a Lebesgue point, i.e., it satisfies

$$(1.1) \quad \lim_{r \rightarrow 0} \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |u - u(x_0)| dx = 0.$$

For Sobolev functions the condition (1.1) holds with a higher integrability exponent outside a smaller exceptional set. The following result is by now well known and widely used: Let $1 < p \leq n$ and let u be a locally L^p -integrable function whose distributional partial derivatives are locally L^p -integrable. Then

$$(1.2) \quad \lim_{r \rightarrow 0} \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |u - u(x_0)|^s dx = 0$$

for all $x_0 \in \mathbf{R}^n$ outside a set of p -capacity zero whenever $s > 0$ is a finite exponent not greater than the Sobolev conjugate of p . This result appeared first in [5] and [14]. Nowadays the proof can be found in standard textbooks on Sobolev functions; see e.g. [4] or [15].

As the main result of this paper we improve the condition (1.2) by showing that the integral averages in (1.2) fulfill the following Wiener integral type condition:

Theorem 1.1. *Let $u \in W^{1,p}(\mathbf{R}^n)$ be p -quasicontinuous, and let $1 < q < p$. If $1 < p < n$, then for q -q.e. $x_0 \in \mathbf{R}^n$ and for all $0 < s \leq \frac{np}{n-p}$,*

$$\int_0^1 \left(\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |u - u(x_0)|^s dx \right)^{\frac{p}{s(p-1)}} \frac{dr}{r} < \infty.$$

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If $p = n$, then for q -q.e. $x_0 \in \mathbf{R}^n$ and for all $s > 0$,

$$\int_0^1 \left(\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |u - u(x_0)|^s dx \right)^{\frac{p}{s(p-1)}} \frac{dr}{r} < \infty.$$

Notice that Sobolev functions are locally Hölder continuous if $p > n$ (see [4], p. 143). In this case our theorem trivially holds for all $x_0 \in \mathbf{R}^n$. This is why we assume above that $1 < p \leq n$. We explain in Remark 3.1 why the convergence of the Wiener type integrals in Theorem 1.1 is a stronger condition than the condition (1.2). Notice that the property (1.2) holds outside a set of Sobolev p -capacity zero. Hence the exceptional set for Theorem 1.1 is larger than the exceptional set for (1.2). On the other hand, by choosing a sequence $1 < q_i < p$ with $\lim_{i \rightarrow \infty} q_i = p$ we easily obtain from well-known results on Hausdorff measure and capacity that the Hausdorff dimension of the exceptional set in Theorem 1.1 is at most $n - p$ (see e.g. [4] or [15]).

This paper is organized as follows. In section 2 we briefly recall the necessary concepts and known results that are necessary for understanding the proof of our main result. In section 3 we prove Theorem 1.1 and make some supplementary comments. The new feature in our proof is that we use fine topological arguments in order to be able to use a version of the Sobolev-Poincaré inequality that fits into our considerations.

2. BACKGROUND

We denote by $|E|$ the n -dimensional Lebesgue measure of the set $E \subset \mathbf{R}^n$, $n \geq 2$. In integrals dx (resp. dr) refers to integration with respect to n -dimensional (resp. 1-dimensional) Lebesgue measure.

Let $\Omega \subset \mathbf{R}^n$ be an open set, and let $p \geq 1$. The Sobolev space $W^{1,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ for which $|\nabla u| \in L^p(\Omega)$. Here and elsewhere, ∇u is the distributional gradient of u .

The first auxiliary lemma is an easy consequence of the Sobolev-Poincaré inequality. Since its proof is not easily found in the literature, we give a proof here. Our proof is the same as in [9], Lemma 2.1, where the authors however study Sobolev functions on metric spaces.

Lemma 2.1. *Let $u \in W^{1,p}(B)$, where $1 < p < n$ and $B = B(x_0, r)$. Suppose that*

$$|\{x \in B : |u(x)| > 0\}| \leq \kappa |B|$$

for some $0 < \kappa < 1$, and let $1 \leq s \leq \frac{np}{n-p}$. Then there is a constant c , depending only on n , p , s and κ , such that

$$\left(\frac{1}{|B|} \int_B |u|^s dx \right)^{\frac{1}{s}} \leq cr \left(\frac{1}{|B|} \int_B |\nabla u|^p dx \right)^{\frac{1}{p}}.$$

Proof. By the Minkowski inequality,

$$\left(\int_B |u|^s dx \right)^{\frac{1}{s}} \leq \left(\int_B |u - u_B|^s dx \right)^{\frac{1}{s}} + |u_B| |B|^{\frac{1}{s}},$$

where u_B denotes the integral average $u_B = \frac{1}{|B|} \int_B u \, dx$. By the Sobolev-Poincaré inequality ([15], Corollary 1.64),

$$\left(\int_B |u - u_B|^s \, dx \right)^{\frac{1}{s}} \leq c |B|^{\frac{1}{s} - \frac{1}{p}} r \left(\int_B |\nabla u|^p \, dx \right)^{\frac{1}{p}},$$

where c depends only on n, p and s . By the Hölder inequality,

$$\begin{aligned} |u_B| |B|^{\frac{1}{s}} &\leq |B|^{\frac{1}{s}-1} \int_{\{x \in B \mid |u(x)| > 0\}} |u| \, dx \\ &\leq \left(\int_B |u|^s \, dx \right)^{\frac{1}{s}} \left(\frac{|\{x \in B \mid |u(x)| > 0\}|}{|B|} \right)^{1-\frac{1}{s}} \leq \left(\int_B |u|^s \, dx \right)^{\frac{1}{s}} \kappa^{1-\frac{1}{s}}. \end{aligned}$$

The assertion follows by combining the inequalities. \square

2.1. Capacity. We need two different capacities. Let $p > 1$, and let $E \subset \mathbf{R}^n$. The Sobolev p -capacity of E is defined by

$$C_p(E) = \inf_u \int_{\mathbf{R}^n} (|u|^p + |\nabla u|^p) \, dx,$$

where the infimum is taken over all $u \in W^{1,p}(\mathbf{R}^n)$ such that $u \geq 1$ on an open neighbourhood of E . The Sobolev p -capacity is studied in many books (see e.g. [4], [8], [15]). The Bessel $(1, p)$ -capacity of E is defined as

$$B_{1,p}(E) = \inf \{ \|f\|_p^p : f \in L_+^p(\mathbf{R}^n), g_1 * f \geq 1 \text{ in } E \}.$$

We refer to [1] and [16] for the properties of the Bessel kernel g_1 and the Bessel capacity $B_{1,p}$. We only need the fact that $g_1(r)$ behaves like r^{1-n} as $r \rightarrow 0^+$ ([1], p. 11). It is known that the capacities C_p and $B_{1,p}$ are equivalent; see e.g. [10], Remark 1.13, i.e., there is a constant $c > 0$, depending only on n and p , such that

$$c^{-1} C_p(E) \leq B_{1,p}(E) \leq c C_p(E)$$

for all $E \subset \mathbf{R}^n$. This implies that all concepts we relate to the capacity are the same for C_p and $B_{1,p}$.

We next introduce the notions of p -quasicontinuity and p -fine continuity. These closely related concepts are the key tools for understanding the precise behaviour of Sobolev functions. A function $f : \Omega \rightarrow [-\infty, +\infty]$ is p -quasicontinuous in Ω if for each $\varepsilon > 0$ there is an open set G such that $C_p(G) < \varepsilon$ and $f|_{\Omega \setminus G}$ is finite and continuous. For $p \leq n$, there are Sobolev functions that do not have a continuous representative. The lack of continuity can be partially compensated by the fact that any function $u \in W^{1,p}(\mathbf{R}^n)$ has a p -quasicontinuous representative ([15], Theorem 2.20). It is even possible to show that any function $u \in W^{1,p}(\mathbf{R}^n)$ has a representative that is Hölder continuous outside a set of arbitrary small q -capacity, $1 < q < p$. This is the content of the second auxiliary lemma.

Lemma 2.2. *Let $1 < p \leq n$, and let $u \in W^{1,p}(\mathbf{R}^n)$ be p -quasicontinuous. Then for each $\varepsilon > 0$ and for each $1 < q < p$ there are $E \subset \mathbf{R}^n$ and constants $\alpha > 0$, $C_0 > 0$ such that $C_q(E) < \varepsilon$ and*

$$|u(x) - u(y)| \leq C_0 |x - y|^\alpha$$

for all $x, y \in \mathbf{R}^n \setminus E$.

Lemma 2.2 was first proved in [12], Theorem 7. A different, more general proof is given in [6].

In order to define the p -fine topology, we first recall the concept of p -thinness. A set $E \subset \mathbf{R}^n$ is called p -thin at $x_0 \in \mathbf{R}^n$ if

$$(2.1) \quad \int_0^1 \left(\frac{C_p(E \cap B(x_0, r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.$$

The integral in (2.1) is called the *Wiener integral*. The Wiener integral is the key tool for characterizing the regular boundary points of the Dirichlet problem for the p -Laplacian; see [8] and [11]. A point $x \in U \subset \mathbf{R}^n$ is called a p -fine interior point of U if $\mathbf{R}^n \setminus U$ is p -thin at x . This definition gives rise to a topology called the p -fine topology. We denote by \overline{E}^p the p -fine closure of E . The role of the p -fine topology for nonlinear elliptic PDEs is carefully studied in [15]. The next lemma is proved in [15], Theorem 2.140 (the original proof goes back to [3] and [7]; see [2], Proposition 2).

Lemma 2.3. *Let $p > 1$, and let $E \subset \mathbf{R}^n$. Then*

$$C_p(E) = C_p(\overline{E}^p).$$

Our ideas essentially rely on the following result, due to Meyers. If $1 < p < n$ and $u \in W^{1,p}(\mathbf{R}^n)$, then by [13], Theorem 2.1,

$$\int_0^\infty \left(g_1^p(4r)r^{n(p-1)} \int_{B(x_0,r)} |\nabla u|^p dx \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty \quad \text{for } p\text{-q.e. } x_0 \text{ in } \mathbf{R}^n.$$

Here and elsewhere we use the phrase p -q.e. to express that some pointwise property holds outside a set of Sobolev p -capacity zero. Since the Bessel kernel $g_1(r)$ is comparable to r^{1-n} as $r \rightarrow 0^+$, we obtain the following lemma:

Lemma 2.4. *Let $1 < p < n$, and let $u \in W^{1,p}(\mathbf{R}^n)$. Then*

$$\int_0^1 \left(r^{p-n} \int_{B(x_0,r)} |\nabla u|^p dx \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty \quad \text{for } p\text{-q.e. } x_0 \text{ in } \mathbf{R}^n.$$

Lemma 2.4 can be used to show that any Sobolev function $u \in W^{1,p}(\mathbf{R}^n)$ has a representative that is p -finely continuous outside a set E with $C_p(E) = 0$ ([16], pp. 124–125). This is much more difficult to prove than the existence of a p -quasicontinuous representative.

3. THE PROOF OF THEOREM 1.1

For the reader’s convenience we first briefly discuss the key idea behind the improvements of the Lebesgue theorem for Sobolev functions; see [4], pp. 160–162, or [15], pp. 90–92.

Let $x_0 \in \mathbf{R}^n$ be a Lebesgue point of a function $u \in W^{1,p}(\mathbf{R}^n)$. By the Minkowski inequality,

$$(3.1) \quad \begin{aligned} & \left(\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |u - u(x_0)|^p dx \right)^{\frac{1}{p}} \\ & \leq \left(\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |u - u_{B(x_0, r)}|^p dx \right)^{\frac{1}{p}} + |u(x_0) - u_{B(x_0, r)}|, \end{aligned}$$

and by the Sobolev-Poincaré inequality,

$$\left(\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |u - u_{B(x_0, r)}|^p dx \right)^{\frac{1}{p}} \leq cr^{p-n} \int_{B(x_0, r)} |\nabla u|^p dx.$$

By [15], Corollary 1.19, $\lim_{r \rightarrow 0} r^{p-n} \int_{B(x_0, r)} |\nabla u|^p dx = 0$ for p -q.e. x_0 . Hence (1.2) will follow once it is shown that p -q.e. x_0 in \mathbf{R}^n is a Lebesgue point.

Why doesn't the estimate (3.1) yield the proof of Theorem 1.1? Lemma 2.4 guarantees that the first term on the right-hand side of (3.1) does not cause any harm, but there is no easy way to deal with the second term. To overcome this difficulty, instead of writing the estimate (3.1) we divide the set $B(x_0, r)$ into two parts. Because of the quasi Hölder continuity of Lemma 2.2 the "good" part does not cause any harm, and the fine topological arguments guarantee that we are able to use Lemma 2.1 to deal with the "bad" part.

Proof of Theorem 1.1. Notice that $u(x_0)$ is finite p -q.e. in \mathbf{R}^n , since u is p -quasicontinuous. Assume first that $1 < p < n$, $0 < s \leq \frac{np}{n-p}$, and put $v(x) = |u(x) - u(x_0)|$. By Lemma 2.2, we are able to choose $E_i \subset \mathbf{R}^n$ such that $C_q(E_i) < \frac{1}{i}$ and u is α -Hölder continuous in $\mathbf{R}^n \setminus E_i$. Set

$$E = \left\{ x \in \mathbf{R}^n : \int_0^1 \left(r^{p-n} \int_{B(x_0, r)} |\nabla u|^p dx \right)^{\frac{1}{p-1}} \frac{dr}{r} = \infty \right\} \cup \bigcap_{i=1}^{\infty} \overline{E}_i^q,$$

where \overline{E}_i^q denotes the q -fine closure of E_i . Since the sets of p -capacity zero are of q -capacity zero ([15], Lemma 2.14), we infer that $C_q(E) = 0$ by using Lemmas 2.3 and 2.4 together with the fact that the capacity is a monotone countably subadditive set function. Let $x_0 \notin E$ and choose an index i such that $x_0 \notin \overline{E}_i^q$. Then there are constants $C_0 > 0$ and $\alpha > 0$ such that

$$(3.2) \quad |u(x) - u(x_0)| \leq C_0|x - x_0|^\alpha$$

for all $x \in \mathbf{R}^n \setminus \overline{E}_i^q$. Let $C_0 > 0$ and $\alpha > 0$ be the constants satisfying (3.2), and define $U = \mathbf{R}^n \setminus \overline{E}_i^q$, $B_r = B(x_0, r)$. We divide our integral into two parts by writing

$$\left(\frac{1}{|B_r|} \int_{B_r} v^s dx \right)^{\frac{p}{s}} \leq 2^{\frac{p}{s}} \left(\frac{1}{|B_r|} \int_{B_r \cap U} v^s dx \right)^{\frac{p}{s}} + 2^{\frac{p}{s}} \left(\frac{1}{|B_r|} \int_{B_r \setminus U} v^s dx \right)^{\frac{p}{s}}.$$

The first term on the right-hand side is trivially estimated as

$$\left(\frac{1}{|B_r|} \int_{B_r \cap U} v^s dx \right)^{\frac{p}{s}} \leq C_0^p r^{\alpha p}.$$

To estimate the second term, observe that

$$\begin{aligned} \left(\frac{1}{|B_r|} \int_{B_r \setminus U} v^s dx \right)^{\frac{2}{s}} &\leq \left(\frac{1}{|B_r|} \int_{B_r \setminus U} [(v - C_0 r^\alpha)^+ + C_0 r^\alpha]^s dx \right)^{\frac{2}{s}} \\ &\leq 2^p \left(\frac{1}{|B_r|} \int_{B_r \setminus U} (v - C_0 r^\alpha)^s + (C_0 r^\alpha)^s dx \right)^{\frac{2}{s}} \\ &\leq c \left(\frac{1}{|B_r|} \int_{B_r} ((v - C_0 r^\alpha)^+)^s dx \right)^{\frac{2}{s}} + c C_0^p r^{\alpha p} \end{aligned}$$

for some constant $c > 0$ depending only on p and s . Here we use the notation $f^+ = \max(f, 0)$. Since u is continuous in $\mathbf{R}^n \setminus \overline{E}_i^q$ and \overline{E}_i^q is q -thin at x_0 , we infer from [15], Corollary 2.51, that

$$\lim_{r \rightarrow 0} \frac{|B_r \setminus U|}{|B_r|} = 0.$$

Hence for small radii r we may apply Lemma 2.1 with $\kappa = \frac{1}{2}$, $p \leq s \leq \frac{np}{n-p}$, and conclude that

$$(3.3) \quad \left(\frac{1}{|B_r|} \int_{B_r} ((v - C_0 r^\alpha)^+)^s dx \right)^{\frac{2}{s}} \leq cr^{p-n} \int_{B_r} |\nabla v|^p dx$$

for all r small enough. Here the constant c depends only on n, p and s . For $0 < s < p$, we have

$$\left(\frac{1}{|B_r|} \int_{B_r} ((v - C_0 r^\alpha)^+)^s dx \right)^{\frac{1}{s}} \leq \left(\frac{1}{|B_r|} \int_{B_r} ((v - C_0 r^\alpha)^+)^p dx \right)^{\frac{1}{p}}$$

by the Hölder inequality. Hence the estimate (3.3) holds for all $0 < s \leq \frac{np}{n-p}$. It follows that

$$\left(\frac{1}{|B_r|} \int_{B_r} v^s dx \right)^{\frac{2}{s}} \leq c C_0^p r^{\alpha p} + cr^{p-n} \int_{B_r} |\nabla v|^p dx$$

for some constant $c > 0$ depending only on n, p and s . Since $|\nabla v| \leq |\nabla u|$ a.e. in \mathbf{R}^n , we easily infer the assertion of the theorem in the case $1 < p < n$.

The borderline case $p = n$ can be treated similarly by using some easy tricks. We only point out the differences in the arguments. We consider an arbitrary exponent t with $q < t < n$ and denote

$$E = \left\{ x \in \mathbf{R}^n : \int_0^1 \left(r^{t-n} \int_{B(x_0,r)} |\nabla u|^t dx \right)^{\frac{1}{p-1}} \frac{dr}{r} = \infty \right\} \cup \bigcap_{i=1}^\infty \overline{E}_i^q.$$

For $s = \frac{nt}{n-t}$ we proceed as in the case $1 < p < n$. Now we apply Lemma 2.1 for $p = t$, and the inequality (3.3) takes the form

$$(3.4) \quad \begin{aligned} \left(\frac{1}{|B_r|} \int_{B_r} ((v - C_0 r^\alpha)^+)^s dx \right)^{\frac{2}{s}} &\leq \left(\frac{1}{|B_r|} \int_{B_r} ((v - C_0 r^\alpha)^+)^s dx \right)^{\frac{1}{s}} \\ &\leq cr^{t-n} \int_{B_r} |\nabla v|^t dx. \end{aligned}$$

Here the first inequality holds just because we are free to assume that the integral average on the left-hand side of (3.4) is less than 1. The case $0 < s < \frac{nt}{n-t}$ is treated by using the Hölder inequality. \square

Remark 3.1. Let $u \in W^{1,p}(\mathbf{R}^n)$, $1 < p \leq n$, and let s be as in Theorem 1.1. Then the integral condition

$$\int_0^1 \left(\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |u - u(x_0)|^s dx \right)^{\frac{p}{s(p-1)}} \frac{dr}{r} < \infty$$

implies the standard Lebesgue point property

$$\lim_{r \rightarrow 0} \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |u - u(x_0)|^s dx = 0.$$

This is easily seen by using the fact that

$$\int_0^1 \left(\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |u - u(x_0)|^s dx \right)^{\frac{p}{s(p-1)}} \frac{dr}{r} < \infty$$

if and only if

$$\sum_{i=1}^{\infty} \left(2^{ni} \int_{B(x_0, 2^{-i})} |u - u(x_0)|^s dx \right)^{\frac{p}{s(p-1)}} < \infty.$$

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