ZEROS OF FUNCTIONS WITH FINITE DIRICHLET INTEGRAL

STEFAN RICHTER, WILLIAM T. ROSS, AND CARL SUNDBERG

Abstract. In this paper, we refine a result of Nagel, Rudin, and Shapiro (1982) concerning the zeros of holomorphic functions on the unit disk with finite Dirichlet integral.

This is a remark about the zeros of functions $f = \sum_{n \geq 0} a_n z^n$ holomorphic on the unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ that have finite Dirichlet integral

$$D(f) := \frac{1}{\pi} \int_U |f'|^2 dA = \sum_{n=0}^{\infty} |a_n|^2,$$

where $dA$ is Lebesgue measure in the plane. Clearly such functions belong to the classical Hardy space $H^2$, and so the zeros $(z_n)_{n \geq 1} \subset U$ of $f$ (repeated according to multiplicity) satisfy the Blaschke condition $\sum_{n \geq 0} (1 - |z_n|^2) < \infty$ [4, p. 18]. However, not every Blaschke sequence are the zeros of a holomorphic $f$ with $D(f) < \infty$ [2].

In 1962, Shapiro and Shields [6] improved a result of Carleson [3] and showed that if

$$\sum_{n=1}^{\infty} \frac{1}{1 - \log(1 - |z_n|)} < \infty,$$

then there is a nontrivial holomorphic $f$ on $U$ with $D(f) < \infty$ such that $f(z_n) = 0$ for all $n$.

This condition does not completely characterize the zero sets of analytic functions with finite Dirichlet integral. For example, if $(z_n)_{n \geq 0} \subset (0, 1)$ is a Blaschke sequence for which (11) fails, then $f = (1 - z)^2 B$ has finite Dirichlet integral, where $B$ is the Blaschke product with zeros $(z_n)_{n \geq 0}$. Nevertheless, in the converse direction, Nagel, Rudin, and Shapiro [5] proved that if $(r_n)_{n \geq 0} \subset (0, 1)$ is such that

$$\sum_{n=0}^{\infty} \frac{1}{1 - \log(1 - r_n)} = \infty,$$

then there is a sequence of angles $(\theta_n)_{n \geq 0}$ such that the sequence $(r_n e^{i\theta_n})_{n \geq 0}$ is not the zeros of any nontrivial holomorphic function $f$ on $U$ with $D(f) < \infty$. They do this by first noting that when $D(f) < \infty$, the limit

$$\lim_{z \to e^{i\theta}, z \in \Omega_{1, \theta}} f(z)$$

Received by the editors October 22, 2002 and, in revised form, May 6, 2003.

2000 Mathematics Subject Classification. Primary 30C15; Secondary 30C85.

©2004 American Mathematical Society
exists for almost every $e^{i\theta}$, where $\Omega_{e^{i\theta}}$ is the exponential contact region

$$\Omega_{e^{i\theta}} := \{ re^{i\phi} : 1 - r^2 > e^{-|\phi - \theta|} \}.$$

Beginning at $z = 1$, lay down arcs $I_n \subset \partial U$ of length

$$\frac{1}{-\log(1 - r_n)}$$

end-to-end (repeatedly traversing the unit circle). Since $\sum_{n \geq 1} |I_n| = \infty$, by hypothesis, each $e^{i\theta} \in \partial U$ will be contained in infinitely many of the intervals $(I_n)_{n \geq 0}$. Let $e^{i\theta_n}$ be the center of the interval $I_n$, and note that simple geometry shows that for every $e^{i\theta}$, the exponential contact region $\Omega_{e^{i\theta}}$ contains infinitely many of the points $r_n e^{i\theta_n}$. Thus if $f$ has finite Dirichlet integral and $f(r_n e^{i\theta_n}) = 0$ for all $n$, the above limit result says that the boundary function for $f$ will vanish almost everywhere on $\partial U$, forcing $f$ to be identically zero. This argument actually shows that the sequence $(r_n e^{i\theta_n})_{n \geq 0}$ cannot be the zeros of a nontrivial harmonic function $f$ on $U$ with finite Dirichlet integral (where $|f'|^2$ is replaced by $|\nabla f|^2$ in the definition of the Dirichlet integral).

In this note, we refine this result and show that for analytic functions the angles $\theta_n$ can be chosen so that the zeros $(r_n e^{i\theta_n})_{n \geq 0}$ need not accumulate at every point of the circle, but instead accumulate at a single point.

**Theorem 2.** Suppose $(r_n)_{n \geq 0} \subset (0, 1)$ with $r_n \to 1$ and

$$\sum_{n = 0}^{\infty} \frac{1}{-\log(1 - r_n)} = \infty.$$

Then there are angles $(\theta_n)_{n \geq 0}$ such that $\text{clos}(r_n e^{i\theta_n})_{n \geq 0} \cap \partial U = \{1\}$ and such that if $f$ is holomorphic on $U$ with $D(f) < \infty$ and $f(r_n e^{i\theta_n}) = 0$ for all $n$, then $f$ is identically the zero function.

Our proof is based on the following lemma. In order to make our construction easier, we will work in the upper half plane.

**Lemma 3.** Let $J \subset \mathbb{R}$ be a finite open interval with center $x_0$ and $0 < y_0 < |J|$. Set

$$S := \{ x + iy : x \in J, \ 0 < y < |J| \}$$

and $\lambda_0 := x_0 + iy_0$. Suppose $f$ is holomorphic on $S$ with

$$\int_S |f'|^2 dA < \infty$$

and $f(\lambda_0) = 0$. If

$$E = \{ x \in J : |f(x)| \geq 1 \},$$

and $|E| \geq \frac{1}{2} |J|$, then

$$\int_S |f'|^2 dA \geq \frac{c}{\log(|J|/y_0)},$$

where $c$ is a universal constant.

**Proof.** Elementary considerations show that $\omega^S_\lambda(E)$, the harmonic measure of $E$ with respect to $S$ at $\lambda_0$, is bounded below by a universal constant times $y_0/|J|$. Indeed,

$$\omega^S_\lambda(E) \geq \omega^S_\lambda(F),$$
where $F$ is the union of two intervals in the real line of length $\frac{1}{4}|J|$ located at the lower corners of $S$. Let $\psi : S \to U$ be the conformal map that takes the centroid of $S$ to the origin and the line segment containing $\lambda_0$ and $x_0$ to the positive real axis. Thus $\psi(x_0) = 1$ and $\psi(\lambda_0) = r$ with $1 - r \approx y_0/|J|$. Then

$$\omega^{S}_{\lambda_0}(F) = |\psi(F)| \approx \omega^{U}_{r}(\psi(F)) = \int_{\psi(F)} \frac{1 - r^2}{|e^t - r|^2}dt.$$  

But since $\psi(F)$ is a fixed distance from the point $z = 1$, the denominator in the above integral does not matter. Thus, since the measure of $\psi(F)$ is fixed, the above integral is comparable to $1 \cdot r \cdot j = y_0$. Then

$$\omega^{F}_{U} = \int_{\psi(F)} \frac{1 - r^2}{|e^t - r|^2}dt.$$  

Let $\varphi : U \to S$ be the conformal map with $\varphi(0) = \lambda_0$, and let $g := f \circ \varphi$. Then $g(0) = 0$, $|g| \geq 1$ on $\varphi^{-1}(E)$, and $|\varphi^{-1}(E)| = \omega^{S}_{\lambda_0}(E) \geq cy_0/|J|$. This means that $g$ is a “test function” for the logarithmic capacity of $\varphi^{-1}(E)$ [11, Theorem 2], and so

$$\int_{U} |g'|^2 dA \geq c \text{cap}(\varphi^{-1}(E)) \geq \frac{c}{\log(|J|/y_0)}.$$  

Here we are using the well-known fact that if $W \subset \partial U$ with $a = |W|$, then

$$\text{cap}(W) \geq \frac{c}{\log(1/a)}.$$  

Finally, note that

$$\int_{U} |g'|^2 dA = \int_{S} |f'|^2 dA.$$  

$\square$

We are now ready to prove our main theorem. To make the construction easier, we work in the upper half plane and replace the sequence $(r_n)_{n \geq 0}$ with a sequence $(y_n)_{n \geq 0} \subset (0, 1)$ with $y_n \to 0$ and such that

$$\sum_{n=0}^{\infty} \frac{1}{\log(1/y_n)} = \infty.$$  

We will construct a sequence $(x_n + iy_n)_{n \geq 0}$ in the upper half plane whose closure intersects the real axis only at $x = 0$ and such that the only holomorphic function $f$ in the upper half plane with finite Dirichlet integral for which $f(x_n + iy_n) = 0$ for all $n$ is the zero function.

Assuming that $y_n \searrow 0$, we can find

$$1 \leq n_1 < m_1 < n_2 < m_2 < \cdots$$  

such that, whenever $n \geq n_k$,  

$$y_n \log \frac{1}{y_n} < \frac{1}{k^2}e^{-2k^2}$$  

and

$$ke^{2k^2} < \sum_{n=n_k}^{m_k} \frac{1}{\log(1/y_n)} = ke^{2k^2} + 1.$$  

For each $k$, lay out intervals

$$J_{n_k}, J_{n_k+1}, \cdots, J_{m_k}.$$
on the real axis end-to-end starting at \( x = 0 \) and such that
\[
|J_n| = \frac{1}{k^2 e^{2k^2} \log(1/y_n)}, \quad n_k \leq n \leq m_k.
\]
Then
\[
\log \frac{|J_n|}{y_n} = \log \frac{1}{k^2 e^{2k^2} y_n \log(1/y_n)} = \log(1/y_n) - k^2 - 2k^2 - \log \log(1/y_n) < \log(1/y_n).
\]

Let \( x_n \) be the center of \( J_n \), and set \( \lambda_n := x_n + iy_n \) and
\[
S_n := \{ x + iy : x \in J_n, 0 < y < |J_n| \}.
\]
Suppose that \( f \) is holomorphic on the upper half plane with finite Dirichlet integral and such that \( f(\lambda_n) = 0 \) for all \( n_k \leq n \leq m_k \). Set
\[
A_k := \{ n : n_k \leq n \leq m_k \text{ and } |f| \geq e^{-k^2} \text{ on a set } E_n \subset J_n \text{ with } |E_n| \geq \frac{1}{2}|J_n| \},
\]
\[
B_k := \{ n : n_k \leq n \leq m_k, n \notin A_k \}.
\]
Apply Lemma 3 to see that if \( n \in A_k \), then
\[
\int_{S_n} |f'|^2 dA \geq ce^{-2k^2} \frac{1}{\log(|J_n|/y_n)} \geq ce^{-2k^2} \frac{1}{\log(1/y_n)}
\]
and if \( n \in B_k \), then
\[
\int_{J_n} \log \frac{1}{|f|} dx \geq \frac{1}{2} |J_n| k^2 = \frac{1}{2} e^{-2k^2} \frac{k^2}{\log(1/y_n)}.
\]
We conclude that
\[
\sum_{n \in A_k} \int_{S_n} |f'|^2 dA + \sum_{n \in B_k} \int_{J_n} \log \frac{1}{|f|} dx \geq ce^{-2k^2} \sum_{n = n_k}^{m_k} \frac{1}{\log(1/y_n)} \geq ck.
\]
Thus by the log-integrability of \( f \) on the boundary [4, p. 17], \( f \) must be the zero function.

It follows that the set
\[
\bigcup_{k=1}^{\infty} (\lambda_n)_{n_k \leq n \leq m_k}
\]
cannot be the zeros of a holomorphic function with finite Dirichlet integral. Choose the remaining points (from the unused \( y_n \)'s) on the imaginary axis to obtain a sequence \( (\lambda_n)_{n \geq 0} \) that is not the zero set of a function with finite Dirichlet integral. Finally, since
\[
\sum_{n = n_k}^{m_k} |J_n| = \frac{1}{k^2 e^{2k^2}} \sum_{n = n_k}^{m_k} \frac{1}{\log(1/y_n)} \leq \frac{ke^{2k^2} + 1}{k^2 e^{2k^2}} \to 0, \quad k \to \infty,
\]
it follows that the closure of the sequence \( (\lambda_n)_{n \geq 0} \) intersects the real axis only at \( x = 0 \).
References


Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996
E-mail address: richter@math.utk.edu

Department of Mathematics and Computer Science, University of Richmond, Richmond, Virginia 23173
E-mail address: wross@richmond.edu

Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996
E-mail address: sundberg@math.utk.edu