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ZEROS OF FUNCTIONS WITH FINITE DIRICHLET INTEGRAL

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Abstract. In this paper, we refine a result of Nagel, Rudin, and Shapiro (1982) concerning the zeros of holomorphic functions on the unit disk with finite Dirichlet integral.

This is a remark about the zeros of functions $f = \sum_{n\geqslant 0} a_n z^n$ holomorphic on the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ that have finite *Dirichlet integral*

$$D(f) := \frac{1}{\pi}\int_U |f'|^2 dA = \sum_{n=0}^{\infty} n|a_n|^2,$$

where dA is Lebesgue measure in the plane. Clearly such functions belong to the classical Hardy space H^2, and so the zeros $(z_n)_{n\geqslant 1} \subset U$ of f (repeated according to multiplicity) satisfy the Blaschke condition $\sum_{n\geqslant 0}(1 - |z_n|^2) < \infty$ [4, p. 18]. However, not every Blaschke sequence are the zeros of a holomorphic f with $D(f) < \infty$ [2].

In 1962, Shapiro and Shields [6] improved a result of Carleson [3] and showed that if

$$\sum_{n=1}^{\infty} \frac{1}{-\log(1-|z_n|)} < \infty, \tag{1}$$

then there is a nontrivial holomorphic f on U with $D(f) < \infty$ such that $f(z_n) = 0$ for all n.

This condition does not completely characterize the zero sets of analytic functions with finite Dirichlet integral. For example, if $(z_n)_{n\geqslant 0} \subset (0, 1)$ is a Blaschke sequence for which (1) fails, then $f = (1 - z)^2 B$ has finite Dirichlet integral, where B is the Blaschke product with zeros $(z_n)_{n\geqslant 0}$. Nevertheless, in the converse direction, Nagel, Rudin, and Shapiro [5] proved that if $(r_n)_{n\geqslant 0} \subset (0, 1)$ is such that

$$\sum_{n=0}^{\infty} \frac{1}{-\log(1-r_n)} = \infty,$$

then there is a sequence of angles $(\theta_n)_{n\geqslant 0}$ such that the sequence $(r_n e^{i\theta_n})_{n\geqslant 0}$ is not the zeros of any nontrivial holomorphic function f on U with $D(f) < \infty$. They do this by first noting that when $D(f) < \infty$, the limit

$$\lim_{z\to e^{i\theta}, z\in\Omega_{e^{i\theta}}} f(z)$$

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exists for almost every $e^{i\theta}$, where $\Omega_{e^{i\theta}}$ is the *exponential* contact region

$$\Omega_{e^{i\theta}} := \{re^{i\phi} : 1 - r^2 > e^{-\frac{1}{|\theta-\phi|}}\}.$$

Beginning at $z = 1$, lay down arcs $I_n \subset \partial U$ of length

$$\frac{1}{-\log(1-r_n)}$$

end-to-end (repeatedly traversing the unit circle). Since $\sum_{n\geqslant 1} |I_n| = \infty$, by hypothesis, each $e^{i\theta} \in \partial U$ will be contained in infinitely many of the intervals $(I_n)_{n\geqslant 0}$. Let $e^{i\theta_n}$ be the center of the interval I_n, and note that simple geometry shows that for every $e^{i\theta}$, the exponential contact region $\Omega_{e^{i\theta}}$ contains infinitely many of the points $r_n e^{i\theta_n}$. Thus if f has finite Dirichlet integral and $f(r_n e^{i\theta_n}) = 0$ for all n, the above limit result says that the boundary function for f will vanish almost everywhere on ∂U, forcing f to be identically zero. This argument actually shows that the sequence $(r_n e^{i\theta_n})_{n\geqslant 0}$ cannot be the zeros of a nontrivial *harmonic* function f on U with finite Dirichlet integral (where $|f'|^2$ is replaced by $|\nabla f|^2$ in the definition of the Dirichlet integral).

In this note, we refine this result and show that for analytic functions the angles θ_n can be chosen so that the zeros $(r_n e^{i\theta_n})_{n\geqslant 0}$ need not accumulate at *every* point of the circle, but instead accumulate at a *single* point.

Theorem 2. *Suppose* $(r_n)_{n\geqslant 0} \subset (0,1)$ *with* $r_n \to 1$ *and*

$$\sum_{n=0}^{\infty} \frac{1}{-\log(1-r_n)} = \infty.$$

Then there are angles $(\theta_n)_{n\geqslant 0}$ *such that* $\operatorname{clos}(r_n e^{i\theta_n})_{n\geqslant 0} \cap \partial U = \{1\}$ *and such that if* f *is holomorphic on* U *with* $D(f) < \infty$ *and* $f(r_n e^{i\theta_n}) = 0$ *for all* n, *then* f *is identically the zero function.*

Our proof is based on the following lemma. In order to make our construction easier, we will work in the upper half plane.

Lemma 3. *Let* $J \subset \mathbb{R}$ *be a finite open interval with center* x_0 *and* $0 < y_0 < |J|$. *Set*

$$S := \{x + iy : x \in J,\ 0 < y < |J|\}$$

and $\lambda_0 := x_0 + iy_0$. *Suppose* f *is holomorphic on* S *with*

$$\int_S |f'|^2 dA < \infty$$

and $f(\lambda_0) = 0$. *If*

$$E = \{x \in J : |f(x)| \geqslant 1\},$$

and $|E| \geqslant \frac{1}{2}|J|$, *then*

$$\int_S |f'|^2 dA \geqslant \frac{c}{\log(|J|/y_0)},$$

where c *is a universal constant.*

Proof. Elementary considerations show that $\omega^S_{\lambda_0}(E)$, the harmonic measure of E with respect to S at λ_0, is bounded below by a universal constant times $y_0/|J|$. Indeed,

$$\omega^S_{\lambda_0}(E) \geqslant \omega^S_{\lambda_0}(F),$$

where F is the union of two intervals in the real line of length $\frac{1}{4}|J|$ located at the lower corners of S. Let $\psi : S \to U$ be the conformal map that takes the centroid of S to the origin and the line segment containing λ_0 and x_0 to the positive real axis. Thus $\psi(x_0) = 1$ and $\psi(\lambda_0) = r$ with $1 - r \asymp y_0/|J|$. Then

$$\omega^S_{\lambda_0}(F) = |\psi(F)| \asymp \omega^U_r(\psi(F)) = \int_{\psi(F)} \frac{1-r^2}{|e^{it} - r|^2} dt.$$

But since $\psi(F)$ is a fixed distance from the point $z = 1$, the denominator in the above integral does not matter. Thus, since the measure of $\psi(F)$ is fixed, the above integral is comparable to $1 - r \asymp y_0/|J|$.

Let $\varphi : U \to S$ be the conformal map with $\varphi(0) = \lambda_0$, and let $g := f \circ \varphi$. Then $g(0) = 0$, $|g| \geqslant 1$ on $\varphi^{-1}(E)$, and $|\varphi^{-1}(E)| = \omega^S_{\lambda_0}(E) \geqslant c y_0/|J|$. This means that g is a "test function" for the logarithmic capacity of $\varphi^{-1}(E)$ [1, Theorem 2], and so

$$\int_U |g'|^2 dA \geqslant c \operatorname{cap}(\varphi^{-1}(E)) \geqslant \frac{c}{\log(|J|/y_0)}.$$

Here we are using the well-known fact that if $W \subset \partial U$ with $a = |W|$, then

$$\operatorname{cap}(W) \geqslant \frac{c}{\log(1/a)}.$$

Finally, note that

$$\int_U |g'|^2 dA = \int_S |f'|^2 dA.$$

□

We are now ready to prove our main theorem. To make the construction easier, we work in the upper half plane and replace the sequence $(r_n)_{n\geqslant 0}$ with a sequence $(y_n)_{n\geqslant 0} \subset (0,1)$ with $y_n \to 0$ and such that

$$\sum_{n=0}^{\infty} \frac{1}{\log(1/y_n)} = \infty.$$

We will construct a sequence $(x_n + iy_n)_{n\geqslant 0}$ in the upper half plane whose closure intersects the real axis only at $x = 0$ and such that the only holomorphic function f in the upper half plane with finite Dirichlet integral for which $f(x_n + iy_n) = 0$ for all n is the zero function.

Assuming that $y_n \searrow 0$, we can find

$$1 \leqslant n_1 < m_1 < n_2 < m_2 < \cdots$$

such that, whenever $n \geqslant n_k$,

$$y_n \log \frac{1}{y_n} < \frac{1}{k^2} e^{-2k^2}$$

and

$$k e^{2k^2} < \sum_{n=n_k}^{m_k} \frac{1}{\log(1/y_n)} < k e^{2k^2} + 1.$$

For each k, lay out intervals

$$J_{n_k}, J_{n_k+1}, \cdots, J_{m_k}$$

on the real axis end-to-end starting at $x = 0$ and such that

$$|J_n| = \frac{1}{k^2 e^{2k^2} \log(1/y_n)}, \quad n_k \leqslant n \leqslant m_k.$$

Then

$$\begin{aligned} \log \frac{|J_n|}{y_n} &= \log \frac{1}{k^2 e^{2k^2} y_n \log(1/y_n)} \\ &= \log(1/y_n) - \log k^2 - 2k^2 - \log\log(1/y_n) \\ &< \log(1/y_n). \end{aligned}$$

Let x_n be the center of J_n, and set $\lambda_n := x_n + iy_n$ and

$$S_n := \{x + iy : x \in J_n, 0 < y < |J_n|\}.$$

Suppose that f is holomorphic on the upper half plane with finite Dirichlet integral and such that $f(\lambda_n) = 0$ for all $n_k \leqslant n \leqslant m_k$. Set

$$A_k := \{n : n_k \leqslant n \leqslant m_k \text{ and } |f| \geqslant e^{-k^2} \text{ on a set } E_n \subset J_n \text{ with } |E_n| \geqslant \tfrac{1}{2}|J_n|\},$$
$$B_k := \{n : n_k \leqslant n \leqslant m_k, n \notin A_k\}.$$

Apply Lemma 3 to see that if $n \in A_k$, then

$$\int_{S_n} |f'|^2 dA \geqslant ce^{-2k^2} \frac{1}{\log(|J_n|/y_n)} \geqslant c \frac{e^{-2k^2}}{\log(1/y_n)},$$

and if $n \in B_k$, then

$$\int_{J_n} \log \frac{1}{|f|} dx \geqslant \frac{1}{2}|J_n| k^2 = \frac{1}{2} \frac{e^{-2k^2}}{\log(1/y_n)}.$$

We conclude that

$$\sum_{n \in A_k} \int_{S_n} |f'|^2 dA + \sum_{n \in B_k} \int_{J_n} \log \frac{1}{|f|} dx \geqslant ce^{-2k^2} \sum_{n=n_k}^{m_k} \frac{1}{\log(1/y_n)} \geqslant ck.$$

Thus by the log-integrability of f on the boundary [4, p. 17], f must be the zero function.

It follows that the set

$$\bigcup_{k=1}^{\infty} (\lambda_n)_{n_k \leqslant n \leqslant m_k}$$

cannot be the zeros of a holomorphic function with finite Dirichlet integral. Choose the remaining points (from the unused y_n's) on the imaginary axis to obtain a sequence $(\lambda_n)_{n \geqslant 0}$ that is not the zero set of a function with finite Dirichlet integral. Finally, since

$$\sum_{n=n_k}^{m_k} |J_n| = \frac{1}{k^2 e^{2k^2}} \sum_{n=n_k}^{m_k} \frac{1}{\log(1/y_n)} < \frac{ke^{2k^2} + 1}{k^2 e^{2k^2}} \to 0, \quad k \to \infty,$$

it follows that the closure of the sequence $(\lambda_n)_{n \geqslant 0}$ intersects the real axis only at $x = 0$.

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