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# FIXED POINT FORMULA FOR HOLOMORPHIC FUNCTIONS

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Abstract. We show a Lefschetz fixed point formula for holomorphic functions in a bounded domain $\mathcal{D}$ with smooth boundary in the complex plane. To introduce the Lefschetz number for a holomorphic map of $\mathcal{D}$, we make use of the Bergman kernel of this domain. The Lefschetz number is proved to be the sum of the usual contributions of fixed points of the map in $\mathcal{D}$ and contributions of boundary fixed points, these latter being different for attracting and repulsing fixed points.

## 1. Introduction

Let $\mathcal{D}$ be a bounded domain with smooth boundary in the complex plane $\mathbb{C}$, and let $f$ be a holomorphic map of $\mathcal{D}$ that is $C^\infty$ up to the boundary of $\mathcal{D}$. The pull-back operator $f^*$ on differential forms preserves the bidegree and commutes with the Cauchy-Riemann operator $\bar{\partial}$. Hence it induces a homomorphism $Hf^*$ of the cohomology of the complex

$$0 \longrightarrow \mathcal{E}^0(\overline{\mathcal{D}}) \xrightarrow{\bar{\partial}} \mathcal{E}^1(\overline{\mathcal{D}}) \longrightarrow 0, \tag{1.1}$$

where $\mathcal{E}^q(\overline{\mathcal{D}})$ stands for the space of all $(0,q)$-forms in $\mathcal{D}$ with coefficients smooth up to the boundary, $q = 0, 1$.

The cohomology of (1.1) at step $q = 0$ just amounts to the space $\mathcal{A}(\overline{\mathcal{D}})$ of holomorphic functions in $\mathcal{D}$ that are $C^\infty$ up to the boundary. This space is infinite-dimensional. On the other hand, the cohomology of (1.1) at step $q = 1$ vanishes. It follows that the usual definition of the holomorphic Lefschetz number leads to

$$L(f) = \operatorname{Tr} f^*|_{\mathcal{A}(\overline{\mathcal{D}})},$$

the trace of $f^*$ on $\mathcal{A}(\overline{\mathcal{D}})$. Since the space $\mathcal{A}(\overline{\mathcal{D}})$ is of infinite dimension, this trace fails to be defined for all maps $f$. The problem arises of defining a regularised trace of $f^*$ on the space $\mathcal{A}(\overline{\mathcal{D}})$.

To this end one might invoke any right fundamental solution $\Phi$ of the operator $\bar{\partial}$ in $\overline{\mathcal{D}}$. Then the operator $\Pi = 1 - \Phi\bar{\partial}$ is a projection in the space $\mathcal{E}^0(\overline{\mathcal{D}}) = \mathcal{E}(\overline{\mathcal{D}})$ whose range is $\mathcal{A}(\overline{\mathcal{D}})$. The kernel $K_\Pi(\zeta, z)$ of $\Pi$ is a representation of the Dirac

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functional $\delta_z(\zeta)$ on the space of holomorphic functions. The regularised trace of $f^*$ on $\mathcal{A}(\overline{\mathcal{D}})$ is then defined by

$$\operatorname{Tr} f^*|_{\mathcal{A}(\overline{\mathcal{D}})} = \text{p.v.} \int K_\Pi(z, f(z)), \tag{1.2}$$

the principal value referring to fixed points of $f$ on the boundary of $\mathcal{D}$, if there are any.

This agrees on the one hand with the trace of the pull-back operator $f^*$ on $\mathcal{A}(\overline{\mathcal{D}})$ for constant maps $f$ of $\mathcal{D}$, i.e., $f(z) = w_0 \in \mathcal{D}$. Indeed, in the latter case both sides of (1.2) are equal to 1. On the other hand, equality (1.2) readily gives the most elementary result of Lefschetz theory. Namely, if $f$ has no fixed points in $\overline{\mathcal{D}}$, then the holomorphic Lefschetz number of $f$ vanishes, for the integral localises to the set of fixed points.

In this paper we take as $\Phi$ the right fundamental solution of the Cauchy-Riemann operator $\bar{\partial}$ given by the Neumann problem for the complex (1.1). At step 1 the latter problem actually reduces to the Dirichlet problem for the Laplace operator in $\mathcal{D}$. The kernel $K_\Pi(\zeta, z)$ obtained this way is nothing but the Bergman kernel of the domain $\mathcal{D}$. We obtain

**Theorem 1.1.** *Suppose $f$ is a holomorphic map of $\mathcal{D}$ that extends smoothly to the closure of $\mathcal{D}$. If $f$ has only isolated fixed points in $\overline{\mathcal{D}}$, then the holomorphic Lefschetz number of $f$ is*

$$L(f) = \sum_{p \in \operatorname{Fix}(f, \overline{\mathcal{D}})} \mu(p).$$

The local indices $\mu(p)$ are infinitesimal invariants of $f$ at $p$. If $p \in \operatorname{Fix}(f, \mathcal{D})$, then $\mu(p)$ coincides with that appearing in the case of compact Riemannian surfaces. Namely, $\mu(p)$ is the trace of the meromorphic function $1/(1 - f'(z))$ near $z = p$ with respect to the map $z - f(z)$ at 0, cf. §6 in Tsikh [14]. The local indices of boundary fixed points are more artful, cf. §5.

A Lefschetz fixed point formula for closed holomorphic curves was first proved by Eichler in [4]. Atiyah and Bott [1] generalised it to the Dolbeault complex on a compact closed complex manifold. For direct constructions along more classical lines, we refer the reader to Patodi [11], Toledo and Tong [13], et al.

For strictly pseudoconvex domains $\mathcal{D}$ in $\mathbb{C}^n$, a holomorphic Lefschetz formula was proved by Donnelly and Fefferman in [3], working within the framework of $L^2$-cohomology of the Bergman metric. Recall that the Bergman metric, whose Kähler potential is given by the Bergman kernel, is a complete Kähler metric on $\mathcal{D}$. This actually corresponds to the case of a non-compact closed manifold, and so $f$ was assumed to have no fixed points on $\partial\mathcal{D}$. Our results extend to the higher-dimensional situation, too.

Brenner and Shubin [2] showed a fixed point formula for elliptic boundary value problems in Boutet de Monvel's algebra. Their results do not apply to the Cauchy-Riemann system, for the latter admits no boundary value problem satisfying the Lopatinskii condition.

## 2. The Neumann problem

The Neumann problem for complex (1.1) at extreme step 1 consists of finding, for a given $F \in \mathcal{E}^1(\overline{\mathcal{D}})$, a differential form $u \in \mathcal{E}^1(\overline{\mathcal{D}})$ such that

$$\begin{array}{rcll} \bar\partial \bar\partial^* u & = & F & \text{in} \quad \mathcal{D}, \\ n(u) & = & 0 & \text{on} \quad \partial\mathcal{D}, \end{array} \tag{2.1}$$

where $\bar\partial^*$ is the formal adjoint for $\bar\partial$, and $n(u)$ the complex normal part of $u$ on the boundary. Write

$$\begin{array}{rcl} F & = & F_1(z)\, d\bar z, \\ u & = & u_1(z)\, d\bar z; \end{array}$$

then problem (2.1) becomes in fact the Dirichlet problem for the function $u_1(z)$, namely

$$\begin{array}{rcll} -(1/2)\,\Delta u_1 & = & F_1 & \text{in} \quad \mathcal{D}, \\ u_1 & = & 0 & \text{on} \quad \partial\mathcal{D}. \end{array}$$

The latter problem has a unique solution given by

$$u_1(z) = 2\imath \int_{\mathcal{D}} F_1(\zeta)\, G(\zeta, z)\, d\bar\zeta \wedge d\zeta$$

for $z \in \mathcal{D}$, where $G(\zeta, z)$ is the Green function of the Dirichlet problem. It has the form

$$G(\zeta, z) = \frac{1}{2\pi} \log|\zeta - z| - h(\zeta, z) \tag{2.2}$$

where $h(\zeta, z)$ is a smooth function defined away from the boundary diagonal in the product $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$. For a fixed $z \in \mathcal{D}$, this function is harmonic in $\zeta \in \mathcal{D}$, continuous in $\zeta \in \overline{\mathcal{D}}$, and satisfies $h(\zeta, z) = \frac{1}{2\pi} \log|\zeta - z|$ in $\zeta \in \partial\mathcal{D}$. This forces it to be symmetric in $\zeta$ and $z$, whence $G(\zeta, z) = G(z, \zeta)$. It follows that the Neumann operator at step 1 is

$$NF\,(z) = \int_{\mathcal{D}} F(\zeta) \wedge \Big(2\imath\, G(\zeta, z)\, d\zeta d\bar z\Big), \tag{2.3}$$

the integral being over $\zeta \in \mathcal{D}$.

The composition $\Phi = \bar\partial^* N$ gives a right fundamental solution of the Cauchy-Riemann operator in $\overline{\mathcal{D}}$. Indeed, (2.1) implies at once that $\bar\partial \Phi = 1$ on $\mathcal{E}^1(\overline{\mathcal{D}})$, as desired.

**Lemma 2.1.** *When regarded as a map in $\mathcal{L}(\mathcal{E}^1(\overline{\mathcal{D}}), \mathcal{E}(\overline{\mathcal{D}}))$, the operator $\Phi$ has the Schwartz kernel*

$$K_\Phi(\zeta, z) = \Big(-\frac{1}{2\pi\imath}\,\frac{1}{\zeta - z} + 2\imath\,\frac{\partial}{\partial z} h(\zeta, z)\Big) d\zeta.$$

*Proof.* Since

$$\bar\partial^*\,(u_1(z) d\bar z) = -\frac{\partial}{\partial z} u_1(z),$$

it follows from (2.3) that

$$\begin{array}{rcl} \Phi F\,(z) & = & \displaystyle\int_{\mathcal{D}} F(\zeta) \wedge -\frac{\partial}{\partial z}\Big(2\imath\, G(\zeta, z)\, d\zeta\Big) \\ & = & \displaystyle\int_{\mathcal{D}} F(\zeta) \wedge K_\Phi(\zeta, z) \end{array}$$

for all $z \in \mathcal{D}$. □

Note that the second term in $K_{\Phi}(\zeta, z)$ is holomorphic in $z \in \mathcal{D}$ for any fixed $\zeta \in \overline{\mathcal{D}}$.

An exact calculation of the kernel of the $\bar{\partial}$-Neumann operator for a strictly pseudoconvex manifold $\mathcal{D}$ was carried out in the 1980s; cf. [6] and [10].

## 3. Augmented complex

The operator $\Pi = 1 - \Phi\bar{\partial}$ belongs to $\mathcal{L}(\mathcal{E}(\overline{\mathcal{D}}))$ and extends to an orthogonal projection of $L^2(\mathcal{D})$ onto the subspace of $L^2(\mathcal{D})$ consisting of holomorphic functions of class $L^2(\mathcal{D})$. The Schwartz kernel of $\Pi$ is just the Bergman kernel of the domain $\mathcal{D}$. We next compute it.

**Lemma 3.1.** *When regarded as a map in $\mathcal{L}(\mathcal{E}(\overline{\mathcal{D}}))$, the operator $\Pi$ has the Schwartz kernel*

$$K_{\Pi}(\zeta, z) = \Big(2\imath \frac{\partial^2}{\partial\bar{\zeta}\partial z} h(\zeta, z)\Big) d\bar{\zeta} \wedge d\zeta.$$

*Proof.* Let $u \in \mathcal{E}(\overline{\mathcal{D}})$. Combining Lemma 2.1, the Cauchy-Pompey theorem and Stokes' integral formula, we obtain

$$\Pi u\,(z) = -\int_{\partial\mathcal{D}} u(\zeta)\, K_{\Phi}(\zeta, z) + \int_{\mathcal{D}} u(\zeta) \Big(2\imath \frac{\partial^2}{\partial\bar{\zeta}\partial z} h(\zeta, z)\Big) d\bar{\zeta} \wedge d\zeta$$

for all $z \in \mathcal{D}$. The proof of Lemma 2.1 actually shows that

$$K_{\Phi}(\zeta, z) = -\frac{\partial}{\partial z}\Big(2\imath\, G(\zeta, z)\, d\zeta\Big)$$

for $(\zeta, z) \in \overline{\mathcal{D}} \times \mathcal{D}$. Since $G(\zeta, z)$ vanishes whenever $(\zeta, z) \in \partial\mathcal{D} \times \mathcal{D}$, so does $K_{\Phi}(\zeta, z)$. Hence the boundary integral in the formula for $\Pi u$ vanishes, which completes the proof. □

The operators $\{\Pi, \Phi\}$ fit together to give a parametrix of the so-called augmented complex

$$0 \longrightarrow \mathcal{A}(\overline{\mathcal{D}}) \xrightarrow{i} \mathcal{E}(\overline{\mathcal{D}}) \xrightarrow{\bar{\partial}} \mathcal{E}^1(\overline{\mathcal{D}}) \longrightarrow 0, \tag{3.1}$$

where $i$ stands for the embedding operator. This means that they satisfy the fundamental equation

$$\begin{array}{rcll} \Pi i & = & 1 - S_{-1} & \text{on} \quad \mathcal{A}(\overline{\mathcal{D}}), \\ i\Pi + \Phi\bar{\partial} & = & 1 - S_0 & \text{on} \quad \mathcal{E}(\overline{\mathcal{D}}), \\ \bar{\partial}\Phi & = & 1 - S_1 & \text{on} \quad \mathcal{E}^1(\overline{\mathcal{D}}) \end{array} \tag{3.2}$$

up to operators $S_{-1}$, $S_0$ and $S_1$ with smooth kernels on $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$. In fact, the operators $S_0$ and $S_{-1}$ vanish by the very construction, while $S_1$ vanishes on the range of $\bar{\partial}$, i.e., $\bar{\partial}\mathcal{E}(\overline{\mathcal{D}})$.

The pull-back operator $f^*$ defines an endomorphism of (3.1), too, inducing a homomorphism of the cohomology of (3.1). Since this cohomology is finite-dimensional, the Lefschetz number of the latter homomorphism is well defined. We denote it by $L_p(f)$, the sub "$p$" indicating "partial", for the cohomology of (3.1) at steps $-1$ and 0 is zero.

## 4. Lefschetz number

By the above, the total holomorphic Lefschetz number is

$$L(f) = \operatorname{Tr} f^*|_{\mathcal{A}(\overline{\mathcal{D}})} + L_p(f), \tag{4.1}$$

the first term on the right-hand side being the regularised trace (1.2). We first evaluate the partial Lefschetz number $L_p(f)$.

**Lemma 4.1.** *Suppose $f$ is a holomorphic map of $\mathcal{D}$, smooth up to the boundary and having only isolated fixed points in $\overline{\mathcal{D}}$. Then*

$$L_p(f) = -\text{p.v.} \int_{\mathcal{D}} \Delta^*(1 \times f)^* K_{\Pi} - \text{p.v.} \int_{\mathcal{D}} d\varphi(\Phi),$$

*where $\Delta$ is the diagonal map $\overline{\mathcal{D}} \to \overline{\mathcal{D}} \times \overline{\mathcal{D}}$, and $\varphi(\Phi) = -\Delta^*(1 \times f)^* K_{\Phi}$ is a smooth $(1,0)$-form away from the set of fixed points of $f$ in the closure of $\mathcal{D}$.*

*Proof.* Applying $f^*$ to both sides of (3.2), we conclude that the endomorphisms $f^*$ and $f^*S$ of (3.1) are homotopic. Hence it follows that $L_p(f) = L_p(f^*S)$. Since the endomorphism $f^*S$ is smoothing, the alternating sum formula readily yields $L_p(f) = -\operatorname{Tr} f^*S_1$, for $S_{-1} = S_0 = 0$. We now use the fundamental equation (3.2) once again, taking into account that the kernel of the identity operator is supported on the diagonal of $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$. This gives

$$\begin{aligned} L_p(f) &= \lim_{\varepsilon \to 0} \int_{\mathcal{D} \setminus U_\varepsilon} \Delta^*(1 \times f)^* K_{-S_1} \\ &= \lim_{\varepsilon \to 0} \int_{\mathcal{D} \setminus U_\varepsilon} \Delta^*(1 \times f)^* K_{\bar{\partial}\Phi}, \end{aligned}$$

where $U_\varepsilon$ is an $\varepsilon$-neighbourhood of the set of fixed points of $f$ in $\overline{\mathcal{D}}$. This establishes the formula, because $\Delta^*(1 \times f)^* K_{\bar{\partial}\Phi}$ coincides with $-\Delta^*(1 \times f)^* K_{\Pi} - d\varphi(\Phi)$ away from the set $\operatorname{Fix}(f, \overline{\mathcal{D}})$. □

Lemma 4.1 gives some suggestive evidence for defining the regularised trace of $f^*$ on holomorphic functions by (1.2). Indeed, from (4.1) and the lemma it follows that

$$L(f) = -\text{p.v.} \int_{\mathcal{D}} d\varphi(\Phi), \tag{4.2}$$

the formula looking like that for the case of compact closed manifolds; cf. Theorem 6.2.15 in [12].

## 5. Local indices

Given a point $p \in \operatorname{Fix}(f, \overline{\mathcal{D}})$, we write $U(p, \varepsilon)$ for the disk with centre $p$ and radius $\varepsilon > 0$ in $\mathbb{C}$. By (4.2) and Stokes' formula, we get

$$L(f) = -\text{p.v.} \int_{\partial\mathcal{D}} \varphi(\Phi) + \sum_{p \in \operatorname{Fix}(f, \overline{\mathcal{D}})} \mu(p), \tag{5.1}$$

where

$$\mu(p) = \lim_{\varepsilon \to 0} \int_{\mathcal{D} \cap \partial U(p,\varepsilon)} \varphi(\Phi) \tag{5.2}$$

is an infinitesimal invariant of $f$ at $p$. Note that in general $\mu(p)$ is a complex number, and not an integer.

**Lemma 5.1.** *Assume that $p \in \mathcal{D}$ is an isolated fixed point of $f$. If $\varepsilon > 0$ is small enough, then*

$$\mu(p) = \int_{\partial U(p,\varepsilon)} \frac{1}{2\pi\imath} \frac{dz}{z - f(z)}.$$

*Proof.* Combining formula (5.2) and Lemma 2.1, we obtain

$$\mu(p) = \lim_{\varepsilon \to 0} \int_{\partial U(p,\varepsilon)} \frac{1}{2\pi\imath} \frac{dz}{z - f(z)} - \lim_{\varepsilon \to 0} \int_{\partial U(p,\varepsilon)} 2\imath\, h'_z(z, f(z))\, dz,$$

$h'_z(\zeta, z)$ meaning the derivative of $h(\zeta, z)$ in $z$. The differential form in the first integral on the right-hand side is closed in a small punctured neighbourhood of the fixed point $p$. Hence the integral does not depend on $\varepsilon$, provided that $\varepsilon > 0$ is sufficiently small. On the other hand, the differential form in the second integral on the right-hand side is smooth in a neighbourhood of $p$, for $h(\zeta, z)$ is smooth in the product $\mathcal{D} \times \mathcal{D}$. It follows that the second limit is equal to zero, which establishes the formula. □

In particular, if $p \in \mathcal{D}$ is a simple fixed point of $f$, i.e., $f'(p) \neq 1$, then

$$\mu(p) = \frac{1}{1 - f'(p)}, \tag{5.3}$$

as is easy to check by the Cauchy formula. In the general case the integral is evaluated by residue theory.

For boundary fixed points of $f$ the computation of the local index $\mu(p)$ is much more subtle. We will touch only the case of simple fixed points of $f$ on the boundary.

Brenner and Shubin [2] specified attracting and repulsing simple fixed points of $f$ on the boundary. Each simple point $p \in \mathrm{Fix}(f, \partial\mathcal{D})$ is either attracting or repulsing. The contribution of an attracting point $p \in \mathrm{Fix}(f, \partial\mathcal{D})$ to the Lefschetz number $L(f)$ amounts to that of any interior simple fixed point, cf. (5.3), while the repulsing points do not contribute to the Lefschetz number at all. In the non-elliptic case the specification is more tricky. A boundary fixed point $p$ of $f$ is said to be attracting if $|f'(p)| < 1$, and repulsing if $|f'(p)| > 1$. Once again each simple fixed point $p$ of $f$ on the boundary of $\mathcal{D}$ is either attracting or repulsing. Indeed, if $|f'(p)| = 1$, then close to $p$ the map $f$ reduces to a rotation around $p$, and so it cannot keep $\mathcal{D}$ invariant.

**Lemma 5.2.** *Assume that $p \in \partial\mathcal{D}$ is a simple fixed point of $f$. Then*

$$\mu(p) = \begin{cases} \dfrac{1}{2}\, \dfrac{1}{1 - f'(p)}, & \text{if } p \text{ is attracting;} \\ \dfrac{1}{2}\, \dfrac{1}{f'(p)}\, \dfrac{1}{1 - f'(p)}, & \text{if } p \text{ is repulsing.} \end{cases}$$

It is worth pointing out that the contribution of a repulsing fixed point is still smaller in absolute value than the contribution of an attracting fixed point.

*Proof.* The proof follows from calculations of §7 and a familiar construction of the Green function for planar domains $\mathcal{D}$. Namely, given a point $z \in \mathcal{D}$, let $C(\zeta, z)$ be a conformal map of $\mathcal{D}$ onto the unit disk with centre at 0 in $\mathbb{C}$, such that $C(z, z) = 0$. Then

$$G(\zeta, z) = \frac{1}{2\pi} \log |C(\zeta, z)|$$

is the Green function of $\mathcal{D}$. □

## 6. Holomorphic Lefschetz formula

We are now in a position to formulate our fixed point theorem, which makes Theorem 1.1 of the Introduction more precise.

**Theorem 6.1.** *Suppose $f$ is a holomorphic map of $\mathcal{D}$ that extends smoothly to the closure of $\mathcal{D}$. If $f$ has only isolated fixed points in $\overline{\mathcal{D}}$, then the holomorphic Lefschetz number of $f$ is*

$$L(f) = \sum_{p \in \mathrm{Fix}(f,\overline{\mathcal{D}})} \mu(p),$$

*the local indices $\mu(p)$ being infinitesimal invariants of $f$ at $p$ given by formula* (5.2).

*Proof.* By (5.1) it suffices to show that the integral

$$-\mathrm{p.v.} \int_{\partial\mathcal{D}} \varphi(\Phi) = \lim_{\varepsilon\to 0} \int_{\partial\mathcal{D}\setminus \cup_{p\in \mathrm{Fix}(f,\partial\mathcal{D})} U(p,\varepsilon)} K_\Phi(z, f(z))$$

is equal to zero. As has been mentioned in the proof of Lemma 3.1, the kernel $K_\Phi(\zeta, z)$ vanishes for all $(\zeta, z) \in \partial\mathcal{D} \times \mathcal{D}$. Since this kernel is actually $C^\infty$ away from the diagonal in the product $\overline{\mathcal{D}}\times\overline{\mathcal{D}}$, we deduce at once that $K_\Phi(z, f(z))$ vanishes for all $z$ away from the set of fixed points of $f$ on the boundary. This finishes the proof. □

Theorem 6.1 extends obviously to the Dolbeault complex on a strictly pseudo-convex domain in $\mathbb{C}^n$, thus implying the fixed point formula of [3] as a highly special case. But we will not develop this point here.

## 7. Automorphisms of the unit disk

Let $\mathcal{D} = U$ be the unit disk centered at the origin in the complex plane. Then the Green function is

$$G(\zeta, z) = \frac{1}{2\pi} \log|\zeta - z| - \frac{1}{2\pi} \log|1 - \bar{\zeta} z|;$$

cf. (2.2). An easy computation shows that

$$-K_\Phi(\zeta, z) = \Big(\frac{1}{2\pi\imath} \frac{1}{\zeta - z} - \frac{1}{2\pi\imath} \frac{\bar{\zeta}}{1 - \bar{\zeta} z}\Big) d\zeta$$

for $(\zeta, z)$ away from the diagonal in $\overline{U} \times \overline{U}$.

By (5.2), we get

$$\mu(p) = \lim_{\varepsilon\to 0} \int_{U\cap\partial U(p,\varepsilon)} \frac{1}{2\pi\imath} \frac{dz}{z - f(z)} - \lim_{\varepsilon\to 0} \int_{U\cap\partial U(p,\varepsilon)} \frac{1}{2\pi\imath} \frac{\bar{z} dz}{1 - \bar{z} f(z)} \tag{7.1}$$

if $p \in \overline{U}$ is an isolated fixed point of $f$. We restrict our discussion to the case $p \in \partial U$, for the local index of interior fixed points $p \in U$ is computed in Lemma 5.1.

The first limit on the right-hand side of (7.1) is standard in the theory of the Cauchy integral, unless $p$ fails to be simple. This is

$$\lim_{\varepsilon\to 0} \int_{U\cap\partial U(p,\varepsilon)} \frac{1}{2\pi\imath} \frac{dz}{z - f(z)} = \frac{1}{2} \frac{1}{1 - f'(p)}. \tag{7.2}$$

To evaluate the second limit on the right-hand side of (7.1), we use the Taylor formula to write

$$f(z) = p + f'(p)(z - p) + o(|z - p|)$$

in a small neighbourhood of $p$. After changing the variables $z - p = \varepsilon\, p\, \zeta$ with $|\zeta| = 1$, we get

$$\begin{aligned} -\lim_{\varepsilon\to 0}\int_{U\cap\partial U(p,\varepsilon)} \frac{1}{2\pi\imath}\,\frac{\bar z dz}{1-\bar z f(z)} &= \lim_{\varepsilon\to 0}\int_{\substack{\zeta\in\partial U\\ \Re\zeta<0}} \frac{1}{2\pi\imath}\,\frac{(1+\varepsilon\bar\zeta)\,d\zeta}{f'(p)\zeta+\bar\zeta+\varepsilon f'(p)+o(\varepsilon)} \\ &= \lim_{\varepsilon\to 0}\int_{\substack{\zeta\in\partial U\\ \Re\zeta<0}} \frac{1}{2\pi\imath}\,\frac{(\zeta+\varepsilon)\,d\zeta}{f'(p)\zeta^2+1+\varepsilon f'(p)\zeta+o(\varepsilon)}, \end{aligned}$$

for the boundary of $U$ is smooth at $p$. If $|f'(p)| \neq 1$, then we can pass to the limit under the integral sign, thus obtaining

$$\int_{\substack{\zeta\in\partial U\\ \Re\zeta<0}} \frac{1}{2\pi\imath}\,\frac{\zeta\, d\zeta}{f'(p)\zeta^2+1} = \frac{1}{2}\int_{\zeta\in\partial U} \frac{1}{2\pi\imath}\,\frac{\zeta\, d\zeta}{f'(p)\zeta^2+1},$$

the last equality being a consequence of the invariance of the differential form under the transformation $\zeta \mapsto -\zeta$.

In the case $|f'(p)| < 1$ the polynomial $f'(p)\zeta^2 + 1$ is different from zero in the closure of $U$. Hence the latter integral vanishes by the Cauchy theorem.

For $|f'(p)| > 1$ the integral in question is easily evaluated by the residue theorem. It is equal to $1/(2f'(p))$. Summarising, we have

$$(7.3)\qquad -\lim_{\varepsilon\to 0}\int_{U\cap\partial U(p,\varepsilon)} \frac{1}{2\pi\imath}\,\frac{\bar z dz}{1-\bar z f(z)} = \begin{cases} 0, & \text{if } p \text{ is attracting;}\\ \dfrac{1}{2}\,\dfrac{1}{f'(p)}, & \text{if } p \text{ is repulsing.}\end{cases}$$

Combining the equalities (7.1), (7.2), and (7.3), we arrive at the formula of Lemma 5.2, namely

$$\mu(p) = \begin{cases} \dfrac{1}{2}\,\dfrac{1}{1-f'(p)}, & \text{if } p \text{ is attracting;}\\ \dfrac{1}{2}\,\dfrac{1}{f'(p)}\,\dfrac{1}{1-f'(p)}, & \text{if } p \text{ is repulsing.}\end{cases}$$

**Example 7.1.** Consider the family of linear-fractional automorphisms of the unit disk $U$, given by

$$f(z) = \frac{z-a}{1-\bar a z},$$

where $a \in U$ is different from zero. The map $f$ has two fixed points $p = \pm a/|a|$, both points belonging to the boundary. Since

$$f'\Big(\pm\frac{a}{|a|}\Big) = \frac{1\pm|a|}{1\mp|a|},$$

the point $+a/|a|$ is repulsing and the point $-a/|a|$ is attracting. Hence it follows that

$$\begin{aligned} \mu\Big(+\frac{a}{|a|}\Big) &= -\frac{1}{4}\,\frac{1-|a|}{1+|a|}\,\frac{1-|a|}{|a|}, \\ \mu\Big(-\frac{a}{|a|}\Big) &= \frac{1}{4}\,\frac{1+|a|}{|a|}, \end{aligned}$$

and so

$$L(f) = \frac{1}{1+|a|}.$$

Note that $L(f)$ tends to 1 when $a \to 0$, while single local indices $\mu(\pm a/|a|)$ have no limit values for $a \to 0$.

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## References

1. M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes. 1*, Ann. Math. **86** (1967), no. 2, 374–407. MR **35:**3701
2. A. V. Brenner and M. A. Shubin, *The Atiyah-Bott-Lefschetz theorem for manifolds with boundary*, Funkts. Analiz **15** (1981), no. 4, 67–68; English transl., Funct. Anal. Appl. **15** (1981), 286–287. MR **83a:**58083
3. H. Donnelly and C. Fefferman, *Fixed point formula for the Bergman kernel*, Amer. J. Math. **108** (1986), no. 5, 1241–1258. MR **87k:**32043
4. M. Eichler, *Eine Verallgemeinerung der Abelschen Integrale*, Math. Z. **67** (1957), 267–298. MR **19:**740a
5. B. Fedosov, *On G-trace and G-index in deformation quantization*, Lett. Math. Phys. **52** (2000), 29–49. MR **2001k:**53175
6. R. Harvey and J. Polking, *The $\bar\partial$-Neumann kernel in the ball in $\mathbb{C}^n$*, Proc. Sympos. Pure Math. **41** (1984), 117–136. MR **85j:**32034
7. H. Hopf, *Über die algebraische Anzahl von Fixpunkten*, Math. Z. **29** (1929), 493–524.
8. L. Hörmander, *The analysis of linear partial differential operators. III. Pseudodifferential operators*, Springer-Verlag, Berlin, 1985. MR **87d:**35002a
9. S. Lefschetz, *Intersections and transformations of complexes and manifolds*, Trans. Amer. Math. Soc. **28** (1926), 1–49.
10. I. Lieb and R. M. Range, *Integral representations on Hermitian manifolds: the $\bar\partial$-Neumann solution of the Cauchy-Riemann equations*, Bull. Amer. Math. Soc. **11** (1984), 355–358. MR **85h:**32004
11. V. K. Patodi, *Holomorphic Lefschetz fixed point formula*, Bull. Amer. Math. Soc. **79** (1973), 825–828. MR **47:**5906
12. N. Tarkhanov, *Complexes of differential operators*, Kluwer Academic Publishers, Dordrecht, 1995. MR **96k:**58205
13. D. Toledo and Y. L. Tong, *Duality and intersection theory in complex manifolds. 2: The holomorphic Lefschetz formula*, Ann. of Math. **108** (1978), 519–538. MR **80j:**32045
14. A. K. Tsikh, *Multidimensional residues and their applications*, Amer. Math. Soc., Providence, RI, 1992. MR **93g:**32005

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