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MAXIMAL INVARIANT SUBSPACES FOR $A^2_\alpha(D)$

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Abstract. We find the maximal invariant subspaces for M_z on $\mathbb{C}^n$-valued Bergman-type spaces.

In this note, we present an alternative view to that of Hedenmalm, [5] or [6], to characterize the maximal invariant subspaces for M_z on $A^2(D)$, the Bergman space on the unit disk. More generally, this method can be applied to certain weighted $\mathbb{C}^n$-valued Bergman spaces. Our point of view is based on the model theory results of Livšic [7], Schwartz [9], and others, and clearly indicates the considerable gap between the theorems proven below and the general invariant subspace problem.

Take any sequence $\{w_n\}_{n=0}^\infty$ of positive weights, and for some $p \geq 1$,

$$\sum_{n=1}^\infty |1 - \frac{w_{n+1}}{w_n}|^p < \infty. \tag{1}$$

Note that from (1) it follows that

$$\operatorname*{Lim}_{n\to\infty} w_n^{\frac{1}{n}} = \operatorname*{Lim}_{n\to\infty} \frac{w_{n+1}}{w_n} = 1. \tag{2}$$

For E a Hilbert space and D the unit disk, define

$$H_{w,E}(D) = \{\underline{f} : D \to E \mid \underline{f} \text{ is } E\text{-valued analytic on } D$$
$$\text{and for } \underline{f}(z) = \sum_{n=0}^\infty a_n z^n,\ \|\underline{f}\|_H^2 \overset{def}{=} \sum_{n=0}^\infty \|a_n\|_E^2 w_n < \infty\}.$$

Condition (2) tells us that $H_{w,E}(D)$ is a reproducing kernel Hilbert space of E-valued functions on D. It is easy to check that for $|z|, |w| < 1$,

$$k_w(z) = \mathcal{K}(z,w) = \sum_{n=0}^\infty \frac{z^n \overline{w}^n}{w_n} I_E$$

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is the reproducing kernel for $H_{w,E}(D)$. That is, for $\underline{f}(z) = \sum_{n=0}^{\infty} a_n z^n \in H_{w,E}(D)$, $e \in E$ and $|w| < 1$,

$$\begin{aligned}\langle \underline{f}(\cdot), \mathcal{K}(\cdot, w)e\rangle_H &= \langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} \frac{\overline{w}^n e}{w_n} z^n \rangle_H \\ &= \sum_{n=0}^{\infty} \langle a_n, \frac{\overline{w}^n e}{w_n}\rangle_E \, w_n \\ &= \langle \sum_{n=0}^{\infty} a_n w^n, e\rangle_E \\ &= \langle \underline{f}(w), e\rangle_E .\end{aligned}$$

For $\varphi : D \to \mathbb{C}$, we denote by M_φ the operator of pointwise multiplication by φ on $H_{w,E}(D)$, whenever this multiplication operator is bounded from $H_{w,E}(D)$ into $H_{w,E}(D)$. By (2),

$$\sup_n \frac{w_{n+1}}{w_n} \overset{def}{=} C < \infty.$$

Thus it follows that $M_z \in B(H_{w,E}(D))$ and $\|M_z\| = C$.

As a special case, for $-1 < \alpha < \infty$ let $A^2_{\alpha,E}(D)$ denote $H_{w,E}(D)$, where the weights are given by $w_n = \frac{n!\,\Gamma(\alpha+1)}{\Gamma(\alpha+n+2)}$. For $\alpha = 0$, we have $w_n = \frac{1}{n+1}$, and so $A^2_{0,E}(D)$, or just $A^2_E(D)$, is E-valued Bergman space. In the case that $w_n = 1$ for all n, we denote $H_{w,E}(D)$ by $H^2_E(D)$, the E-valued Hardy space.

Let $B = M_z$ on $H = H_{w,E}(D)$. If $\mathcal{M}$ is an invariant subspace of B, then if $\mathcal{N}$ is a larger invariant subspace of B with $\mathcal{N} \neq H$, we have

$$0 \subsetneqq \mathcal{N}^\perp \overset{def}{=} \mathcal{N}_* \subsetneqq \mathcal{M}_* \overset{def}{=} \mathcal{M}^\perp$$

and $\mathcal{N}_*, \mathcal{M}_*$ are invariant subspaces for B^*. Thus $\mathcal{N}_*$ is a nontrivial invariant subspace for $B^* \mid_{\mathcal{M}_*}$. Therefore, the content of the following theorem is that (for $\dim E < \infty$) the maximal invariant subspaces of B are the invariant subspaces of codimension 1. These spaces can be identified with $[e\, k_z]^\perp$ for some $e \neq 0$ in E and $z \in D$.

Let $\mathcal{M}_*$ be an invariant subspace for M_z^* on $H_{w,E}(D)$. Denote $M_z^* \mid_{\mathcal{M}_*}$ by A^*.

Theorem 1. *If* $\dim \mathcal{M}_* > 1$ *and* $\dim E < \infty$ *(say* $\dim E = N$*), then* A^* *has a nontrivial invariant subspace.*

Proof. We have $I_{\mathcal{M}_*} - A\,A^* = P_{\mathcal{M}_*}(I - M_z M_z^*) P_{\mathcal{M}_*}$. Now if $u_n(z) = \frac{z^n}{\sqrt{w_n}}$ and if $\{\underline{e}_n\}_{n=0}^N$ is an orthonormal basis for E, then $\{u_n \underline{e}_m\}_{n=0,m=0}^{\infty,N}$ is an orthonormal basis for $H_{w,E}(D)$.

It is easy to check that M_z^* is an E-valued weighted shift and that

$$M_z^*(u_n \underline{e}_m) = \begin{cases} 0, & n = 0, \\ \sqrt{\frac{w_n}{w_{n-1}}}\, u_{n-1}\, \underline{e}_m, & n > 0. \end{cases}$$

Thus, $I - M_z M_z^*$ is the diagonal operator whose diagonal entries are $(1 - \frac{w_n}{w_{n-1}}) I_E$. This operator has Schatten p-class norm ($p \geq 1$) equal to

$$(1 + \sum_{n=1}^{\infty} |1 - \frac{w_n}{w_{n-1}}|^p)^{\frac{1}{p}} \dim E,$$

which is finite by hypothesis (1). Thus, since the Schatten p-class, $\mathcal{C}_p$, is an ideal, we have

$$I_{\mathcal{M}_*} - A\,A^* \in \mathcal{C}_p.$$

Now by a classical result essentially due to Livšic [7] or Schwartz [9] (see Radjavi and Rosenthal [8], p. 107), it follows that A^* has a nontrivial invariant subspace. □

For a good discussion of this type of invariant subspace results and some generalizations, see Radjavi and Rosenthal [8]. The basic idea is that $I - AA^*$ belonging to some Schatten p-class with $p \geq 1$ gives growth restrictions on the resolvent of A^*, which leads to nontrivial invariant subspaces of A^*.

The main interest in a theorem like Theorem 1 stems from the following classical results. To show that every bounded linear operator on a Hilbert space (of dimension greater than 1) has a nontrivial invariant subspace, it suffices to show that (A) holds. Similarly for (B).

(A): For any subspace $\mathcal{M}_*$, invariant for M_z^* on $H^2_E(D)$ with $\dim \mathcal{M}_* > 1$, show that $M_z^* \mid_{\mathcal{M}_*}$ has a nontrivial invariant subspace.

Of course, $\dim E = \infty$ is the unknown case. (A) comes from the model theory approach, pioneered by Livšic [7], Sz.-Nagy and Foias [10], and de Branges and Rovnyak [4].

(B): For any pair of invariant subspaces of M_z on $A^2(D)$, $\mathcal{N} \subset \mathcal{M}$, with $\dim(\mathcal{M} \ominus \mathcal{N}) > 1$, show that $P_{\mathcal{M}\ominus\mathcal{N}} M_z P_{\mathcal{M}\ominus\mathcal{N}}$ has a nontrivial invariant subspace.

This approach comes from the beautiful theory developed by C. Apostol, S. Brown, H. Bercovici, B. Chevreau, C. Pearcy, and C. Foias. The main references for this work are [1] and [2]. For an exposition of this material, see [3].

The case of (A) when $\dim E < \infty$ falls under Theorem 1, but this case has long been known to hold. See [8]. A special case of (B) occurs when $\mathcal{M} = A^2(D)$. Then finding a nontrivial invariant subspace of $P_{\mathcal{N}^\perp} M_z P_{\mathcal{N}^\perp}$ is equivalent to finding a nontrivial invariant subspace for $(P_{\mathcal{N}^\perp} M_z P_{\mathcal{N}^\perp})^* = M_z^* \mid_{\mathcal{N}^\perp}$. This is the problem considered and solved by Hedenmalm [5], using modern factorization results on Bergman space. Clearly, this result falls under Theorem 1, but more general ones do also. For example,

Theorem 2. *Let $\mathcal{N}_* \subset \mathcal{M}_*$ be invariant subspaces of M_z^* on $A^2_E(D)$ with*

$$\dim(\mathcal{M}_* \ominus \mathcal{N}_*) > 1, \qquad \dim E < \infty, \qquad \dim(A^2_E(D) \ominus \mathcal{M}_*) < \infty.$$

Then $P_{\mathcal{M}_*\ominus\mathcal{N}_*} M_z^* P_{\mathcal{M}_*\ominus\mathcal{N}_*}$ has a nontrivial invariant subspace.

Proof. This is just a modification of the proof of Theorem 1. □

In conclusion, we see that, for the theorems discussed, all the operators A considered are such that $I - A\,A^*$ belongs to $\mathcal{C}_p$ for some p. This is a strong restriction limiting the scope of the invariant subspace results.

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