

## NOTE ON THE STABILITY OF PRINCIPAL BUNDLES

DONGHOON HYEON AND DAVID MURPHY

(Communicated by Michael Stillman)

ABSTRACT. We compare various notions of stability for principal bundles, and show that over a compact Riemann surface of genus greater than 2, there exist principal  $\mathrm{SL}(2)$ -bundles that are Ad-stable.

### 1. INTRODUCTION

The moduli space of stable principal  $G$ -bundles over a compact Riemann surface was constructed by A. Ramanathan in his thesis [6]. The method was generalized recently by the first named author in [5], where he gave a construction of a quasi-projective moduli scheme for principal bundles over a projective scheme of arbitrary dimension. However, the moduli constructed in [5] may fail to include some bundles that we would like it to have. For instance, over a compact Riemann surface, the stability condition used in [5] is strictly stronger than the stability employed by Ramanathan. It has thus been of concern to the authors whether a principal bundle that is stable in the sense of [5] exists at all. The purpose of this article is to give an answer to this question, and to give a detailed account of the relation between the various stability conditions.

In section 2, we introduce the stability conditions for principal bundles used by A. Ramanathan, D. Hyeon (*Ad-stability*), and D. Mumford (for  $G = \mathrm{GL}(n)$ ). We give an elementary proof that for  $G = \mathrm{GL}(n)$ , the stability condition defined by Ramanathan is equivalent to the slope stability of Mumford. We also show that the Ad-stability is strictly stronger than Ramanathan's stability, and that there are no Ad-stable principal  $G$ -bundles if  $G$  is not semisimple. (This is essentially proved in [6].)

In section 3, we briefly recall the main result in [4], and using it, construct examples of Ad-stable  $\mathrm{SL}(2)$ -bundles. Furthermore, we give an example of an  $\mathrm{SL}(2)$ -bundle that is stable in the sense of Ramanathan, but is not Ad-stable.

Unless otherwise stated,  $X$  will be a compact Riemann surface of genus  $g$ . The letter  $G$  will be reserved to denote a connected reductive complex holomorphic Lie group, whose Lie algebra will be denoted  $\mathfrak{g}$ . For any vector bundle  $E$  over  $X$ , the degree  $\mathrm{deg}(E)$  of  $E$  is defined by the Riemann-Roch formula

$$\chi(E) = \mathrm{deg}(E) + \mathrm{rk}(E) \cdot (1 - g),$$

and the slope  $\mu(E)$  is the quantity  $\frac{\mathrm{deg}(E)}{\mathrm{rk}(E)}$ .

---

Received by the editors February 18, 2002 and, in revised form, February 18, 2003.  
2000 *Mathematics Subject Classification*. Primary 14D20.

2. STABILITY NOTIONS FOR PRINCIPAL BUNDLES

**Definition 1.** (1) Let  $E$  be a principal  $G$ -bundle over  $X$ , and  $\rho : G \rightarrow H$  a homomorphism of holomorphic complex Lie groups. Let  $G$  act diagonally on  $E \times H$  by the natural right action on  $E$ , and through  $\rho$  on  $H$ . It is well known that the quotient bundle  $(E \times H)/G$  exists as a principal  $H$ -bundle, and we denote it by  $E_\rho(H)$ , or by  $E(H)$  when the representation used is clear. In a similar fashion, if  $F$  is any  $G$ -space, let  $E(F)$  denote the fibre bundle  $(E \times F)/G$  with fibre  $F$ , where  $G$  acts diagonally on  $E \times F$ .

(2) A reduction  $\sigma$  of the structure group of a principal  $G$ -bundle  $E$  to a subgroup  $P$  of  $G$  is a section  $\sigma : X \rightarrow E/P$  of the fibre bundle  $E/P = E(G/P)$ . Then  $\sigma^*E$  is a  $P$ -bundle over  $X$  and there is a natural isomorphism of  $G$ -bundles  $\sigma^*E(G) \simeq E$ .

Ramanathan’s definition of stability for principal bundles can be phrased in many different ways (e.g., Theorem 2.2. of [2], and [1]). The original version reads as follows.

**Definition 2.** A principal  $G$ -bundle  $E$  over  $X$  is said to be *semistable* if for every reduction  $\sigma : X \rightarrow E/P$  of the structure group to a maximal parabolic  $P \subset G$ , the following inequality holds:

$$\text{deg } \sigma^*T_{E/P} \geq 0.$$

We call  $E$  *stable* if the above inequality is always strict.

Here,  $T_{E/P}$ , the *tangent bundle over  $E/P$  along the fibre of  $E$* , is defined to be the vector bundle  $E(\mathfrak{g}/\mathfrak{p}) = (E \times (\mathfrak{g}/\mathfrak{p}))/P$  over  $E/P$  associated to the  $P$ -bundle  $E \rightarrow E/P$  and the  $P$  action on  $\mathfrak{g}/\mathfrak{p}$  ( $\mathfrak{p}$  is the Lie algebra of  $P$ ) induced by the adjoint representation.

This stability condition is consistent with the condition defined by Mumford, in the case  $G = \text{GL}(n)$ . This is not hard to see, but we include an elementary proof for the sake of completeness.

Let  $P \subset \text{GL}(n)$  be a parabolic subgroup consisting of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where  $A \in \text{GL}(r)$  and  $C \in \text{GL}(n - r)$ , with respect to a fixed basis of  $\mathbb{C}^n$ . Let  $\sigma : X \rightarrow E/P$  be a reduction of the structure group, and let

$$\begin{pmatrix} h_{\alpha\beta} & b_{\alpha\beta} \\ 0 & q_{\alpha\beta} \end{pmatrix}$$

be the transition functions for the  $P$ -bundle  $\sigma^*E$  with respect to a trivializing cover  $\{U_\alpha\}$ . Let  $F$  be the rank  $r$  subbundle of  $E(\mathbb{C}^n)$  corresponding to  $(h_{\alpha\beta})$ . Then the quotient bundle  $E(\mathbb{C}^n)/F$  corresponds to  $(q_{\alpha\beta})$ , and we have the following natural isomorphism.

**Proposition 1.**

$$\sigma^*T_{E/P} \simeq F^* \otimes (E(\mathbb{C}^n)/F).$$

*Proof.* The adjoint action of  $P$  on the Lie algebra  $\mathfrak{gl}(n)$  is given by the matrix multiplication

$$\begin{pmatrix} h & b \\ 0 & q \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} h^{-1} & b' \\ 0 & q^{-1} \end{pmatrix} = \begin{pmatrix} * & * \\ q \cdot C \cdot h^{-1} & * \end{pmatrix}.$$

Here,  $\begin{pmatrix} h^{-1} & b' \\ 0 & q^{-1} \end{pmatrix}$  is the inverse matrix of  $\begin{pmatrix} h & b \\ 0 & q \end{pmatrix}$ . The places with  $*$  are covered by  $\mathfrak{p}$ , and the lower left corner is the only part relevant to  $\mathfrak{g}/\mathfrak{p}$ . Hence we conclude that the adjoint action of  $P$  on  $\mathfrak{g}/\mathfrak{p}$  is given by

$$\left(\begin{pmatrix} h & b \\ 0 & q \end{pmatrix}, C\right) \mapsto q \cdot C \cdot h^{-1}.$$

Therefore, the transition functions for  $\sigma^*E(\mathfrak{g}/\mathfrak{p})$  are  $({}^T h_{\alpha\beta}^{-1} \otimes q_{\alpha\beta})$ , which correspond to  $F^* \otimes (E(\mathbb{C}^n)/F)$ . (See, for instance, section 0.5 of [3].)  $\square$

**Corollary 1.** *A principal  $GL(n)$ -bundle  $E$  over  $X$  is semistable (resp. stable) if and only if for every subbundle  $F$  of  $E(\mathbb{C}^n)$ ,  $\mu(F) \leq \mu(E(\mathbb{C}^n))$  (resp.  $\mu(F) < \mu(E(\mathbb{C}^n))$ ) holds, i.e.,  $E(\mathbb{C}^n)$  is semistable (resp. stable) in the sense of Mumford.*

*Proof.* We shall prove the statement for the “semistable” part only. One proves the “stable” part by simply replacing every inequality by a strict inequality.

Suppose that  $E(\mathbb{C}^n)$  is semistable in the sense of Mumford, and let  $\sigma : X \rightarrow E/P$  be a reduction of the structure group to a maximal parabolic subgroup  $P$  of  $G$ . Then  $P$  corresponds to a two-step flag

$$0 = V_0 \subset V_1 \subset V_2 = \mathbb{C}^n.$$

Let  $F = \sigma^*(E(V_1))$ . By Proposition 1,  $\sigma^*T_{E/P} \simeq F^* \otimes (E(\mathbb{C}^n)/F)$ . Since  $E(\mathbb{C}^n)$  is Mumford semistable, we have  $\mu(F) \leq \mu(E(\mathbb{C}^n))$ , which is equivalent to  $\mu(F) \leq \mu(E(\mathbb{C}^n)/F)$ . This is in turn equivalent to  $\deg F^* \otimes (E(\mathbb{C}^n)/F) \geq 0$ , since

$$\begin{aligned} (*) \quad \deg F^* \otimes (E(\mathbb{C}^n)/F) &= -\deg F \cdot \text{rk}(E(\mathbb{C}^n)/F) + \text{rk } F \cdot \deg E(\mathbb{C}^n)/F \\ &= (\mu(E(\mathbb{C}^n)/F) - \mu(F)) \cdot \text{rk}(E(\mathbb{C}^n)/F) \cdot \text{rk}(F). \end{aligned}$$

Therefore,  $E$  is semistable as a principal  $GL(n)$ -bundle.

Conversely, let  $E$  be a semistable principal  $GL(n)$ -bundle. Any vector subbundle  $F$  of  $E(\mathbb{C}^n)$  is of the form  $\sigma^*(E(V))$  for some reduction  $\sigma$  of the structure group to a parabolic subgroup  $P$  corresponding to a flag  $0 = V_0 \subset V \subset \mathbb{C}^n$ . Since  $E$  is stable as a principal  $GL(n)$ -bundle, we have

$$\deg \sigma^*(T_{E/P}) = \deg(F^* \otimes (E(\mathbb{C}^n)/F)) \geq 0.$$

By (\*), we know that this implies  $\mu(F) \leq \mu(E(\mathbb{C}^n))$ .  $\square$

Now we introduce a very important lemma that rephrases the stability condition for principal bundles in terms of the well-known stability condition for vector bundles. This suggests that one may deduce the existence of a moduli space of principal bundles from the existence of a moduli space of vector bundles, which is exactly what Ramanathan did in his thesis [6].

**Lemma 1** ([6]). *A holomorphic principal  $G$ -bundle  $E$  over  $X$  is semistable if and only if the associated vector bundle  $\text{Ad}(E) = (E \times \mathfrak{g})/G$  is a semistable vector bundle.*

In fact, this is taken as the definition of semistability in [2]. Based on this lemma, we make the following definition.

**Definition 3.** A holomorphic principal  $G$ -bundle over  $X$  is said to be *Ad-stable* if the associated vector bundle  $\text{Ad}(E)$  is a stable vector bundle.

In [5], the first named author constructed a quasi-projective moduli scheme for Ad-stable principal  $G$ -bundles over a projective scheme of arbitrary dimension. (The term “Ad-stable” is not used in [5].) It is natural to ask what the relationship is between Ad-stability and stability according to Ramanathan.

**Proposition 2.** *If a principal  $G$ -bundle  $E$  over a compact Riemann surface  $X$  is Ad-stable, then it is stable in the sense of Ramanathan.*

*Proof.* Suppose that  $E$  is an Ad-stable  $G$ -bundle. Since  $G$  is reductive, there exists a nondegenerate  $G$ -invariant bilinear form on  $\mathfrak{g}$ . Via this form,  $\text{Ad}(E) = E(\mathfrak{g})$  is dual to itself and hence is of degree zero. Let  $P$  be a parabolic subgroup of  $G$  and let  $\mathfrak{p}$  be its Lie algebra. Also, let  $\sigma : X \rightarrow E/P$  be a reduction of the structure group to  $P$ . We have the exact sequence of  $P$ -modules (the module structures are given by the adjoint representation)

$$0 \rightarrow \mathfrak{p} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{p} \rightarrow 0$$

and the exact sequence of vector bundles

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \sigma^*E(\mathfrak{p}) & \longrightarrow & \sigma^*E(\mathfrak{g}) & \longrightarrow & \sigma^*E(\mathfrak{g}/\mathfrak{p}) \longrightarrow 0 \\
 & & & & \parallel & & \parallel \\
 & & & & \text{Ad}(E) & & \sigma^*T_{E/P}
 \end{array}$$

Since  $\text{Ad}(E)$  is a stable vector bundle of degree zero, it follows that  $\text{deg } \sigma^*T_{E/P} > 0$ , and we are done. □

However, as remarked in [2],  $\text{Ad}(E)$  can be strictly semistable even if  $E$  is stable in the sense of Ramanathan. For example, for any  $n > 1$ , no  $\text{GL}(n)$ -bundle is Ad-stable, since  $\text{Ad}(E) = \text{End}(E(\mathbb{C}^n))$  has the trivial line subbundle generated by the trivial section  $Id$  defined by  $Id_x(v) = v$  for all  $x \in X$  and  $v \in E(\mathbb{C}^n)_x$ . In fact, we know that

$$\text{End}(E(\mathbb{C}^n)) = \text{End}_o(E(\mathbb{C}^n)) \oplus \mathcal{O}_X,$$

where  $\text{End}_o(E(\mathbb{C}^n))$  is the bundle of trace-free endomorphisms. This decomposition follows from the fact that there exists a split exact sequence of  $\text{GL}(n)$ -modules

$$(**) \quad 0 \longrightarrow \mathfrak{sl}(n) \longrightarrow \mathfrak{gl}(n) \begin{array}{c} \xrightarrow{Tr} \\ \xleftarrow{\iota} \end{array} \mathbb{C} \longrightarrow 0,$$

where the module structures are given by the adjoint representation. Here  $Tr$  is the trace map and  $\iota$  is defined by  $\iota(z) = \frac{z}{n}1$ ,  $1$  being the identity matrix in  $\mathfrak{gl}(n)$ . More generally,

**Lemma 2.** *There are no Ad-stable principal  $G$ -bundles if  $G$  is not semisimple.*

*Proof.* Let  $\mathfrak{z}$  denote the center of  $\mathfrak{g}$ . Since we are in characteristic 0, note that  $G$  being semisimple, the connected center  $Z^o(G)$  being trivial, and  $\mathfrak{z} = 0$  are all equivalent. Suppose that there exists an Ad-stable  $G$ -bundle  $E$ . By Proposition 2,  $E$  is stable. So we can use Proposition 3.2 of [6] to conclude that  $H^0(X, E(\mathfrak{g})) = \mathfrak{z}$ . If  $\mathfrak{z} \neq 0$ , i.e., if  $G$  is not semisimple, then  $E(\mathfrak{g})$  has a global section  $\phi : \mathcal{O}_X \rightarrow E(\mathfrak{g})$  that generates  $\mathcal{O}_X$  as a subsheaf of  $E(\mathfrak{g})$ . This contradicts the assumption that  $E$  is Ad-stable. □

This shows that the construction in [5] is effective only for semisimple groups. However, even in the semisimple case, stability and Ad-stability are not equivalent. In section 4, we shall encounter an example of a stable  $\mathrm{SL}(2)$ -bundle that is not Ad-stable.

### 3. AD-STABLE $\mathrm{SL}(2)$ -BUNDLES OVER A RIEMANN SURFACE

First we recall the main result in [4]. The fundamental group of  $X$  is generated by  $2g$  elements  $a_1, b_1, \dots, a_g, b_g$  with the single relation

$$\prod_{i=1}^g [a_i, b_i] = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

We denote this group by  $\Gamma_g$ .

Let  $Y$  denote the universal cover of  $X$ . It is a principal  $\Gamma_g$ -bundle over  $X$  (or, a Galois covering of  $X$  with Galois group  $\Gamma_g$ ). Therefore, any rational representation  $\rho : \Gamma_g \rightarrow \mathrm{GL}(n)$  produces a rank  $n$  vector bundle  $(Y \times \mathbb{C}^n)/\Gamma_g$  over  $X$ . The main result of [4] is the following theorem.

**Theorem 1.** *Let  $X$  be a compact Riemann surface of genus  $\geq 2$ . Then a holomorphic vector bundle of degree zero on  $X$  is stable if and only if it arises from an irreducible unitary representation of the fundamental group of  $X$ .*

We need the following basic lemma, which can easily be seen when one considers the transition functions of the bundles involved.

**Lemma 3.** *Let  $E$  be a principal  $G_1$ -bundle, and let  $\rho_1 : G_1 \rightarrow G_2$  and  $\rho_2 : G_2 \rightarrow G_3$  be homomorphisms of holomorphic complex Lie groups. Then we have a natural isomorphism*

$$(E_{\rho_1}(G_2))_{\rho_2}(G_3) \simeq E_{\rho_2 \circ \rho_1}(G_3).$$

A principal  $\mathrm{SL}(n)$ -bundle can be thought of as a rank  $n$  vector bundle with trivial determinant. Therefore we can immediately deduce the following proposition from Theorem 1 and Lemma 3.

**Proposition 3.** *Let  $X$  be a compact Riemann surface of genus  $\geq 2$ . Then a holomorphic principal  $\mathrm{SL}(n)$ -bundle  $E$  is Ad-stable if and only if it arises from a representation  $\rho : \Gamma_g \rightarrow \mathrm{SL}(n)$  such that  $\mathrm{ad}(\rho)$ , the composition of the adjoint representation  $\mathrm{ad} : \mathrm{SL}(n) \rightarrow \mathrm{GL}(\mathfrak{sl}(n))$  and  $\rho$ , is irreducible and unitary.*

*Remark 1.* Such a  $\rho$  is necessarily irreducible and unitary by Lemma 6, which will be shown later in this article. Therefore we conclude that a principal  $\mathrm{SL}(n)$ -bundle is stable if it is Ad-stable. Of course, this also follows from Proposition 2.

We let  $\mathrm{Ad}$  (resp.  $\mathrm{ad}$ ) denote the adjoint representation  $\mathrm{GL}(n) \rightarrow \mathrm{GL}(\mathfrak{gl}(n))$  (resp.  $\mathrm{SL}(n) \rightarrow \mathrm{GL}(\mathfrak{sl}(n))$ ). Hence for any  $x \in \mathrm{SL}(n)$ ,  $\mathrm{Ad}(x)$  denotes the adjoint action of  $x$  on  $\mathfrak{gl}(n)$ , and  $\mathrm{ad}(x)$  denotes the adjoint action on  $\mathfrak{sl}(n) \subset \mathfrak{gl}(n)$ . Let  $\{e_1, \dots, e_n\}$  denote the standard basis of  $\mathbb{C}^n$ . Then the standard basis for  $\mathfrak{gl}(n)$  consists of the matrices  $e_{ij} = e_i \otimes e_j^*$ , the matrix with 1 at the  $ij$ -th place and 0 elsewhere.

**Definition 4.** Let  $G$  be a group and  $\rho : G \rightarrow \mathrm{GL}(n)$  a representation. Then  $\rho$  is said to be *unitary* if  $\rho(G) \subset \mathrm{U}(n)$ .

**Lemma 4.** *Let  $x \in U(n)$ . With respect to the basis  $\{e_i \otimes e_j^*\}$ , we have*

$$(\text{Ad}(x))_{kl\ ij} = x_{ki}\bar{x}_{lj}.$$

That is,

$$\text{Ad}(x)(e_{ij}) = \sum_{k,l} x_{ki}\bar{x}_{lj} e_{kl}.$$

*Proof.* Let  $x' = \bar{x}^T$ . The adjoint action is given by

$$\begin{aligned} \text{Ad}(x)(e_{ij}) &= x \cdot e_i \otimes e_j^* \cdot x' \\ &= \begin{pmatrix} 0 & 0 & \cdots & 0 & x_{1i} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & x_{2i} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & x_{ni} & 0 & \cdots & 0 \end{pmatrix} \cdot x' \\ &= \begin{pmatrix} x_{1i}x'_{j1} & x_{1i}x'_{j2} & \cdots & x_{1i}x'_{jn} \\ x_{2i}x'_{j1} & x_{2i}x'_{j2} & \cdots & x_{2i}x'_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ x_{ni}x'_{j1} & x_{ni}x'_{j2} & \cdots & x_{ni}x'_{jn} \end{pmatrix}. \end{aligned}$$

Hence the element at the  $kl$ th place of  $\text{Ad}(x)(e_{ij})$  is

$$x_{ki}x'_{jl} = x_{ki}\bar{x}_{lj}.$$

□

**Lemma 5.** *Let  $x \in \text{SL}(n)$ .  $\text{Ad}(x)$  is unitary if and only if  $\text{ad}(x) \in \text{GL}(\mathfrak{sl}(n))$  is unitary.*

*Proof.* Recall the split exact sequence (\*\*) of  $\text{GL}(n)$ -modules. From it, we see that with respect to any unitary basis containing  $\frac{1}{\sqrt{n}}(e_{11} + \cdots + e_{nn})$ ,  $\text{Ad}(x)$  is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \text{ad}(x) \end{pmatrix},$$

and it follows immediately that  $\text{Ad}(x)$  is unitary if and only if  $\text{ad}(x)$  is so. □

**Lemma 6.** *Let  $x \in \text{GL}(n)$ . Then  $\text{Ad}(x)$  is unitary if and only if  $x$  is so.*

*Proof.* We have

$$\begin{aligned} (\text{Ad}(x) \cdot \overline{\text{Ad}(x)}^T)_{ij\ kl} &= \sum_{r,s} (\text{Ad}(x))_{ij\ rs} \cdot \overline{(\text{Ad}(x))^T}_{rs\ kl} \\ &= \sum_{r,s} (\text{Ad}(x))_{ij\ rs} \cdot \overline{\text{Ad}(x)}_{kl\ rs} \\ &= \sum_{r,s} x_{ir}\bar{x}_{js} \cdot \overline{x_{kr}\bar{x}_{ls}} \\ &= \sum_{r,s} x_{ir}\bar{x}_{js} \cdot \overline{x_{kr}x_{ls}} \\ &= (x \cdot \bar{x}^T)_{ik} \cdot (x \cdot \bar{x}^T)_{lj}. \end{aligned}$$

Therefore,  $\text{Ad}(x)$  is unitary if and only if  $(x x) = x \cdot \bar{x}^T$  is diagonal and  $(x x)_{ii} \cdot (x x)_{jj} = 1$  for any  $i, j$ . Thus  $(x x)_{ii}^2 = 1$ , but  $(x x)_{ii} = \sum_s x_{is}\bar{x}_{is} \geq 0$ , and it follows that  $(x x)$  is the identity matrix. □

Combining Lemma 5 and Lemma 6, we obtain

**Lemma 7.** *Let  $x \in \text{SU}(n)$ . Then  $\text{ad}(x)$  is unitary.*

**Example 1.** In the rank 2 case ( $x \in U(2)$ ),

$$\text{Ad}(x) = \begin{pmatrix} x_{11}\bar{x}_{11} & x_{11}\bar{x}_{12} & x_{12}\bar{x}_{11} & x_{12}\bar{x}_{12} \\ x_{11}\bar{x}_{21} & x_{11}\bar{x}_{22} & x_{12}\bar{x}_{21} & x_{12}\bar{x}_{22} \\ x_{21}\bar{x}_{11} & x_{21}\bar{x}_{12} & x_{22}\bar{x}_{11} & x_{22}\bar{x}_{12} \\ x_{21}\bar{x}_{21} & x_{21}\bar{x}_{22} & x_{22}\bar{x}_{21} & x_{22}\bar{x}_{22} \end{pmatrix},$$

and with respect to the unitary basis  $\{e_{12}, e_{21}, \frac{1}{\sqrt{2}}(e_{11} - e_{22})\}$ ,

$$\text{ad}(x) = \begin{pmatrix} x_{11}\bar{x}_{22} & x_{12}\bar{x}_{21} & \sqrt{2}x_{11}\bar{x}_{21} \\ x_{21}\bar{x}_{12} & x_{22}\bar{x}_{11} & \sqrt{2}x_{21}\bar{x}_{11} \\ \sqrt{2}x_{11}\bar{x}_{12} & \sqrt{2}x_{12}\bar{x}_{11} & |x_{11}|^2 - |x_{12}|^2 \end{pmatrix}.$$

**Proposition 4.** For any integer  $g > 2$ , there exists an irreducible unitary representation  $\rho$  of  $\Gamma_g$  of rank 2 such that  $\text{ad}(\rho)$  is also irreducible and unitary.

*Proof.* Let  $z$  be a complex number satisfying  $z^2 \neq \bar{z}^2$ , and set  $r = |z|$ . Let

$$A_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, B_2 = \frac{1}{r\sqrt{2}} \begin{pmatrix} z & -\bar{z} \\ z & \bar{z} \end{pmatrix}, A_2 = B_2^{-1}.$$

Also, let  $A_3 = A_1$ ,  $B_3 = B_1$ , and  $A_i = B_i = 1$  for  $i > 3$ . Then  $[A_1 B_1] = -1$ ,  $[A_2 B_2] = 1$ ,  $[A_3 B_3] = -1$  and hence  $\prod_{i=1}^g [A_i B_i] = 1$ .

With respect to the unitary basis  $\{e'_1 = e_{12}, e'_2 = e_{21}, e'_3 = \frac{1}{\sqrt{2}}(e_{11} - e_{22})\}$ , we have

$$\begin{aligned} \text{ad}(A_1) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ \text{ad}(B_1) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \text{ad}(B_2) &= \frac{1}{2r^2} \begin{pmatrix} z^2 & -\bar{z}^2 & \sqrt{2}r^2 \\ -z^2 & \bar{z}^2 & \sqrt{2}r^2 \\ -\sqrt{2}z^2 & -\sqrt{2}\bar{z}^2 & 0 \end{pmatrix}. \end{aligned}$$

We need to show that they do not stabilize any proper subspace. The eigenvectors of  $\text{ad}(A_1)$  are  $\mathbb{C}(e'_1 + e'_2)$ ,  $\mathbb{C}(e'_1 - e'_2) + \mathbb{C} \cdot e'_3$ . Those of  $\text{ad}(B_1)$  are  $\mathbb{C} \cdot e'_3$  and  $\mathbb{C} \cdot e'_1 + \mathbb{C} \cdot e'_2$ . Hence common eigenvectors for the two are  $\mathbb{C} \cdot e'_3$ ,  $\mathbb{C} \cdot (e'_1 - e'_2)$  and  $\mathbb{C} \cdot (e'_1 + e'_2)$ . But none of them is an eigenvector for  $\text{ad}(B_2)$ .

We need to show that there is no 2-dimensional subspace stabilized by the above three transformations. Assume that  $W$  is such a subspace. Since  $\text{ad}(B_2)$  does not stabilize  $\text{Span}\{e'_1, e'_2\}$ ,  $W$  contains an element  $w = (a, b, c) = ae'_1 + be'_2 + ce'_3$  with  $c \neq 0$ . Consider the following cases.

1.  $a = b = 0$  : Then  $W$  is spanned by  $e'_3$  and  $\lambda e'_1 + \mu e'_2$  for some  $\lambda, \mu \in \mathbb{C}$ . For  $\text{ad}(A_1)$  to stabilize  $\text{Span}\{e'_3, \lambda e'_1 + \mu e'_2\}$ , either  $\lambda = \mu$  or  $\lambda = -\mu$ . But  $\text{Span}\{e'_3, e'_1 + e'_2\}$  and  $\text{Span}\{e'_3, e'_1 - e'_2\}$  are not stabilized by  $\text{ad}(B_2)$ , since

$$\begin{aligned} \text{ad}(B_2)(e'_1 + e'_2) &= \frac{1}{2r^2}(z^2 + \bar{z}^2, -z^2 + \bar{z}^2, -\sqrt{2}(z^2 - \bar{z}^2)) \\ &\notin \text{Span}\{e'_1 + e'_2, e'_3\}, \end{aligned}$$

and

$$\text{ad}(B_2)(e'_3) = \frac{1}{2r^2}(\sqrt{2}r^2, \sqrt{2}r^2, 0) \notin \text{Span}\{e'_1 - e'_2, e'_3\}.$$

2.  $a \neq 0$  : We can assume that  $w$  is not an eigenvector of  $\text{ad}(A_1)$ . (Otherwise  $w$  is of the form  $(1, -1, c)$  and  $W$  is spanned by  $(1, -1, c)$  and  $\text{ad}(B_1)(w) = (-1, 1, c)$ . That is,  $W = \text{Span}\{e'_1 - e'_2, e'_3\}$ ; but we have already seen in case 1 that this is not stabilized by  $\text{ad}(B_2)$ .) Therefore,  $W$  is spanned by  $w$  and  $w' = \text{ad}(A_1)(w) = (b, a, -c)$ . Since  $\text{ad}(B_1)(w) = (-a, -b, c)$  has to be in  $W$ , there are  $x, y \in \mathbb{C}$  such that

$$(-a, -b, c) = xw + yw' = (xa + yb, xb + ya, (x - y)c).$$

From this we get  $x - y = 1$  and  $(x + 1)^2 - y^2 = 0$ , which forces  $x = 0, y = -1$ , and  $a = b$ . But then  $W$  is spanned by  $w = (a, a, c)$  and  $\text{ad}(B_1)(w) = (-a, -a, c)$ . So  $W = \text{Span}\{e'_1 + e'_2, e'_3\}$ , which we have already seen is not stabilized by  $\text{ad}(B_2)$ .

3.  $a = 0, b \neq 0$  : Then  $W$  is spanned by  $w = (0, b, c)$  and  $w' = \text{ad}(A_1)(w) = (b, 0, -c)$ . Also,

$$\text{ad}(B_1)(w) = (0, b, -c) = xw + yw' = (yb, xb, (x - y)c)$$

implies  $y = 0$  and forces  $(0, -b, c) = (0, b, c)$ . Since  $b \neq 0$ , this is impossible.  $\square$

Combining this with Proposition 3, we obtain

**Proposition 5.** *There exist Ad-stable  $\text{SL}(2)$ -bundles over any compact Riemann surface of genus  $> 2$ .*

**Example 2** (A unitary irreducible representation  $\rho$  whose adjoint  $\text{Ad}(\rho)$  is not irreducible). Let

$$A_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and let all  $B_i$  and  $A_j, j > 2$ , be 1. Clearly  $\prod_{i=1}^g [A_i B_i] = 1$ . They form an irreducible set, since they do not have any common eigenvector.

Now, let us take a look at the adjoint representation  $\text{Ad}(\rho) : \Gamma_g \rightarrow \text{GL}(\mathfrak{gl}(2))$ . We use the formula

$$(\text{Ad}(A))_{kl \ ij} = A_{ki} \overline{A_{lj}}$$

with respect to the standard basis, and get

$$\text{Ad}(A_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\text{Ad}(A_2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

With respect to  $\{e_{11} - e_{22}, e_{12}, e_{21}\}$ , one has

$$\text{ad}(A_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \text{ad}(A_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

which is certainly reducible.

*Remark 2.* By Theorem 1 and Proposition 2, this example provides us with a stable  $\text{SL}(2)$ -bundle that is *not* Ad-stable.

## REFERENCES

- [1] B. Balaji and C. S. Seshadri, *Semistable principal bundles, I*, J. Algebra **258** (2002), 321-347. MR **2003m**:14050
- [2] R. Friedman and J. W. Morgan, *Holomorphic principal bundles over elliptic curves*, Preprint (1998), math.AG/9811130.
- [3] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, Inc. (1978). MR **80b**:14001
- [4] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) **82** (1965), 540-567. MR **32**:1725
- [5] D. Hyeon, *Principal bundles over a projective scheme*, Trans. Amer. Math. Soc., **354** (2002), 1899-1908. MR **2003d**:14013
- [6] A. Ramanathan, *Moduli for principal bundles over algebraic curves I, II*, Proc. Indian Acad. Sci. (Math. Sci.), **106** (1996), 301-328 and 421-449. MR **98b**:14009a; MR **98b**:14009b
- [7] A. Ramanathan, *Stable principal bundles on a compact Riemann surface*, Math. Ann. **213** (1975), 129-152. MR **51**:5979

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 W. GREEN STREET, URBANA, ILLINOIS 61801

*Current address:* Department of Mathematics, Rice University, 6100 Main St., Houston, Texas 77005

*E-mail address:* [hyeon@math.rice.edu](mailto:hyeon@math.rice.edu)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 W. GREEN STREET, URBANA, ILLINOIS 61801

*E-mail address:* [dcmurphy@math.uiuc.edu](mailto:dcmurphy@math.uiuc.edu)