PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 8, Pages 2321–2326
S 0002-9939(04)07391-5

THE NUMERICAL RANGE OF A NILPOTENT OPERATOR ON A HILBERT SPACE

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(Communicated by Joseph A. Ball)

Abstract. We prove that the numerical range $W(N)$ of an arbitrary nilpotent operator N on a complex Hilbert space H is a circle (open or closed) with center at 0 and radius not exceeding $\|N\| \cos \frac{\pi}{n+1}$, where n is the power of nilpotency of N.

The purpose of this note is to describe the numerical range of a nilpotent operator on a Hilbert space.

Let H be a complex Hilbert space, with inner product $\langle , \rangle$, and denote by $B(H)$ the algebra of bounded linear operators on H. The numerical range $W(T)$ of an operator $T \in B(H)$ is defined by

$$W(T) = \{\langle Tx, x\rangle : x \in (H)_1\},$$

where $(H)_1 = \{x \in H : \|x\| = 1\}$ is the unit sphere in H. The numerical radius $w(T)$ of an operator $T \in B(H)$ is given by

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

Haagerup and Harpe [HH] proved the following sharp estimate for the numerical radius of a nilpotent operator $N \in B(H)$:

$$w(N) \leq \|N\| \cos \frac{\pi}{n+1}, \tag{1}$$

where n is the power of nilpotency of N. By the Toeplitz-Hausdorff theorem (see [H, GR]) the numerical range $W(N)$ is convex. Then, to prove that $W(N)$ is a disk centered at zero, it suffices to show its circularity, i.e., that each point λ is contained in $W(N)$ with the whole circle $|z| = |\lambda|$, which is essentially contained in the paper [LT] by Li and Tsing.

The present paper presents a new proof of this known result; namely, we give a different proof of circularity of $W(T)$ using the Sz.-Nagy-Foiaş model.

The main result of the paper is

Theorem. *Let H be a complex Hilbert space and let $N \in B(H)$ be a nilpotent operator with power of nilpotency n. Then the numerical range $W(N)$ of the operator N is a circle (open or closed) with center at* 0 *and radius not exceeding $\|N\| \cos \frac{\pi}{n+1}$.*

Received by the editors December 22, 2002 and, in revised form, April 28, 2003.

2000 *Mathematics Subject Classification.* Primary 47A12; Secondary 15A45, 42A05.

Key words and phrases. Numerical range, numerical radius, nilpotent operator, model operator.

As noticed already, by the Toeplitz-Hausdorff theorem and by the inequality (1) of Haagerup and Harpe, it remains to establish circularity of $W(N)$, which will be done using the Sz.-Nagy-Foiaş model.

Let us recall that the characteristic function Θ_T of the contraction $T \in B(H)$ is defined by

$$\Theta_T(\lambda) = \left[-T + \sum_{m=1}^{\infty} \lambda^m D_{T^*} (T^*)^{m-1} D_T \right] \Big| \mathcal{D}_T \quad (\lambda \in \mathbb{D}),$$

where

$$D_T = (I - T^*T)^{\frac{1}{2}}, \quad D_{T^*} = (I - TT^*)^{\frac{1}{2}},$$
$$\mathcal{D}_T = \overline{D_T H}, \qquad \mathcal{D}_{T^*} = \overline{D_{T^*} H},$$

and the series is norm convergent.

Let E_1 and E_2 be Hilbert spaces, $B(E_1, E_2)$ the space of bounded linear operators from E_1 to E_2, and $H^\infty(B(E_1, E_2))$ the space of bounded analytic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ taking values in $B(E_1, E_2)$. We call a function $\Theta \in H^\infty(B(E_1, E_2))$ inner if $\Theta(\zeta)^* \Theta(\zeta) = I_{E_1}$ for almost all $\zeta \in \mathbb{T} = \partial\mathbb{D}$, and $*-$inner if $\Theta(\zeta)\Theta(\zeta)^* = I_{E_2}$ for almost all $\zeta \in \mathbb{T}$. A function that is inner and $*-$inner simultaneously is called a two-sided inner function, where I_{E_i} is an identity operator in $E_i, i = 1, 2$. It is known that $\Theta_T \in H^\infty(B(\mathcal{D}_T, \mathcal{D}_{T^*}))$.

Recall that a contraction $T \in B(H)$ is said [SF] to be of class C_0 if T is completely nonunitary and there is a nonzero function f in H^∞ such that $f(T) = 0$. Then there is a unique (up to a constant factor of modulus one) nonconstant inner function u, called the minimal function of T, such that $u(T) = 0$. It is a well-known result of Sz.-Nagy and Foiaş [SF] that each $T \in C_0$ is unitary equivalent to its model operator $M_\Theta = P_\Theta S_E | \mathcal{K}_\Theta$, acting on the model space $\mathcal{K}_\Theta = H^2(E) \ominus \Theta H^2(E)$. Here $\Theta = \Theta_T \in H^\infty(B(E))$ is the characteristic function of the contraction T (a two-sided inner function); E is a Hilbert space with $\dim E = \dim (I - T^*T) H; H^2(E)$ is the Hardy space of E-valued functions consisting of all Taylor series $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n, z \in \mathbb{D}$, where $\widehat{f}(n) \in E, n \geq 0$, $\sum_{n=0}^{\infty} \left\| \widehat{f}(n) \right\|_E^2 < +\infty; P_\Theta = I - \Theta P_+ \Theta^*$ is the orthogonal projection of $H^2(E)$ onto $\mathcal{K}_\Theta$, where P_+ is the Riesz projector of $L^2(E)$ onto $H^2(E)$, and $S_E f = zf$ is the shift operator on $H^2(E)$.

If $T \in C_0$ and $u \in H^\infty$ is its minimal function, then by the theorem on scalar multiples (see, e.g., [SF, N]) there exists $\Omega \in H^\infty(B(E))$ such that

$$\Theta_T \Omega = \Omega \Theta_T = u I_E.$$

In particular, if T is a nilpotent contraction, $T^n = 0$, $n \geq 2$, and Θ is its characteristic function, then there exists $\Omega \in H^\infty(B(E))$ such that

$$\Theta \Omega = \Omega \Theta = z^n I_E.$$

Therefore, it follows from the inclusion $\Theta \Omega H^2(E) \subset \Theta H^2(E)$ that

$$H^2(E) \ominus \Theta H^2(E) \subset H^2(E) \ominus \Theta \Omega H^2(E) = H^2(E) \ominus z^n H^2(E)$$

or

$$\mathcal{K}_\Theta \subset H^2(E) \ominus z^n H^2(E).$$

Consequently, every $x \in \mathcal{K}_\Theta$ has the form

$$x(z) = \sum_{m=0}^{n-1} \widehat{x}(m) z^m. \tag{2}$$

Then for each $x \in (\mathcal{K}_\Theta)_1$ we have

$$\begin{aligned}\langle M_\Theta x, x\rangle &= \langle P_\Theta z x, x\rangle \\ &= \left\langle z \sum_{m=0}^{n-1} \widehat{x}(m) z^m, \sum_{m=0}^{n-1} \widehat{x}(m) z^m \right\rangle \\ &= \left\langle \sum_{m=0}^{n-1} \widehat{x}(m) z^{m+1}, \sum_{m=0}^{n-1} \widehat{x}(m) z^m \right\rangle \\ &= \sum_{m=0}^{n-2} \left\langle \widehat{x}(m), \widehat{x}(m+1) \right\rangle_E\end{aligned}$$

(E can be identified with a subspace of constant functions in $H^2(E)$). That is,

$$\langle M_\Theta x, x\rangle = \sum_{m=0}^{n-2} \left\langle \widehat{x}(m), \widehat{x}(m+1) \right\rangle_E . \tag{3}$$

Thus, the numerical range $W(M_\Theta)$ of the operator M_Θ is the set

$$W(M_\Theta) = \left\{ \sum_{m=0}^{n-2} \left\langle \widehat{x}(m), \widehat{x}(m+1) \right\rangle_E : x \in (\mathcal{K}_\Theta)_1 \right\}. \tag{4}$$

Proof of the Theorem. We need only to prove the particular case of $\|N\| \le 1$, and since $\left\| \frac{N}{\|N\|} \right\| = 1$, the general result will follow by repeated application of the particular case. Then it is obvious that $W(N) = W(M_\Theta)$ and $w(N) = w(M_\Theta)$, where M_Θ is the model operator of the contraction N and Θ is its characteristic function. Now we prove that the corresponding set (4) is a circular set. In fact, by virtue of the equality (2) we have that

$$y_\zeta(z) \overset{\text{def}}{=} \sum_{m=0}^{n-1} \overline{\zeta}^m \widehat{x}(m) z^m \in \mathcal{K}_{z^n I_E}$$

for each $x = \sum\limits_{m=0}^{n-1} \widehat{x}(m) z^m \in \mathcal{K}_\Theta$ and $\zeta \in \mathbb{T}$, and therefore, for all $h \in H^2(E)$,

$$0 = \langle y_\zeta, z^n h\rangle = \langle y_\zeta, \Omega\Theta h\rangle = \langle \Omega^* y_\zeta, \Theta h\rangle = \langle P_+ \Omega^* y_\zeta, \Theta h\rangle ,$$

which implies that $\Omega^* y_\zeta \in L^2(E) \ominus \Theta H^2(E)$ and $P_+ \Omega^* y_\zeta \in \mathcal{K}_\Theta$. Since $L^2(E) \ominus \Theta H^2(E) = \mathcal{K}_\Theta \oplus H^2_-(E)$, we have that $\Omega^* y_\zeta = P_+ \Omega^* y_\zeta + h_-$, where $h_- \in H^2_-(E)$.

Now let the point $0 \neq \lambda \in W(M_\Theta)$ be realized in the element $x = \sum\limits_{m=0}^{n-1} \widehat{x}(m) z^m \in (\mathcal{K}_\Theta)_1$, i.e., $\lambda = \langle M_\Theta x, x\rangle$. Then, by virtue of (3), for any $t \in [0, 2\pi]$ we have

$$\begin{aligned} e^{it}\lambda = e^{it}\langle M_\Theta x, x\rangle &= e^{it}\sum_{m=0}^{n-2}\left\langle \widehat{x}(m), \widehat{x}(m+1)\right\rangle_E \\ &= \sum_{m=0}^{n-2}\left\langle e^{-imt}\widehat{x}(m), e^{-i(m+1)t}\widehat{x}(m+1)\right\rangle_E \\ &= \sum_{m=0}^{n-2}\left\langle \widehat{y_\zeta}(m), \widehat{y}_\zeta(m+1)\right\rangle_E = \langle zy_\zeta, y_\zeta\rangle, \end{aligned}$$

where

$$y_\zeta(z) = \sum_{m=0}^{n-1}\widehat{y_\zeta}(m) z^m \in (\mathcal{K}_{z^n I_E})_1,$$

$$\widehat{y_\zeta}(m) = \overline{\zeta}^m\widehat{x}(m) = e^{-imt}\widehat{x}(m) \quad (m \geq 0).$$

On the other hand,

$$\begin{aligned} \langle M_\Theta P_+\Omega^* y_\zeta, P_+\Omega^* y_\zeta\rangle &= \langle P_\Theta z P_+\Omega^* y_\zeta, P_+\Omega^* y_\zeta\rangle = \langle zP_+\Omega^* y_\zeta, P_+\Omega^* y_\zeta\rangle \\ &= \langle zP_+\Omega^* y_\zeta, \Omega^* y_\zeta\rangle = \langle z(\Omega^* y_\zeta - h_-), \Omega^* y_\zeta\rangle \\ &= \langle z\Omega^* y_\zeta, \Omega^* y_\zeta\rangle - \langle zh_-, \Omega^* y_\zeta\rangle \\ &= \langle zy_\zeta, y_\zeta\rangle - \langle zh_-, \Omega^* y_\zeta\rangle \\ &= e^{it}\langle M_\Theta x, x\rangle - \langle zh_-, \Omega^* y_\zeta\rangle = e^{it}\lambda - \langle zh_-, \Omega^* y_\zeta\rangle. \end{aligned}$$

Thus

$$e^{it}\lambda = \left\|P_+\Omega_\zeta^* y\right\|^2\left\langle M_\Theta\frac{P_+\Omega^* y_\zeta}{\|P_+\Omega^* y_\zeta\|}, \frac{P_+\Omega^* y_\zeta}{\|P_+\Omega^* y_\zeta\|}\right\rangle + \langle zh_-, \Omega^* y_\zeta\rangle,$$

or

$$e^{it}\lambda = \left(1 - \|h_-\|^2\right)\left\langle M_\Theta\frac{P_+\Omega^* y_\zeta}{\|P_+\Omega^* y_\zeta\|}, \frac{P_+\Omega^* y_\zeta}{\|P_+\Omega^* y_\zeta\|}\right\rangle + \langle zh_-, \Omega^* y_\zeta\rangle. \tag{5}$$

We shall prove that there exists $x_0 \in (\mathcal{K}_\Theta)_1$ such that

$$\langle zh_-, \Omega^* y_\zeta\rangle = \|h_-\|^2\langle M_\Theta x_0, x_0\rangle, \tag{6}$$

and so formulas (5) and (6) will give the desired inclusion $e^{it}\lambda \in W(M_\Theta)$, since $\|h_-\|^2 < 1$ and $W(M_\Theta)$ is convex.

First recall that if F_1, F_2 are two subspaces of H, the angle between F_1 and F_2 is the number $\alpha = \alpha_{F_1,F_2}, 0 \leq \alpha \leq \frac{\pi}{2}$, such that

$$\cos\alpha = \sup_{x_1 \in F_1, x_2 \in F_2}\frac{|\langle x_1, x_2\rangle|}{\|x_1\|\,\|x_2\|}.$$

Since $\ker M_\Theta \neq \{0\}$, then $0 \in W\left(\|h_-\|^2 M_\Theta\right)$. From the convexity of numerical range it follows that $W\left(\|h_-\|^2 M_\Theta\right)$ contains the line segment $\left[0, \|h_-\|^2\lambda\right]$, and moreover, $\overline{\mathbb{D}}_r \overset{\text{def}}{=} \{z : |z| \leq r\} \subset W\left(\|h_-\|^2 M_\Theta\right)$ for some $r, 0 < r < \|h_-\|^2|\lambda|$. Let $L_1 \subset L^2(E)$ and $L_2 \subset L^2(E)$ be the subspaces such that $zh_- \in L_1$ and $\Omega^* y_\zeta \in L_2$. Recall that two contractive functions $\Theta_1 \in H^\infty(B(E_1, E_2)), \Theta_2 \in$

$H^\infty\left(B\left(E_1', E_2'\right)\right)$ coincide (see [SF], Chapter 5) if there exist unitary operators $\tau_1 : E_1 \to E_1'$ and $\tau_2 : E_2 \to E_2'$ such that $\Theta_2(z) = \tau_2 \Theta_1(z) \tau_1^{-1}, |z| < 1$. It is well known that two completely nonunitary contractions are unitarily equivalent if and only if their characteristic functions coincide. Therefore, we can assume the function Ω and the subspaces L_1, L_2 are such that $\|h_-\| \cos\alpha_{L_1,L_2} \le r$. Then we have

$$\begin{aligned} |\langle zh_-, \Omega^* y_\zeta\rangle| &= \|zh_-\| \frac{|\langle zh_-, \Omega^* y_\zeta\rangle|}{\|zh_-\| \, \|\Omega^* y_\zeta\|} \\ &\le \|h_-\| \sup_{x_1 \in L_1, x_2 \in L_2} \frac{|\langle x_1, x_2\rangle|}{\|x_1\| \, \|x_2\|} \\ &= \|h_-\| \cos\alpha_{L_1,L_2} \le r, \end{aligned}$$

which shows that $\langle zh_-, \Omega^* y_\zeta\rangle \in \overline{\mathbb{D}}_r$. Since $\overline{\mathbb{D}}_r \subset W\left(\|h_-\|^2 M_\Theta\right)$, there exists $x_0 \in (\mathcal{K}_\Theta)_1$ such that

$$\langle zh_-, \Omega^* y_\zeta\rangle = \left\langle \|h_-\|^2 M_\Theta x_0, x_0 \right\rangle = \|h_-\|^2 \langle M_\Theta x_0, x_0\rangle .$$

Consequently, $e^{it}\lambda \in W(M_\Theta)$ for each $t \in [0, 2\pi]$, which means that $W(M_\Theta)$ is a circular set. On the other hand, since the numerical range is convex, $W(M_\Theta)$ is a circle (open or closed) with center at zero, which completes the proof. □

Corollary. *If $N \in B(H)$ is a compact nilpotent operator, then $W(N)$ is a closed disk centered at 0 and with radius not greater than $\|N\| \cos\frac{\pi}{n+1}$.*

Proof. It follows from the result of Lancaster [L] that if $\mathcal{K} \in B(H)$ is compact, then $W(\mathcal{K})$ is closed if and only if $0 \in W(\mathcal{K})$. It remains only to apply the theorem. □

Example 1. *Let $(V_0 f)(x) = \int_{-x}^{x} f(t)\,dt$ be a skew-symmetric Volterra operator acting on the space $L^2(-1, 1)$. Then its numerical range $W(V_0)$ is a closed circle with center at 0 and radius $\frac{2}{\pi}$, i.e., $W(V_0) = \overline{\mathbb{D}}_{\frac{2}{\pi}}$.*

Proof. Observe that the set of values of the operator V_0 is contained in the set of all odd functions from the space $L^2(-1, 1)$ and for the odd function $f, V_0 f = 0$. This means that $V_0^2 = 0$, i.e., V_0 is a nilpotent operator with the power of nilpotency 2. Then by our theorem, $W(V_0)$ is a circle with center at zero. Since V_0 is a compact operator and $0 \in W(V_0)$, by the corollary, $W(V_0)$ is a closed set. So, $W(V_0)$ is a closed circle. Now we calculate its radius. By inequality (1), $w(V_0) \le \frac{\|V_0\|}{2}$. On the other hand, $w(V_0) \ge \frac{\|V_0\|}{2}$, and hence $w(V_0) = \frac{\|V_0\|}{2}$. Since $\|V_0\| = \frac{4}{\pi}$ (see [H]), we have that $w(V_0) = \frac{2}{\pi}$. This completes the proof. □

The following example is easy to verify.

Example 2. *Consider the diagonal operator $D = \operatorname{diag}\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$ and $N = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}$ acting in ℓ^2. Then N is not compact, $N^2 = 0$, and $W(N)$ is the open disk centered at 0 with radius $\frac{1}{2}$, i.e., $W(N) = \mathbb{D}_{\frac{1}{2}}$.*

Remark. Recall that the k-numerical range $W_k(A)$ of the operator $A \in B(H)$ is defined by

$$W_k(A) = \{\lambda \in \mathbb{C} : \lambda = \sum_{i=1}^{k} \langle Ax_i, x_i \rangle, \quad \text{where } \{x_1, ..., x_k\} \text{ are } k \text{ orthonormal vectors in } H\}.$$

When $k = 1$, this reduces to the classical numerical range $W(A)$. The most fundamental property of the k-numerical range is its convexity (see [H]). Using this property and arguments similar to those in the proof of the theorem, we can also obtain the analogous characterization for the k-numerical range $W_k(N)$ of a nilpotent operator $N \in B(H)$, which is also contained among other general results of the paper [LT].

I am grateful to the referee for his constructive suggestions and for pointing out the reference [LT].

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