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THERE ARE $2^{\aleph_0}$ MANY H-DEGREES IN THE RANDOM REALS

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ABSTRACT. We prove that there are $2^{\aleph_0}$ many H-degrees in the random reals.

1. Introduction

We work in Cantor space 2^ω with topology whose basic clopen sets are $I_\sigma = \{\sigma\alpha : \alpha \in 2^\omega\}$, having Lebesgue measure $2^{-|\sigma|}$.

In this paper, we will denote the prefix-free Kolmogorov complexity of a string σ by $H(\sigma)$. Schnorr proved that a real x is Martin-Löf random iff $H(x \upharpoonright n) > n+O(1)$ for every n. This characterization allows us to describe the relative randomness of reals by measuring their initial segment complexity. Following Downey, Hirschfeldt and Nies, [2], we say that $A \leq_H B$ iff there is a constant c such that for all n, $H(A \upharpoonright n) \leq H(B \upharpoonright n) + c$. (This notion is essentially implicit in the work of Solovay [7].) Then this allows us to calibrate the reals by a natural measure of relative randomness. This notion is a pre-ordering, and the equivalence classes are called H-degrees. The structure of the reals under this pre-ordering is poorly understood.

It is somewhat understood for the computably enumerable reals, which are those that are limits of computable sequences of rationals. Equivalently, a real is c.e. iff it is the measure of the domain of a prefix-free machine. These are the natural analogs of c.e. sets in classical computability theory. Downey, Hirschfeldt, Nies and Stephan ([2] and [3]) have shown that for the c.e. reals, the H-degrees form a dense upper semi-lattice with arithmetic addition inducing a join operation, and following from a result Kučera and Slaman ([5]) that the H-degree of Chaitin's Ω is the top degree. Recently Downey and Wu [4] have shown that there are minimal pairs in this structure. It is not even known that it is not a lattice.

The structure of the H-degrees of the class of all reals is wide open, with even fundamental questions such as the one we address here being open. How many of them are there? Moreover, how many H-degrees are there in the random reals? In this paper we will answer this fundamental question.

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In the realm of c.e. reals, all random reals look alike in a very strong sense. The analytic analog of m-reducibility is *Solovay* or *domination* reducibility, where $\alpha \leq_S \beta$ iff there are a partial computable function f and a constant c such that, for all rationals $q < \beta$, $f(q) \downarrow < \alpha$, and $c(\beta - q) > \alpha - f(q)$. Solovay [7] proved that $\alpha \leq_S \beta$ implies that $\alpha \leq_H \beta$. Kučera and Slaman proved the following analog of Myhill's theorem for c.e. reals.

Theorem 1.1 (Kučera and Slaman [5]). *If a c.e. real α is Martin-Löf and random, then α is Solovay complete.*

Since Solovay reducibility implies H-reducibility, Theorem 1.1 means that there is only one H-degree in the c.e. random reals. Does the Kučera-Slaman phenomenon hold for all random reals? The point here is that most treatments of randomness are "zero-one" in the sense that all one is concerned with is whether the real in question is random or not. Thus it is conceivable that all random reals occupy a single H-degree. The question becomes clearly nontrivial when we look at the cardinality of a single degree. Despite the fact that the trivial H-degree (the degree consisting of reals with initial segment complexity identical with 1^ω) is countable, there are $2^{\aleph_0}$ many reals $\beta \leq_H \Omega$ ([8]).

In 1975, Solovay [7] proved a result, a consequence of which is that there exist at least two H-degrees in the random reals, and a refined version of Solovay's result can be found in Yu, Ding and Downey [8].

Theorem 1.2 (Solovay [7], Yu, Ding and Downey [8]). *There exist at least two H-degrees of random sets. Indeed there exists a random α such that*

$$\limsup_{n \in \mathbb{N}} (H(\alpha \upharpoonright n) - H(\Omega \upharpoonright n)) = \infty.$$

Yu, Ding and Downey proved that there are many H-degrees in the random reals.

Theorem 1.3 (Yu, Ding and Downey [8]). *For every real x, $\mu(\{y \mid y \leq_H x\}) = 0$, and so there are at least $\aleph_1$ many H-degrees in the random reals*

In this paper, we settle the cardinality of the H-degrees of the collection of random reals by the following theorem. A crucial ingredient in the proof is Theorem 1.3 above.

Theorem 1.4. *There exist $2^{\aleph_0}$ many H-degrees in the random reals. Indeed, these may be chosen to be an antichain.*

We denote the interval $\{\sigma x : x \in 2^\omega\}$ by I_σ for any $\sigma \in 2^{<\omega}$. For $\sigma, \tau \in 2^{<\omega}$ (or 2^ω), $\sigma \subset \tau$ means that σ is an initial segment of τ and $\sigma < \tau$ means that there exists $\sigma' \in 2^{<\omega}$ so that $\sigma'^\frown \langle 0 \rangle \subseteq \sigma$ but $\sigma'^\frown \langle 1 \rangle \subseteq \tau$. Our notation is relatively standard, and the reader may refer to the forthcoming book [1].

2. Proof of Theorem 1.4

We need some facts about Cantor space.

Fact 2.1. For any set $A \subseteq 2^\omega$ with $\mu(A) > 0$, there is a closed set $C \subseteq A$ so that $\mu(C) > 0$.

Fact 2.2. For any set $A \subseteq 2^\omega$ with $\mu(A) > 0$, there is a real $x \in A$ so that for any $\sigma \subset x$, $\mu(A \cap I_\sigma) > 0$.

Proof. If not, then for every $x \in A$ there must be a $\sigma_x \subset x$ so that $\mu(A \cap I_{\sigma_x}) = 0$. Since $I = \bigcup_{x \in A} I_{\sigma_x} \supseteq A$, this implies $\mu(A) \le \mu(I \cap A) \le \sum_{x \in A} \mu(I_{\sigma_x} \cap A) = 0$, a contradiction. □

We now turn to the proof of the main result.

We will construct $2^{\aleph_0}$ many random H-degrees. We denote the set $\{x : x$ is a random real$\}$ by *Ran*. Fact 2.1 says that there is a closed set $C \subseteq Ran$ with $\mu(C) > 0$. From now on, the closed set C is fixed.

We construct a tree, stage by stage, $(\bigcup_s T_s, \subset) = (T, \subset) \subseteq (2^{<\omega}, \subset)$ and sets $\{D_\sigma\}_{\sigma \in T}$ so that for every $s > 0$ the following hold:

(i) $D_\varnothing = C$ and $D_\sigma \subseteq I_\sigma$ for every $\sigma \in T_s$;

(ii) $\mu(D_\sigma) > 0$ for every $\sigma \in T_s$;

(iii) $D_\sigma \supseteq D_\tau$ if $\sigma, \tau \in T_s$ and $\sigma \subset \tau$;

(iv) $\forall s \forall \sigma \in T_s \exists \tau_0 \exists \tau_1 [\tau_0 \in T_{s+1} \& \tau_1 \in T_{s+1} \& \tau_0, \tau_1 \supset \sigma \& \tau_0 \not\subseteq \tau_1 \& \tau_1 \not\subseteq \tau_0]$;

(v) $\forall s \forall \tau_0, \tau_1 \in T_{s+1} - T_s [|\tau_0| = |\tau_1| \& (\tau_0 < \tau_1 \longrightarrow \exists n (H(\tau_0 \restriction n) + s \leqslant H(\tau_1 \restriction n))]$.

Note 1. T may not be closed under initial segments, and so is different from the trees in the usual sense.

Note 2. The argument is more or less similar to the Cantor-Bendixson argument used in descriptive set theory, by which they proved that every uncountable closed real set has a nonempty perfect subset. (The reader can refer to Moschovakis's book [6].) The reader may mind that we replace "countable" by "has measure 0" and "uncountable" by "has measure larger than 0" here.

If this construction is successful, we set $R = [T] = \{x : x$ is an infinite path in $T\}$. By (i), (iii) and (iv), $|R| = 2^{\aleph_0}$ and $R \subseteq C$ since C is closed. But for every pair of reals $x_0 < x_1 \in R$, by (v), $\limsup_n (H(x_1 \restriction n) - H(x_0 \restriction n)) = \infty$. This will finish the proof.

We begin to construct these sets.

At stage 0, set $D_\varnothing = C$.

At stage $s+1$, suppose that we constructed the tree $(\bigcup_{t<s+1} T_t, \subset)$ and the sets $\{D_\sigma\}_{\sigma \in \cup_{t<s+1} T_t}$ that satisfy (i)–(v). Enumerate $T_s - T_{s-1}$ as $\{\sigma_i\}_{i \le n}$ with $\sigma_i < \sigma_j$ if $i < j$. We proceed by induction on i to define appropriate extensions $\tau_{i0}, \tau_{i1} \supset \sigma_i$ and reals $x_{ij} \in I_{\tau_{ij}} \cap D_{\sigma_i}$. We then use these to define T_{s+1}.

Since $\mu(D_{\sigma_0}) > 0$, we can find two strings $\tau_{00}, \tau_{01} \supset \sigma_0$ with $\tau_{00} < \tau_{01}$ so that $\mu(D_{\sigma_0}) > \mu(I_{\tau_{00}} \cap D_{\sigma_0}), \mu(I_{\tau_{01}} \cap D_{\sigma_0}) > 0$. By Fact 2.2, there is a real $x_{00} \in I_{\tau_{00}} \cap D_{\sigma_0}$ so that for any $\tau \subset x_{00}$, $\mu(I_{\tau_{00}} \cap D_{\sigma_0} \cap I_\tau) > 0$. By Theorem 1.3, $\mu(\{y : y \le_H x_{00}\}) = 0$, and so $\mu(\{y : y \not\le_H x_{00}\} \cap I_{\tau_{01}} \cap D_{\sigma_0}) > 0$. Thus by Fact 2.2, select a real $x_{01} \in I_{\tau_{01}} \cap D_{\sigma_0}$ so that $\limsup_n (H(x_{01} \restriction n) - H(x_{00} \restriction n)) = \infty$.

Suppose for every $i < k \le n$, we can find intervals $I_{\tau_{i0}}$ and $I_{\tau_{i1}}$ ($\tau_{i0} < \tau_{i1}$ and $\tau_{i0}, \tau_{i1} \supset \sigma_i$) with $\mu(D_{\sigma_i}) > \mu(I_{\tau_{i0}} \cap D_{\sigma_i}), \mu(I_{\tau_{i1}} \cap D_{\sigma_i}) > 0$ and reals $x_{i0} < x_{i1}$ with $x_{i0} \in I_{\tau_{i0}} \cap D_{\sigma_i}$ and $x_{i1} \in I_{\tau_{i1}} \cap D_{\sigma_i}$ so that for any $\tau \subset x_{ij}$, $\mu(I_{\tau_{ij}} \cap D_{\sigma_i} \cap I_\tau) > 0$ ($j = 0, 1$) and for any $x_{ij} < x_{i'j'}$, $\limsup_n (H(x_{ij} \restriction n) - H(x_{i'j'} \restriction n)) = \infty$.

Since $\mu(D_{\sigma_k}) > 0$, there exist two strings $\tau_{k0}, \tau_{k1} \supset \sigma$ with $\tau_{k0} < \tau_{k1}$ so that

$$\mu(D_{\sigma_k}) > \mu(I_{\tau_{k0}} \cap D_{\sigma_k}), \mu(I_{\tau_{k1}} \cap D_{\sigma_k}) > 0.$$

Let $j = 0$. By Theorem 1.3, $\mu(\{y : y \le_H x_{ij'}\}) = 0$ for every $i < k$ and $j' = 0, 1$, and so

$$\mu(\bigcup_{i<k, j'=0,1} \{y : y \not\le_H x_{ij'}\} \cap I_{\tau_{kj}} \cap D_{\sigma_{kj}}) > 0.$$

By Fact 2.2, there is a real $x_{kj} \in I_{\tau_{kj}} \cap D_{\sigma_k}$ so that $\mu(I_{\tau_{kj}} \cap D_{\sigma_k} \cap I_\tau) > 0$ and $x_{kj} \not\leq_H x_{ij'}$ for any $\tau \subset x_{kj}$ (i.e., $\limsup_n (H(x_{kj} \restriction n) - H(x_{ij'} \restriction n)) = \infty$) for every $i < k$ and $j' = 0, 1$. If $j = 1$, then reasoning as above, by Theorem 1.3 and Fact 2.2, there is a real $x_{kj} \in I_{\tau_{kj}} \cap D_{\sigma_k}$ so that for any $\tau \subset x_{kj}$, $\mu(I_{\tau_{kj}} \cap D_{\sigma_k} \cap I_\tau) > 0$ and $\limsup_n (H(x_{kj} \restriction n) - H(x_{ij'} \restriction n)) = \infty$ for any $i < k$ and $j' = 0, 1$, and $\limsup_n (H(x_{kj} \restriction n) - H(x_{k(1-j)'} \restriction n)) = \infty$.

Thus there are $2n$ many reals $\{x_{ij}\}_{0 \leq i \leq n, 0 \leq j \leq 1}$ so that

(1) $\limsup_n (H(x_{ij} \restriction n) - H(x_{i'j'} \restriction n)) = \infty$ for any $x_{ij} > x_{i'j'}$;

(2) $x_{ij} \in I_{\tau_{ij}} \cap D_{\sigma_i}$;

(3) for every $\tau \subset x_{ij}$, $\mu(I_{\tau_{ij}} \cap D_{\sigma_i} \cap I_\tau) > 0$ $(j = 0, 1)$.

By (1), for every pair of $x_{ij} > x_{i'j'}$, there is a number $m_{iji'j'}$ so that

$$H(x_{ij} \restriction m_{iji'j'}) - H(x_{i'j'} \restriction m_{iji'j'}) \geq s.$$

Define $m = \max\{m_{iji'j'} : 0 \leq i, i' \leq n \text{ and } 0 \leq j, j' \leq 1\} + 1$. Set $\sigma_{ij} = x_{ij} \restriction m$ and $D_{\sigma_{ij}} = D_{\sigma_i} \cap I_{\sigma_{ij}}$. By (2) and (3), $\mu(D_{\sigma_i}) > \mu(D_{\sigma_{ij}}) > 0$ for any $0 \leq i \leq n$ and $0 \leq j \leq 1$. Define $T_{s+1} = T_s \cup \{\sigma_{ij} : i \leq n, j \leq 1\}$.

Now it is easy to check (i)–(v).

This concludes the proof that there are $2^{\aleph_0}$ many H-degrees.

The extension to making the degrees all H-incomparable is not hard (but tedious), by a trivial extension of our argument. To do this we would redo the process back the other way at every stage.

References

[1] Rodney G. Downey and Denis Hirschfeldt. *Algorithmic Randomness and Complexity*, in preparation.

[2] Rodney Downey, Denis Hirschfeldt, and André Nies, *Randomness, computability and density*, SIAM Journal of Computing **31** (2002), 1169–1183. MR **2003f:**03057

[3] Rodney G. Downey, Denis R. Hirschfeldt, André Nies and Frank Stephan, Trivial reals. *Electronic Notes in Theoretical Computer Science (ENTCS)*, 2002.

[4] Rod Downey and Guohua Wu, personal communication.

[5] Antonin Kučera and Theodore A. Slaman, *Randomness and recursive enumerability*, SIAM J. Comput. **31** (2001), 199–211. MR **2002k:**68078

[6] Yiannis N. Moschovakis, *Desciptive Set Theory*, North-Holland, Amsterdam, 1980. MR **82e:**03002

[7] Robert M. Solovay, *Draft of paper (or series of papers) on Chaitin's work*, unpublished notes, May, 1975, 215 pages.

[8] Liang Yu, Decheng Ding, and Rod Downey. *The complexity of the random reals*, submitted.

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