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THE EQUATIONS DETERMINING INTERMEDIATE INTEGRALS FOR MONGE-AMPÈRE PDE

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ABSTRACT. In this note we will find the differential equations determining the intermediate integrals for Monge-Ampère equations in an arbitrary number of variables.

An intermediate integral of a PDE system $\mathcal{E}$ is another PDE system of lower order, whose solutions are also solutions of $\mathcal{E}$. The Goursat method for integrating classical Monge-Ampère (second-order) equations is based on the search of intermediate integrals [2]. In this paper we will obtain a computable criterion which determines when a first-order PDE is an intermediate integral of a (generalized) Monge-Ampère equation in an arbitrary number of variables. The derivation of this criterion relies only on well-known properties of first-order PDE systems and some elementary linear algebra.

1. PRELIMINARIES: FIRST-ORDER PDE IN A SINGLE UNKNOWN

In this section we will recall a few notions about first-order PDEs. For further information refer to [3] and [5].

Let M be an n-dimensional manifold and J^1M the bundle of 1-jets of functions $M \to \mathbb{R}$. J^1M is a contact manifold endowed with the canonical 1-form θ which in appropriate local coordinates $\{x_i, z, p_i\}$, $1 \leq i \leq n$, has the expression $\theta = dz - \sum_{i=1}^n p_i dx_i$.

Definition 1.1. A *first-order PDE in a single unknown* in n variables is given by means of a hypersurface $\mathcal{E}_F = \{F = 0\} \subset J^1M$, $F \in \mathcal{C}^\infty(J^1M)$. A (generalized) *solution* of the equation $\mathcal{E}_F$ is an n-dimensional solution of the exterior differential system generated by F and θ.

From now on, we will take a fixed function $F \in \mathcal{C}^\infty(J^1M)$, and we will consider the family of equations $\mathcal{E}_{F+const}$ (one for each chosen constant).

Definition 1.2. A point $e \in J^1M$ is called *non-singular* for F if d_eF and θ_e are linearly independent. Let us denote the set of non-singular points of F by U_F. A solution Z of $\mathcal{E}_{F+const}$ is said to be *non-singular* if $Z \subset U_F$.

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In the following two lemmas we will recall some well-known facts which will be needed in the next section.

Lemma 1.3. *Let $e \in U_F$, and let us denote by $C_{F,e}$ the linear span of the tangent spaces T_eZ, where Z is a solution of $\mathcal{E}_{F-F(e)}$. Then,*

$$C_{F,e} = \{D \in T_eJ^1M \mid i_D\theta_e = i_D d_e F = 0\}. \tag{1}$$

Lemma 1.4. *There is a unique contact tangent field $\overline{X}_F$ such that $i_{\overline{X}_F}\theta = F$. In addition, $X_F = F\overline{X}_1 - \overline{X}_F$ generates the characteristic system of the Pfaff system $\{\theta, dF\}$ and also*

1. $X_F F = i_{X_F} dF = 0$,
2. $i_{X_F}\theta = 0$,
3. $(X_F)_e \neq 0$ *for every* $e \in U_F$,
4. X_F *is tangent to any solution of each equation* $\mathcal{E}_{F+const}$.

2. Intermediate integrals for Monge-Ampère equations

In this section we will recall the definition of Monge-Ampère equations and prove our main result.

Definition 2.1. Let ω be a differential n-form on J^1M. The *Monge-Ampère equation* defined by ω is given by the exterior differential system

$$\mathcal{E}_\omega \overset{\text{def}}{=} \{\theta, \omega\}.$$

Thus, an n-dimensional submanifold $Z \subset J^1M$ is a solution of $\mathcal{E}_\omega$ if

$$\omega|_Z = 0, \qquad \theta|_Z = 0.$$

Definition 2.2. A function $F \in \mathcal{C}^\infty(J^1M)$ is said to be an *intermediate integral* of $\mathcal{E}_\omega$ if the non-singular solutions of $\mathcal{E}_{F+const}$ are also solutions of $\mathcal{E}_\omega$ for every constant *const*.

Let ω be a differential n-form on J^1M and $F \in \mathcal{C}^\infty(J^1M)$. On a neighborhood of each point in U_F we choose a coframe $\{\theta, dF, \sigma_1, \ldots, \sigma_{2n-1}\}$. This way we can write

$$\omega = \theta \wedge \lambda + dF \wedge \mu + \omega' \tag{2}$$

where ω' does not contain θ or dF. Moreover, $\omega|_Z = \omega'|_Z$ when Z is a solution of $\mathcal{E}_{F+const}$.

Theorem 2.3. *A function $F \in \mathcal{C}^\infty(J^1M)$ is an intermediate integral of $\mathcal{E}_\omega$ if and only if*

$$\theta \wedge dF \wedge i_{X_F}\omega = 0 \tag{3}$$

on the open set U_F.

Proof. Without loss of generality, we can work over an open subset $V \subset U_F$ where the above coframe was defined.

From (2) and items (1)–(2) of Lemma 1.4 we derive that (3) is equivalent to

$$i_{X_F}\omega' = 0. \tag{4}$$

Let us assume that (3) (or (4)) holds. If Z is a solution of $\mathcal{E}_{F+const}$ we can restrict X_F to Z (item (4) of Lemma 1.4). Then

$$i_{X_F|_Z}\omega'|_Z = (i_{X_F}\omega')|_Z = 0.$$

On the other hand, the inner contraction of a non-vanishing n-form by a non-vanishing tangent field (item (3) of Lemma 1.4) on an n-dimensional manifold cannot be zero at any point. As a result we obtain $\omega'|_Z = 0$ and therefore, $\omega|_Z = 0$. In other words, F is an intermediate integral of $\mathcal{E}_\omega$.

Conversely, let F be an intermediate integral of $\mathcal{E}_\omega$. This way

$$(i_{X_F}\omega')|_Z = i_{X_F|_Z}\omega'|_Z = i_{X_F|_Z}\omega|_Z = 0$$

for every solution Z of $\mathcal{E}_{F+const}$.

By the definition of $C_{F,e}$, we can also obtain $(i_{X_F}\omega')|_{C_{F,e}} = 0$, for each point $e \in V$. In addition, $C_{F,e}$ is the incident subspace to θ_e and d_eF (Lemma 1.3). Recall, also, that ω'_e does not contain θ_e or d_eF (in the coframe we are using). In conclusion, $(i_{X_F}\omega')_e = 0$ for every e. This way, $i_{X_F}\omega' = 0$ and thus (3) holds. $\square$

Remarks 2.4. 1) Condition (3) is a first-order PDE for the unknown function F. As a consequence, we can find solutions of $\mathcal{E}_\omega$ by solving first-order equations: equation (3) and $\mathcal{E}_{F+const}$.

2) Any contact manifold is locally equivalent to J^1M for an appropriate M. Thus, all the results in this paper are locally valid for arbitrary contact manifolds. As a matter of fact, if a global contact structure 1-form θ exists, the field X_F can be defined and the result can be directly translated to this case.

3) A criterion was also given by Zil'bergleĭt [6] who uses the more sophisticated framework of Lychagin [4] on contact geometry: effective forms, $\mathfrak{sl}(2)$-representations, Hodge-Lepage expansions, operators $\top$ and $\perp$, etc. In addition, the criterion in [6], unlike Theorem 2.3, must be applied only on "effective" forms ω.

4) We can define an intermediate integral of $\mathcal{E}_\omega$ in a more general sense, that is, as a function F such that all the solutions of $\mathcal{E}_F$ are also solutions of $\mathcal{E}_\omega$ (i.e., just $const = 0$ is considered in Definition 2.2). Then, Theorem 2.3 should be modified by requiring $\theta \wedge dF \wedge i_{X_F}\omega = 0$ only at the points of the hypersurface $\{F = 0\} \subset J^1M$.

5) Let us consider the case $\dim M = 2$, $\omega = \omega^1 \wedge \omega^2$. If dF belongs to the characteristic system $\{\theta, \omega^1, \omega^2\}$, then $\theta \wedge dF \wedge \omega = 0$. By applying Theorem 2.3 we can see that F is an intermediate integral of $\mathcal{E}_\omega$. This way, we recover the criterion given for example in [1] (where a detailed analysis on the possibility of singular solutions was done). However, this result states just a sufficient condition. For example, it is not difficult to check that $F = p_1 + x_2$ is an intermediate integral for the equation

$$\frac{\partial^2 z}{\partial x_1^2}\frac{\partial^2 z}{\partial x_2^2} - \left(\frac{\partial^2 z}{\partial x_1 \partial x_2}\right)^2 + 1 = 0,$$

which is obtained from $\omega = \omega^1 \wedge \omega^2$, $\omega^1 = dp_2 + dx_1$, and $\omega^2 = -dp_1 + dx_2$. Nevertheless, $dF \notin \{\theta, \omega^1, \omega^2\}$ (on the other hand, $i_{X_F}\omega = 0$ and definitely Theorem 2.3 can be applied).

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