

THE EQUATIONS DETERMINING INTERMEDIATE INTEGRALS FOR MONGE-AMPÈRE PDE

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ABSTRACT. In this note we will find the differential equations determining the intermediate integrals for Monge-Ampère equations in an arbitrary number of variables.

An intermediate integral of a PDE system \mathcal{E} is another PDE system of lower order, whose solutions are also solutions of \mathcal{E} . The Goursat method for integrating classical Monge-Ampère (second-order) equations is based on the search of intermediate integrals [2]. In this paper we will obtain a computable criterion which determines when a first-order PDE is an intermediate integral of a (generalized) Monge-Ampère equation in an arbitrary number of variables. The derivation of this criterion relies only on well-known properties of first-order PDE systems and some elementary linear algebra.

1. PRELIMINARIES: FIRST-ORDER PDE IN A SINGLE UNKNOWN

In this section we will recall a few notions about first-order PDEs. For further information refer to [3] and [5].

Let M be an n -dimensional manifold and J^1M the bundle of 1-jets of functions $M \rightarrow \mathbb{R}$. J^1M is a contact manifold endowed with the canonical 1-form θ which in appropriate local coordinates $\{x_i, z, p_i\}$, $1 \leq i \leq n$, has the expression $\theta = dz - \sum_{i=1}^n p_i dx_i$.

Definition 1.1. A first-order PDE in a single unknown in n variables is given by means of a hypersurface $\mathcal{E}_F = \{F = 0\} \subset J^1M$, $F \in \mathcal{C}^\infty(J^1M)$. A (generalized) solution of the equation \mathcal{E}_F is an n -dimensional solution of the exterior differential system generated by F and θ .

From now on, we will take a fixed function $F \in \mathcal{C}^\infty(J^1M)$, and we will consider the family of equations $\mathcal{E}_{F+const}$ (one for each chosen constant).

Definition 1.2. A point $e \in J^1M$ is called *non-singular* for F if $d_e F$ and θ_e are linearly independent. Let us denote the set of non-singular points of F by U_F . A solution Z of $\mathcal{E}_{F+const}$ is said to be *non-singular* if $Z \subset U_F$.

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In the following two lemmas we will recall some well-known facts which will be needed in the next section.

Lemma 1.3. *Let $e \in U_F$, and let us denote by $C_{F,e}$ the linear span of the tangent spaces $T_e Z$, where Z is a solution of $\mathcal{E}_{F-F(e)}$. Then,*

$$(1) \quad C_{F,e} = \{D \in T_e J^1 M \mid i_D \theta_e = i_D d_e F = 0\}.$$

Lemma 1.4. *There is a unique contact tangent field \overline{X}_F such that $i_{\overline{X}_F} \theta = F$. In addition, $X_F = F\overline{X}_1 - \overline{X}_F$ generates the characteristic system of the Pfaff system $\{\theta, dF\}$ and also*

- (1) $X_F F = i_{X_F} dF = 0,$
- (2) $i_{X_F} \theta = 0,$
- (3) $(X_F)_e \neq 0$ for every $e \in U_F,$
- (4) X_F is tangent to any solution of each equation $\mathcal{E}_{F+const}.$

2. INTERMEDIATE INTEGRALS FOR MONGE-AMPÈRE EQUATIONS

In this section we will recall the definition of Monge-Ampère equations and prove our main result.

Definition 2.1. Let ω be a differential n -form on $J^1 M$. The Monge-Ampère equation defined by ω is given by the exterior differential system

$$\mathcal{E}_\omega \stackrel{\text{def}}{=} \{\theta, \omega\}.$$

Thus, an n -dimensional submanifold $Z \subset J^1 M$ is a solution of \mathcal{E}_ω if

$$\omega|_Z = 0, \quad \theta|_Z = 0.$$

Definition 2.2. A function $F \in C^\infty(J^1 M)$ is said to be an *intermediate integral* of \mathcal{E}_ω if the non-singular solutions of $\mathcal{E}_{F+const}$ are also solutions of \mathcal{E}_ω for every constant $const$.

Let ω be a differential n -form on $J^1 M$ and $F \in C^\infty(J^1 M)$. On a neighborhood of each point in U_F we choose a coframe $\{\theta, dF, \sigma_1, \dots, \sigma_{2n-1}\}$. This way we can write

$$(2) \quad \omega = \theta \wedge \lambda + dF \wedge \mu + \omega'$$

where ω' does not contain θ or dF . Moreover, $\omega|_Z = \omega'|_Z$ when Z is a solution of $\mathcal{E}_{F+const}$.

Theorem 2.3. *A function $F \in C^\infty(J^1 M)$ is an intermediate integral of \mathcal{E}_ω if and only if*

$$(3) \quad \theta \wedge dF \wedge i_{X_F} \omega = 0$$

on the open set U_F .

Proof. Without loss of generality, we can work over an open subset $V \subset U_F$ where the above coframe was defined.

From (2) and items (1)–(2) of Lemma 1.4 we derive that (3) is equivalent to

$$(4) \quad i_{X_F} \omega' = 0.$$

Let us assume that (3) (or (4)) holds. If Z is a solution of $\mathcal{E}_{F+const}$ we can restrict X_F to Z (item (4) of Lemma 1.4). Then

$$i_{X_F|_Z} \omega'|_Z = (i_{X_F} \omega')|_Z = 0.$$

On the other hand, the inner contraction of a non-vanishing n -form by a non-vanishing tangent field (item (3) of Lemma 1.4) on an n -dimensional manifold cannot be zero at any point. As a result we obtain $\omega'|_Z = 0$ and therefore, $\omega|_Z = 0$. In other words, F is an intermediate integral of \mathcal{E}_ω .

Conversely, let F be an intermediate integral of \mathcal{E}_ω . This way

$$(i_{X_F}\omega')|_Z = i_{X_F|_Z}\omega'|_Z = i_{X_F|_Z}\omega|_Z = 0$$

for every solution Z of $\mathcal{E}_{F+const}$.

By the definition of $C_{F,e}$, we can also obtain $(i_{X_F}\omega')|_{C_{F,e}} = 0$, for each point $e \in V$. In addition, $C_{F,e}$ is the incident subspace to θ_e and d_eF (Lemma 1.3). Recall, also, that ω'_e does not contain θ_e or d_eF (in the coframe we are using). In conclusion, $(i_{X_F}\omega')_e = 0$ for every e . This way, $i_{X_F}\omega' = 0$ and thus (3) holds. \square

Remarks 2.4. 1) Condition (3) is a first-order PDE for the unknown function F . As a consequence, we can find solutions of \mathcal{E}_ω by solving first-order equations: equation (3) and $\mathcal{E}_{F+const}$.

2) Any contact manifold is locally equivalent to J^1M for an appropriate M . Thus, all the results in this paper are locally valid for arbitrary contact manifolds. As a matter of fact, if a global contact structure 1-form θ exists, the field X_F can be defined and the result can be directly translated to this case.

3) A criterion was also given by Zil'bergleit [6] who uses the more sophisticated framework of Lychagin [4] on contact geometry: effective forms, $\mathfrak{sl}(2)$ -representations, Hodge-Lepage expansions, operators \top and \perp , etc. In addition, the criterion in [6], unlike Theorem 2.3, must be applied only on "effective" forms ω .

4) We can define an intermediate integral of \mathcal{E}_ω in a more general sense, that is, as a function F such that all the solutions of \mathcal{E}_F are also solutions of \mathcal{E}_ω (i.e., just $const = 0$ is considered in Definition 2.2). Then, Theorem 2.3 should be modified by requiring $\theta \wedge dF \wedge i_{X_F}\omega = 0$ only at the points of the hypersurface $\{F = 0\} \subset J^1M$.

5) Let us consider the case $\dim M = 2$, $\omega = \omega^1 \wedge \omega^2$. If dF belongs to the characteristic system $\{\theta, \omega^1, \omega^2\}$, then $\theta \wedge dF \wedge \omega = 0$. By applying Theorem 2.3 we can see that F is an intermediate integral of \mathcal{E}_ω . This way, we recover the criterion given for example in [1] (where a detailed analysis on the possibility of singular solutions was done). However, this result states just a sufficient condition. For example, it is not difficult to check that $F = p_1 + x_2$ is an intermediate integral for the equation

$$\frac{\partial^2 z}{\partial x_1^2} \frac{\partial^2 z}{\partial x_2^2} - \left(\frac{\partial^2 z}{\partial x_1 \partial x_2} \right)^2 + 1 = 0,$$

which is obtained from $\omega = \omega^1 \wedge \omega^2$, $\omega^1 = dp_2 + dx_1$, and $\omega^2 = -dp_1 + dx_2$. Nevertheless, $dF \notin \{\theta, \omega^1, \omega^2\}$ (on the other hand, $i_{X_F}\omega = 0$ and definitely Theorem 2.3 can be applied).

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