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A LIMIT-POINT CRITERION FOR A CLASS OF STURM-LIOUVILLE OPERATORS DEFINED IN L^p SPACES

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ABSTRACT. Using a recent result of Chernyavskaya and Shuster we show that the maximal operator determined by $M[y] = -y'' + qy$ on $[a, \infty)$, $a > -\infty$, where $q \geq 0$ and the mean value of q computed over all subintervals of $\mathbb{R}$ of a fixed length is bounded away from zero, shares several standard "limit-point at ∞" properties of the L^2 case. We also show that there is a unique solution of $M[y] = 0$ that is in all $L^p[a, \infty)$, $p = [1, \infty]$.

1. Introduction

In [3] Chernyavskaya and Shuster have determined necessary and sufficient conditions for the symmetric differential expression $M[y] = -y'' + qy$ where $q \geq 0$ and q is locally Lebesgue integrable to be "correctly solvable" in $L^p(\mathbb{R})$. This concept means:

(i) for every $f \in L^p(\mathbb{R})$, $p \in [1, \infty]$, there is a unique solution $y_p \in L^p(\mathbb{R})$ of $M[y] = f$;
(ii) y_p satisfies the inequality

$$\|y_p\|_{p,\mathbb{R}} \leq c(p)\|M[y_p]\|_{p,\mathbb{R}}. \tag{1.1}$$

A main result of [3] is

Theorem A. *M is correctly solvable in $L^p(\mathbb{R})$, $p \in [1, \infty]$, if and only if there exists $a \in (0, \infty)$ such that*

$$q_0(a) := \inf_{x \in \mathbb{R}} \int_{x-a}^{x+a} q\,dt > 0. \tag{1.2}$$

Two corollaries to Theorem A are:

(i) M is correctly solvable in $L^p(\mathbb{R})$, $p \in [1, \infty]$, if $q \geq k > 0$ on $\mathbb{R}$;
(ii) M is not correctly solvable in $L^p(\mathbb{R})$ for any $p \in [1, \infty)$ if $q \in L(-\infty, 0)$ or $q \in L(0, \infty)$.

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We assume throughout that M satisfies (1.2) and shall then show that the correct solvability of M implies two interesting conclusions holding for $p \in [1, \infty]$, one of which generalizes the fact that the restriction of M to the half-line $I = [a, \infty)$ is limit-point at ∞ in $L^2(I_a)$.

Before stating them however, we need some preliminaries. If X is a Banach space with norm $\|(\cdot)\|$, e.g. $L^p(\mathbb{R})$, $p \in [1, \infty]$, and $T : X \to X$ is an operator with dense domain $\mathcal{D} \subset X$ we write $R(T)$ and $N(T)$ for the range and null space of T. If X^* is the dual of X, e.g. $L^{p'}(I_a)$, $p' = p/p - 1$, and $[x, x^*]$ signifies $\bar{x^*}(x)$ for $x \in X$ and $x^* \in X^*$, we consider the set of pairs $G(T^*) := (z, z') \in X^* \times X^*$ such that

$$[T(y), z] = [y, z']. \tag{1.3}$$

The density of $\mathcal{D}$ implies that $G(T^*)$ determines an operator T^* called the *adjoint* of T such that $T^*(z) = z'$. If $T : X^* \to X^*$ has a domain $\mathcal{D}^*$ that is *total* over X (i.e., $[x, x^*] = 0$ for all $x \in X$ and $x^* \in \mathcal{D}^* \implies x = 0$), then the set of pairs $(z, z') \in X \times X$ satisfying (1.3) also determines an operator, which we also denote by T^*, and call the adjoint of T in X. In either case T^* is closed and

$$[T(y), z] = [y, T^*(z)] \tag{1.4}$$

for all $y \in \mathcal{D}$, $z \in \mathcal{D}^*$. Furthermore, if M is a subspace of X and M^* is a subspace of X^*, then

$$M^{\perp} := \{x^* \in X^* : [x, x^*] = 0, \quad \forall x \in M\},$$
$${}^{\perp}M^* := \{x \in X : [x, x^*] = 0, \quad \forall x^* \in M^*\},$$

and if X is reflexive, i.e., the natural mapping of X to $(X^*)^* \equiv X^{**}$ is an isomorphism, then $(M^*)^{\perp} = {}^{\perp}M$. If we set $G(T) := \{(y, T(y)) : y \in \mathcal{D}\}$ and $G(-T) := \{(y, -T(y)) : y \in \mathcal{D}\}$, then $G(T^*) = G(-T)^{\perp} \subset X^* \times X^*$.

It follows from these definitions and (1.3) (see, e.g., Kato, [7, Problem 5.27, p. 168], Rudin, [11, Theorem 4.7] and Goldberg, [5, Theorem IV.1.3] that when T is closed,

$$\begin{aligned} R(T)^{\perp} &= N(T^*), & {}^{\perp}R(T^*) &= N(T), \\ \overline{R(T)} &= {}^{\perp}N(T^*), & \overline{R(T^*)} &= N(T)^{\perp}. \end{aligned} \tag{1.5}$$

We are interested in the operators determined by M but on the half-line $I_a = [a, \infty)$, $-\infty < a$, rather than on $\mathbb{R}$. This parallels a common situation in the Hilbert space theory of M and allows study of the L^p solutions of $M[y] = 0$ on I_a, which obviously cannot exist on $\mathbb{R}$ if M is correctly solvable.

If $AC_{\mathrm{loc}}(I_a)$ denotes the functions that are locally absolutely continuous on I_a, we set

$$\{y, z\}(t) := y'(t)\bar{z}(t) - y(t)\bar{z}'(t)$$

for $y, z \in AC_{\mathrm{loc}}(I_a)$ and define the following operators and domains in $L^p(I_a)$.

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Definition. For $p \in [1,\infty]$ assume that $q \in L^\infty_{\text{loc}}(I_a) \subset L_{\text{loc}}(I_a)$[1] and let $T'_{0,p}$, T_p, and $T_{0,p}$ be the operators determined by M on

$$\begin{aligned} \mathcal{D}'_{0,p} &:= \{y \in C_0^\infty(I_a)\}, \\ \mathcal{D}_p &:= \{y \in L^p(I_a) : y' \in AC_{\text{loc}}(I_a); M[y] \in L^p(I_a)\}, \\ \mathcal{D}_{0,p} &:= \{y \in \mathcal{D}_p : y(a) = y'(a) = 0; \lim_{t\to\infty}\{y,z\}(t) = 0, \forall z \in \mathcal{D}_p\}. \end{aligned}$$

We call $T'_{0,p}$, $T_{0,p}$ respectively the "preminimal" and "minimal" operators, and T_p the "maximal" operator determined by M. These operators have the following properties.

Theorem B. *For the cases $p \in (1,\infty)$, $p = 1$, or $p = \infty$ set $p' := p/(p-1)$, $p' = \infty$, or $p' = 1$. Then*

(i) *$T_{0,p}$ and T_p are closed operators;*
(ii) *$[T_p(y), z] = \lim_{t\to\infty}\{y,z\}(t) - \{y,z\}(a) + [y, T_{p'}]$;*
(iii) *$T_p^* = T_{0,p'}$ and $T_{0,p}^* = T_{p'}$;*
(iv) *$R(T_{0,p})^\perp = N(T_{p'})$ and ${}^\perp N(T_{p'}) = \overline{R(T_{0,p})}$;*
(v) *$N(T_{0,p}) = \{0\}$.*

Moreover, for $p \in (1,\infty]$, $T'_{0,p}$ is closable and $\overline{T'_{0,p}} = T_{0,p}$.

Proofs of (i)–(v), the last statement, as well as more general results may be found in one of [5, Chapter VI], [10], or [2]. (iv) is a consequence of the fact that solutions of $M[y] = 0$ under prescribed initial conditions are unique. The L^2 theory is throughly treated in [8].

We are now in a position to state our two principal results.

Theorem 1. *If $q \in L^\infty_{loc}$ and satisfies (1.2) and $p \in [1,\infty]$, then M is "p limit-point" (pLP) at ∞ in the sense that*

(i) *$\dim\left(\frac{\mathcal{D}_p}{\mathcal{D}_{0,p}}\right) = 2$ and $\dim\left(\frac{R(T_p)}{R(T_{0,p})}\right) = 1$;*
(ii) *$\dim N(T_p) = 1$;*
(iii) *for all $y \in \mathcal{D}_p$ and $z \in \mathcal{D}_{p'}$ we have $\lim_{t\to\infty}\{y,z\}(t) = 0$.*

Theorem 2. *If y_1 denotes the principal or "small" solution of $M[y] = 0$, then $y_1 \in L^p(I_a)$ for all $p \in [1,\infty]$.*

Remarks. (i) Theorem 1 is an extension to all $p \in [1,\infty]$ and to a more general q of the fact that M is LP at ∞ when $q \geq k > 0$ and $p = 2$.

(ii) In the case when $q \geq k > 0$ a direct argument can be given to show that $M[y] = 0$ has exponentially growing and exponentially decaying solutions. Read [9] has extended this by showing that the same is true if

$$\liminf_{x\to\infty} \int_x^{x+a} q^{1/2}\, dt > aL \tag{1.6}$$

for positive constants a and L. Clearly (1.6) implies (1.2) of Theorem A for the extension $\hat{q}_1$ of q obtained by setting $\hat{q}_1(t) = 1$ for $t < a$. In this case or if q

[1] By extending the procedure of Naimark [8, §17], used to define maximal and minimal operators in the L^2 case, it is almost certain that q need only be locally integrable. However, in this case, $\mathcal{D}'_{0,p}$ need not contain any member of $C_0^\infty(I_a)$ and has to be redefined as the subspace of $\mathcal{D}$ having compact support on I_a. The density of $\mathcal{D}'_{0,p}$ must then be shown by an independent argument. Because of its technical complications we will not pursue this approach here.

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is bounded away from zero, Theorem 2 is trivially true since y_1 is exponentially decreasing. Under the condition (1.2), however, Theorem 2 seems to be new.

(iii) In the L^2 theory M is said to be *strong limit-point* (SLP) or *Dirichlet* (D) at ∞ if $\lim_{t\to\infty} y'\bar{z} = 0$ for all $y, z \in \mathcal{D}_2$, or if y' and $q^{1/2}y \in L^2(I_a)$ for all $y \in \mathcal{D}_2$. It is known [1] that $D \Rightarrow SLP$ and that both SLP and D hold if $q \geq k > 0$. It would be interesting to see if either SLP or D might be profitably extended to the L^p setting.

2. Proofs of Theorems 1 and 2

Lemma 1. *For $p \in [1,\infty]$, $R(T_p) = L^p(I_a)$ and $T_{0,p}$ has closed range.*

Proof. Since q is defined only on I_a, to apply Theorem A we consider the extension $\hat{q}_1$. Similarly if $f \in L^p(I_a)$, we construct an extension $\hat{f}$ to $\mathbb{R}$ by setting $\hat{f}(t) = 0$ for $t < a$. By Theorem A there is a unique $\hat{y} \in L^p(\mathbb{R})$ such that $M[y] = \hat{f}$. The restriction y of $\hat{y}$ to I_a is evidently in $\mathcal{D}_p$ and satisfies $M[y] = f$, showing that T_p is onto $L^p(I_a)$. If $R(T_0, p)$ is not closed, by the Closed Range Theorem the inverse of $T_{0,p}$ is unbounded. Therefore there are sequences $\{y_n\} \subset \mathcal{D}_{0,p}$, $\{f_n\} \subset R(T_{0,p})$ such that $f_n \to 0$, $\|y_n\| = 1$, and $M[y_n] = f_n$. Let $\hat{y}_n$ and $\hat{f}_n$ be the extensions of y_n and f_n to $\mathbb{R}$ defined as above. Applying Theorem A again there is a unique $\hat{z}_n$ for each $\hat{f}_n$ such that $M[\hat{z}_n] = f_n$ and

$$\begin{aligned}\|\hat{z}_n\|_{p,\mathbb{R}} &\leq c_p\|\hat{f}_n\|_{p,\mathbb{R}}\\ &\leq c_p\|f_n\|_{p,\mathbb{R}}.\end{aligned}$$

Therefore if z_n is the restriction of $\hat{z}_n$ to I_a we have also that $\|z_n\|_{p,I_a} \leq c_p\|f_n\|_{p,I_a}$. Hence since $f_n \to 0$, so does z_n. Because $M[z_n] = f_n = M[y_n]$ we have that $z_n - y_n \in N(T_p)$. Moreover, $1/2 \leq \|z_n - y_n\|_{p,I_a} \leq 3/2$ for sufficiently large n because $\|y_n\|_{p,I_a} = 1$ and $z_n \to 0$. If $N(T_p) = \{0\}$, then $z_n = y_n$, which is impossible. If $N(T_p) = \operatorname{span}\{u\}$ where $\|u\| = 1$, then $z_n - y_n = k_n u$. Since the k_n lie in the bounded interval $[1/2, 3/2]$ for sufficiently large n, there must be a subsequence $k_{n_i} \to k \neq 0$. It follows that $y_{n_i} \to -ku$. Because $T_{0,p}$ is closed, $u \in \mathcal{D}_{0,p}$. But since the null space of $T_{0,p}$ is trivial, $u = 0$, contradicting our assumption. The case $N(T_p) = \operatorname{span}\{u_1, u_2\}$ is handled by a similar argument. □

The following three results are standard, but since some are difficult to find in the literature in the form stated we include proofs.

Lemma 2. *If X is a Banach space, X^* is its dual, and M is a finite-dimensional subspace of X, then*

$$\dim\left(\frac{X^*}{M^\perp}\right) = \dim M.$$

Proof. Let $\{m_i\}$, $i = 1, \ldots, n$, be a basis for M. By the Hahn-Banach theorem we can find $f_i \in X^*$ such that $f_i(m_j) = \delta_{ij}$, $1 \leq i, j \leq n$. We claim that $\{f_i\}$, $i = 1, \ldots, n$, is a linearly independent set mod $M^\perp$. For if $c_1 f_1 + \cdots + c_n f_n \in M^\perp$ where not all the c_i are zero we have that for any particular (and therefore all) c_j that

$$c_j = c_j f_j(m_j) = \left(\sum_{i=1}^n c_i f_i\right)(m_j) = 0,$$

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which is a contradiction. It follows that

$$\dim M \leq \dim\left(\frac{X^*}{M^\perp}\right).$$

If the inequality is strict we can find $f \in X^*$ such that $S = \{f_1, \ldots, f_n, f\}$ is linearly independent mod $M^\perp$. Consider

$$\psi := f - \sum_{i=1}^{m} f(m_i) f_i.$$

Since $f \notin M^\perp$, not all the $f(m_i) = 0$. However, $\psi(m_i) = 0$ for $i = 1, \ldots, n$, showing that $\psi \in M^\perp$ so that S is linearly dependent mod $M^\perp$ and thus contradicting our assumption. □

Lemma 3. *Suppose that X and its dual X^* are Banach spaces, and let $T_2 \subset T_1$ be densely defined operators $X \to X$ with domains $\mathcal{D}_1$ and $\mathcal{D}_2$, and let $T_1^* \subset T_2^*$ be their adjoints with domains $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$. If $\dim(\mathcal{D}_1/\mathcal{D}_2) = n < \infty$, then*

$$\dim\left(\frac{\mathcal{D}_1}{\mathcal{D}_2}\right) = \dim\left(\frac{\mathcal{D}_2^*}{\mathcal{D}_1^*}\right).$$

Proof. The technique is similar to that of Lemma 2. First note that

$$\dim\left(\frac{\mathcal{D}_1}{\mathcal{D}_2}\right) = \dim\left(\frac{G(-T_1)}{G(-T_2)}\right) \quad \text{and} \quad \dim\left(\frac{\mathcal{D}_2^*}{\mathcal{D}_1^*}\right) = \dim\left(\frac{G(T_2^*)}{G(T_1^*)}\right).$$

Let $\nu_i = (\alpha_i, -T_1(\alpha_i))$, $i = 1, \ldots, n$, be a linearly independent set mod $G(-T_2)$. By the Hahn-Banach Theorem we can find

$$\psi_j = (u_j, v_j) \in (G(-T_2))^\perp \equiv G(T_2^*) \subset X^* \times X^*, \quad j = 1, \ldots, n,$$

such that $\psi_j(\nu_i) \equiv u_j(\alpha_i) + v_j(-T_1(\alpha_i)) = \delta_{ij}$. Now $\{\psi_j\}$, $j = 1, \ldots, n$, is a linearly independent set mod $G(T_1^*)$, for if

$$\eta = \sum_{j=1}^{n} c_j \psi_j \in G(T_1^*),$$

then $\eta(\nu_i) = 0 = c_i$ for $i = 1, \ldots, n$. This shows that $\dim(\mathcal{D}_1/\mathcal{D}_2) \leq \dim(\mathcal{D}_2^*/\mathcal{D}_1^*)$. But if $\{\psi_1, \ldots, \psi_n, \psi\}$, $\psi \in G(T_2^*)$ is linearly independent mod $G(T_1^*)$, this contradicts the fact that $\psi - \sum_{i=1}^{n} d_i \psi_i \in G(T_1^*)$ where $d_i = \psi(\nu_i)$. □

Lemma 4. *Suppose that X is a Banach space and X^* is its dual, and let $T_1 : X^* \to X^*$ be an operator with domain $\mathcal{D}_1$ and T_2 be a one-to-one restriction of T_1 with closed range having domain $\mathcal{D}_2 \subset \mathcal{D}_1$. If $\dim N(T_1) < \infty$ and $\dim {}^\perp R(T_2) \subseteq X < \infty$, then*

$$\dim\left(\frac{\mathcal{D}_1}{\mathcal{D}_2}\right) = \dim N(T_1) + \dim {}^\perp R(T_2).$$

Proof. In Lemma 2 we identify M with ${}^\perp R(T_2)$. Since $R(T_2)$ is closed, $M^\perp = R(T_2)$. We can conclude that

$$\dim\left(\frac{R(T_1)}{R(T_2)}\right) = \dim {}^\perp R(T_2).$$

If $k = \dim {}^\perp R(T_2)$, we can find nonzero elements $f_1, \ldots, f_k$ in $R(T_1)$ that are linearly independent mod $R(T_2)$. Suppose $y_i \in \mathcal{D}_1$ and $M[y_i] = f_i$, $i = 1, \ldots, k$.

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Then it is easily verified that $S_1 = \{y_1, \dots, y_k\}$ is a linearly independent set mod $\mathcal{D}_2$ (for otherwise a linear combination of the f_i would be in $R(T_2)$). Let $S_2 = \{z_1, \dots, z_m\}$ be a basis of $N(T_1)$. We claim that $S_1 \cup S_2$ is linearly independent mod $\mathcal{D}_2$. For assume that $\eta = \eta_1 + \eta_2 \in \mathcal{D}_2$ where

$$\eta_1 = \sum_{i=1}^{k} c_i y_i, \quad \eta_2 = \sum_{i=1}^{m} d_i z_i$$

and not all the coefficients vanish. In particular, under this assumption, since $N(T_1) \cap \mathcal{D}_2 = \{0\}$, not all the $c_i = 0$ vanish, for otherwise $\eta_2 \neq 0$ and $\eta_2 \in \mathcal{D}_2 \cap N(T_1)$. But then

$$T_1(\eta) = T_1(\eta_1) = \sum_{i=1}^{k} c_i f_i \in R(T_2),$$

contradicting the linear independence of the f_i mod $R(T_2)$. It follows that

$$\dim\left(\frac{\mathcal{D}_1}{\mathcal{D}_2}\right) \geq \dim N(T_1) + \dim{}^{\perp} R(T_2) = m + k.$$

Suppose that we can adjoin an element $u \in \mathcal{D}_1$ to $S_1 \cup S_2$ so that $S_3 = (S_1 \cup S_2) \cup \{u\}$ is linearly independent mod $\mathcal{D}_2$. Since $N(T_1)$ is finite dimensional, it is complemented in X, and so u can be written uniquely as a sum $u_1 + u_2$ where $u_1 \in N(T_1)$ and $u_2 \in (\mathcal{D}_1 \backslash N(T_1)) \cup \{0\}$. If η_3 is an arbitrary linear combination of elements of S_3 with a nonzero coefficient d of u, then

$$T_1(\eta_3) = \sum_{i=1}^{k} c_i f_i + d T_1(u_2) \notin R(T_2),$$

implying that $\dim(R(T_1)/R(T_2)) = k + 1$, which is false. □

Lemma 5. *For $p \in [1, \infty]$,*

$$\dim N(T_p) = \dim\left(\frac{R(T_p)}{R(T_{0,p})}\right) = 1,$$
$$\dim\left(\frac{\mathcal{D}_p}{\mathcal{D}_{0,p}}\right) = 2.$$

Proof. Since M is disconjugate on I_a and since $q > 0$, it follows by Corollary 6.4 and Theorem 6.4 of Hartman [6] that there is a fundamental set of positive linearly independent solutions y_1 and y_2 of $M[y] = 0$, called respectively the principal and nonprincipal solutions, such that $y_1' \leq 0$ and $y_2' > 0$ on I_a. Additionally, $\lim_{t\to\infty} y_1/y_2 = 0$. Suppose $y_p \in N(T_p)$ for $p \in [1, \infty)$. We claim that y_p must be a multiple of y_1. For if $y_p = c_1 y_1 + c_2 y_2$ with $c_2 \neq 0$, then

$$\lim_{t\to\infty} |y_p(t)| = \lim_{t\to\infty} |y_2(c_1 y_1/y_2 + c_2)| = |c_2| y_2. \tag{2.1}$$

Since $|y_p(t)|$ becomes arbitrarily close to a nondecreasing positive function, it cannot be in $L^p(I_a)$. This shows that $\dim N(T_p) \leq 1$, for $p \in [1, \infty]$, and if nontrivial, $N(T_p)$ is spanned by $\{y_1\}$.

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We next show that $N(T_p) \neq \{0\}$. If $r \in (1, \infty]$, then $R(T_{0,r})$ is closed by Lemma 1, and from Theorem B (iv) and the fact that $L^r(I_a) = L^{r,}(I_a)^*$,

$$R(T_{0,r}) = {}^{\perp}N(T_{r'}) = N(T_{r'})^{\perp}, \tag{2.2}$$

$$^{\perp}R(T_{0,r} = N(T_{r'}). \tag{2.3}$$

Also, by Lemma 1 (again), $R(T_r) = L^r(I_a)$. We now identify M with $N(T_{r'})$, X with $L^{r'}(I_a)$, and X^* with $L^r(I_a)$. (2.2) and Lemma 2 then give that

$$\dim \left(\frac{R(T_r)}{R(T_{0,r})} \right) = \dim N(T_{r'}) \leq 1. \tag{2.4}$$

Hence if $\dim(R(T_r)/R(T_{0,r})) = 0$, then $N(T_{r'}) = \{0\}$, and by (2.3) and Lemma 4, $\dim(\mathcal{D}_r/\mathcal{D}_{0,r}) = \dim N(T_r)$. However, since we can find C_0^∞ linearly independent functions ϕ_1, ϕ_2 with support in I_a such that $\phi_1(a) = 1 = \phi_2'(a)$ and $\phi_1'(a) = 0 = \phi_2(a)$ it must be the case that $\dim(\mathcal{D}_r/\mathcal{D}_{0,r}) \geq 2$. So $\dim N(T_r) = 2$, which as we have seen from (2.1) is not possible given the properties of y_1 and y_2. This contradiction shows that $\dim N(T_{r'}) = 1$ for $r' \in [1, \infty)$. If $p = r \in (1, \infty]$ by (2.4), then $\dim(R(T_p)/R(T_{0,p})) = 1$ and if we choose $r' = p$, then $\dim N(T_p) = 1$ for $p \in [1, \infty)$. If $p = \infty$, then $y_1 \in N(T_\infty)$ since $y_1 > 0$ and $y' \leq 0$, and so $\dim N(T_\infty) = 1$. Since we have now established that $\dim N(T_p) = \dim N(T_p')$ for $p \in (1, \infty]$ by Lemma 4 and (2.3), $\dim(D_p/D_{0,p}) = 2$. By Lemma 3, $\dim(\mathcal{D}_1/\mathcal{D}_{0,1}) = 2$. Since T_1 is one-to-one on $\mathcal{D}_1 \ominus N(T_1)$ we conclude that $\dim(R(T_1)/R(T_{0,1})) = 1$. The lemma is now established in all cases. □

Proof of Theorem 1. (i) and (ii) is the assertion of Lemma 5. It follows that $\mathcal{D}_p = \mathcal{D}_{0,p} \oplus \operatorname{span}\{\phi_1, \phi_2\}$. Since ϕ_1, ϕ_2 vanish at ∞, $\lim_{t\to\infty}\{y, z\}(t) = 0$, which proves (iii). □

Proof of Theorem 2. This is also obvious from (2.1) and the proof of Lemma 5. □

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