A LIMIT-POINT CRITERION
FOR A CLASS OF STURM-LIOUVILLE OPERATORS
DEFINED IN $L^p$ SPACES

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(Communicated by Carmen C. Chicone)

Abstract. Using a recent result of Chernyavskaya and Shuster we show that the maximal operator determined by
$$M[y] = -y'' + qy$$
on $(a, \infty)$, $a > -\infty$, where $q \geq 0$ and the mean value of $q$ computed over all subintervals of $\mathbb{R}$ of a
fixed length is bounded away from zero, shares several standard “limit-point at $\infty$” properties of the $L^2$ case. We also show that there is a unique solution of $M[y] = 0$ that is in all $L^p[a, \infty)$, $p = [1, \infty]$.

1. Introduction

In [3] Chernyavskaya and Shuster have determined necessary and sufficient conditions for the symmetric differential expression
$$M[y] = -y'' + qy$$
where $q \geq 0$ and $q$ is locally Lebesgue integrable to be “correctly solvable” in $L^p(\mathbb{R})$. This concept means:
(i) for every $f \in L^p(\mathbb{R})$, $p \in [1, \infty]$, there is a unique solution $y_p \in L^p(\mathbb{R})$ of $M[y] = f$;
(ii) $y_p$ satisfies the inequality
$$\|y_p\|_{p, \mathbb{R}} \leq c(p)\|M[y_p]\|_{p, \mathbb{R}}.$$

A main result of [3] is

**Theorem A.** $M$ is correctly solvable in $L^p(\mathbb{R})$, $p \in [1, \infty]$, if and only if there exists $a \in (0, \infty)$ such that
$$q_0(a) := \inf_{x \in \mathbb{R}} \int_{x-a}^{x+a} q \, dt > 0.$$

Two corollaries to Theorem A are:
(i) $M$ is correctly solvable in $L^p(\mathbb{R})$, $p \in [1, \infty]$, if $q \geq k > 0$ on $\mathbb{R}$;
(ii) $M$ is not correctly solvable in $L^p(\mathbb{R})$ for any $p \in [1, \infty]$ if $q \in L(-\infty, 0)$ or $q \in L(0, \infty)$.

Received by the editors December 18, 2002.
2000 Mathematics Subject Classification. Primary 47E05, 34C11, 34B24; Secondary 34C10.
Key words and phrases. Second-order differential operators of symmetric form in $L^p$ spaces, correct solvability, limit-point, $L^p$ solutions.
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We assume throughout that $M$ satisfies (1.2) and shall then show that the correct solvability of $M$ implies two interesting conclusions holding for $p \in [1, \infty]$, one of which generalizes the fact that the restriction of $M$ to the half-line $I = [a, \infty)$ is limit-point at $\infty$ in $L^2(I_a)$.

Before stating them however, we need some preliminaries. If $X$ is a Banach space with norm $\|\cdot\|$, e.g. $L^p(\mathbb{R})$, $p \in [1, \infty]$, and $T : X \to X$ is an operator with dense domain $D \subset X$ we write $R(T)$ and $N(T)$ for the range and null space of $T$. If $X^*$ is the dual of $X$, e.g. $L^p'(I_a)$, $p' = p/p - 1$, and $[x, x^*]$ signifies $x^*(x)$ for $x \in X$ and $x^* \in X^*$, we consider the set of pairs $G(T^*) := (z, z') \in X^* \times X^*$ such that

$$T(y), z = [y, z']$$

The density of $D$ implies that $G(T^*)$ determines an operator $T^*$ called the adjoint of $T$ such that $T^*(z) = z'$. If $T : X^* \to X^*$ has a domain $D^*$ that is total over $X$ (i.e., $[x, x^*] = 0$ for all $x \in X$ and $x^* \in D^* \implies x = 0$), then the set of pairs $(z, z') \in X \times X$ satisfying (1.3) also determines an operator, which we also denote by $T^*$, and call the adjoint of $T$ in $X$. In either case $T^*$ is closed and

$$T(y), z = [y, T^*(z)]$$

for all $y \in D$, $z \in D^*$. Furthermore, if $M$ is a subspace of $X$ and $M^*$ is a subspace of $X^*$, then

$$M^\perp := \{x^* \in X^* : [x, x^*] = 0, \ \forall x \in M\},$$

$$M^* = \{x \in X : [x, x^*] = 0, \ \forall x^* \in M^*\},$$

and if $X$ is reflexive, i.e., the natural mapping of $X$ to $(X^*)^* = X^{**}$ is an isomorphism, then $(M^*)^\perp = M^\perp$. If we set $G(T) := \{(y, T(y)) : y \in D\}$ and $G(-T) := \{(y, -T(y)) : y \in D\}$, then $G(T^*) = G(-T)^\perp \subset X^* \times X^*$.

It follows from these definitions and (1.3) (see, e.g., Kato, [7, Problem 5.27, p. 168], Rudin, [11, Theorem 4.7] and Goldberg, [5, Theorem IV.1.3]) that when $T$ is closed,

$$R(T) = N(T^*)^\perp,$$

$$R(T)^\perp = M^* = N(T^*),$$

(1.5)

$$R(T^*) = N(T^*)^\perp,$$

$$R(T^*)^\perp = M^\perp = N(T).$$

We are interested in the operators determined by $M$ but on the half-line $I_a = [a, \infty)$, $-\infty < a$, rather than on $\mathbb{R}$. This parallels a common situation in the Hilbert space theory of $M$ and allows study of the $L^p$ solutions of $M[y] = 0$ on $I_a$, which obviously cannot exist on $\mathbb{R}$ if $M$ is correctly solvable.

If $AC_{\text{loc}}(I_a)$ denotes the functions that are locally absolutely continuous on $I_a$, we set

$$\{y, z\}(t) := y'(t)\tilde{z}(t) - y(t)\tilde{z}'(t)$$

for $y, z \in AC_{\text{loc}}(I_a)$ and define the following operators and domains in $L^p(I_a)$.
Theorem 1. If is thoroughly treated in [8].

We call \( T \) solutions of found in one of [5, Chapter VI], [10], or [2]. (iv) is a consequence of the fact that \( a \) is the maximal" operator determined by \( \maximal \) operator determined by

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Theorem 2. If extension \( \^ \) point" (yi)

Remarks. (i) Theorem 1 is an extension to all \( M \) of the fact that \( M \) is \( \preminimal \) and \( \minimal \) operators, and \( T \) the "maximal" operator determined by \( M \). These operators have the following properties.

Theorem B. For the cases \( p \in (1, \infty) \), \( p = 1 \), or \( p = \infty \) set \( p' := p/(p-1) \), \( p' = \infty \), or \( p' = 1 \). Then

(i) \( T_{0,p} \) and \( T \) are closed operators;
(ii) \( \left[ T_{p} (y), z \right] = \lim_{t \to \infty} \{ y, z \} (t) - \{ y, z \} (a) + \{ y, T_{p'} \} ;
(iii) \( T_{p} = T_{0,p'} \) and \( T_{0,p} = T_{p'} ;
(iv) \( R(T_{0,p}) \) = \( N(T_{p'}) \) and \( \perp N(T_{p'}) = \overline{R(T_{0,p})} ;
(v) \( N(T_{0,p}) = \{ 0 \} \).

Moreover, for \( p \in (1, \infty) \), \( T_{0,p} \) is closable and \( \overline{T_{0,p}} = T_{0,p} \).

Proofs of (i)--(v), the last statement, as well as more general results may be found in one of [5] Chapter VI, [10], or [2]. (iv) is a consequence of the fact that solutions of \( M[y] = 0 \) under prescribed initial conditions are unique. The \( L^{2} \) theory is thoroughly treated in [8].

We are now in a position to state our two principal results.

Theorem 1. If \( q \in L^{\infty}_{\text{loc}} \) and satisfies (1.2) and \( p \in [1, \infty] \), then \( M \) is "p limit-point" (\( p \)LP) at \( \infty \) in the sense that

(i) \( \dim \left( \frac{D_{p}}{D_{0,p}} \right) = 2 \) and \( \dim \left( \frac{R(T_{p})}{R(T_{0,p})} \right) = 1 \);
(ii) \( \dim N(T_{p}) = 1 \);
(iii) for all \( y \in D_{p} \) and \( z \in D_{p} \) we have \( \lim_{t \to \infty} \{ y, z \} (t) = 0 \).

Theorem 2. If \( y_{1} \) denotes the principal or "small" solution of \( M[y] = 0 \), then \( y_{1} \in L^{p}(I_{a}) \) for all \( p \in [1, \infty] \).

Remarks. (i) Theorem 1 is an extension to all \( p \in [1, \infty] \) and to a more general \( q \) of the fact that \( M \) is \( LP \) at \( \infty \) when \( q \geq k > 0 \) and \( p = 2 \).

(ii) In the case when \( q \geq k > 0 \) a direct argument can be given to show that \( M[y] = 0 \) has exponentially growing and exponentially decaying solutions. Read [9] has extended this by showing that the same is true if

\[
\liminf_{x \to \infty} \int_{x}^{x+a} q^{1/2} dt > aL
\]

for positive constants \( a \) and \( L \). Clearly (1.6) implies (1.2) of Theorem A for the extension \( \hat{q}_{1} \) of \( q \) obtained by setting \( \hat{q}_{1}(t) = 1 \) for \( t < a \). In this case or if \( q \)

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1By extending the procedure of Naimark [8 §17], used to define maximal and minimal operators in the \( L^{2} \) case, it is almost certain that \( q \) need only be locally integrable. However, in this case, \( D_{0,p} \) need not contain any member of \( C_{0}^{\infty}(I_{a}) \) and has to be redefined as the subspace of \( D \) having compact support on \( I_{a} \). The density of \( D_{0,p} \) must then be shown by an independent argument. Because of its technical complications we will not pursue this approach here.
is bounded away from zero, Theorem 2 is trivially true since \( y_1 \) is exponentially decreasing. Under the condition (\[2\]), however, Theorem 2 seems to be new.

(iii) In the \( L^p \) theory \( M \) is said to be strong limit-point (SLP) or Dirichlet (D) at \( \infty \) if \( \lim_{t \to \infty} y(t) = 0 \) for all \( y, z \in D_2 \), or if \( y' \) and \( q^{1/2}y \in L^2(I_a) \) for all \( y \in D_2 \).

It is known \([1]\) that \( D \Rightarrow \text{SLP} \) and that both SLP and D hold if \( q \geq k > 0 \). It would be interesting to see if either SLP or D might be profitably extended to the \( L^p \) setting.


**Lemma 1.** For \( p \in [1, \infty] \), \( R(T_p) = L^p(I_a) \) and \( T_{0,p} \) has closed range.

**Proof.** Since \( q \) is defined only on \( I_a \), to apply Theorem A we consider the extension \( \tilde{q}_1 \). Similarly if \( f \in L^p(I_a) \), we construct an extension \( \tilde{f} \) to \( \mathbb{R} \) by setting \( \tilde{f}(t) = 0 \) for \( t < a \). By Theorem A there is a unique \( \tilde{y} \in L^p(\mathbb{R}) \) such that \( M[\tilde{y}] = \tilde{f} \). The restriction \( y \) of \( \tilde{y} \) to \( I_a \) is evidently in \( D_p \) and satisfies \( M[y] = f \), showing that \( T_p \) is onto \( L^p(I_a) \). If \( R(T_{0,p}) \) is not closed, by the Closed Range Theorem the inverse of \( T_{0,p} \) is unbounded. Therefore there are sequences \( \{y_n\} \subset D_{0,p}, \{f_n\} \subset R(T_{0,p}) \) such that \( f_n \to 0 \), \( \|y_n\| = 1 \), and \( M[y_n] = f_n \). Let \( \tilde{y}_n \) and \( \tilde{f}_n \) be the extensions of \( y_n \) and \( f_n \) to \( \mathbb{R} \) defined as above. Applying Theorem A again there is a unique \( \tilde{z}_n \) for each \( \tilde{f}_n \) such that \( M[\tilde{z}_n] = \tilde{f}_n \) and

\[
\|\tilde{z}_n\|_{p,\mathbb{R}} \leq c_p \|\tilde{f}_n\|_{p,\mathbb{R}} \leq c_p \|f_n\|_{p,\mathbb{R}}.
\]

Therefore if \( z_n \) is the restriction of \( \tilde{z}_n \) to \( I_a \) we have also that \( \|z_n\|_{p,\mathbb{I}_a} \leq c_p \|f_n\|_{p,\mathbb{I}_a} \). Hence since \( f_n \to 0 \), so does \( z_n \). Because \( M[z_n] = f_n = M[y_n] \) we have that \( z_n = y_n \in N(T_p) \).

Moreover, \( 1/2 \leq \|z_n - y_n\|_{p,\mathbb{I}_a} \leq 3/2 \) for sufficiently large \( n \) because \( \|y_n\|_{p,\mathbb{I}_a} = 1 \) and \( z_n \to 0 \). If \( N(T_p) = \{0\} \), then \( z_n \to 0 \), which is impossible. If \( N(T_p) = \text{span}\{u\} \) where \( \|u\| = 1 \), then \( z_n - y_n = k_nu \). Since the \( k_n \) lie in the bounded interval \( [1/2, 3/2] \) for sufficiently large \( n \), there must be a subsequence \( k_{n_i} \to k \neq 0 \). It follows that \( y_{n_i} \to -ku \). Because \( T_{0,p} \) is closed, \( u \in D_{0,p} \). But since the null space of \( T_{0,p} \) is trivial, \( u = 0 \), contradicting our assumption. The case \( N(T_p) = \text{span}\{u_1, u_2\} \) is handled by a similar argument. \( \square \)

The following three results are standard, but since some are difficult to find in the literature in the form stated we include proofs.

**Lemma 2.** If \( X \) is a Banach space, \( X^* \) is its dual, and \( M \) is a finite-dimensional subspace of \( X \), then

\[
\dim \left( \frac{X^*}{M^\perp} \right) = \dim M.
\]

**Proof.** Let \( \{m_i\}, i = 1, \ldots, n \) be a basis for \( M \). By the Hahn-Banach theorem we can find \( f_i \in X^* \) such that \( f_i(m_i) = \delta_{ij}, 1 \leq i, j \leq n \). We claim that \( \{f_i\}, i = 1, \ldots, n \), is a linearly independent set mod \( M^\perp \). For if \( c_1f_1 + \cdots + c_nf_n \in M^\perp \), where not all the \( c_i \) are zero we have that for any particular (and therefore all) \( c_j \) that

\[
c_j = c_j f_j(m_j) = \left( \sum_{i=1}^{n} c_if_i \right)(m_j) = 0,
\]
which is a contradiction. It follows that
\[ \dim M \leq \dim \left( \frac{X^*}{M^\perp} \right). \]

If the inequality is strict we can find \( f \in X^* \) such that \( S = \{f_1, \ldots, f_n, f\} \) is linearly independent mod \( M^\perp \). Consider
\[ \psi := f - \sum_{i=1}^m f(m_i) \varphi_i. \]

Since \( f \notin M^\perp \), not all the \( f(m_i) = 0 \). However, \( \psi(m_i) = 0 \) for \( i = 1, \ldots, n \), showing that \( \psi \in M^\perp \) so that \( S \) is linearly dependent mod \( M^\perp \) and thus contradicting our assumption. \( \square \)

**Lemma 3.** Suppose that \( X \) and its dual \( X^* \) are Banach spaces, and let \( T_2 \subseteq T_1 \) be densely defined operators \( X \rightarrow X \) with domains \( D_1 \) and \( D_2 \), and let \( T_1^* \subseteq T_2^* \) be their adjoints with domains \( D_1^* \) and \( D_2^* \). If \( \dim(D_1/D_2) = n < \infty \), then
\[ \dim \left( \frac{D_1}{D_2} \right) = \dim \left( \frac{D_2}{D_1} \right). \]

**Proof.** The technique is similar to that of Lemma 2. First note that
\[ \dim \left( \frac{D_1}{D_2} \right) = \dim \left( \frac{G(-T_1)}{G(-T_2)} \right) \quad \text{and} \quad \dim \left( \frac{D_2}{D_1} \right) = \dim \left( \frac{G(T_2^*)}{G(T_1^*)} \right). \]

Let \( \nu_i = (\alpha_i, -T_1(\alpha_i)), i = 1, \ldots, n \), be a linearly independent set mod \( G(-T_2) \). By the Hahn-Banach Theorem we can find
\[ \psi_j = (u_j, v_j) \in (G(-T_2))^\perp \equiv G(T_2^*) \subset X^* \times X^*, \quad j = 1, \ldots, n, \]

such that \( \psi_j(\nu_i) = u_j(\alpha_i) + v_j(-T_1(\alpha_i)) = \delta_{ij} \). Now \( \{\psi_j\}, j = 1, \ldots, n \), is a linearly independent set mod \( G(T_1^*) \), for if
\[ \eta = \sum_{j=1}^n c_j \psi_j \in G(T_1^*), \]
then \( \eta(\nu_i) = 0 = c_i \) for \( i = 1, \ldots, n \). This shows that \( \dim(D_1/D_2) \leq \dim(D_2/D_1) \). But if \( \{\psi_1, \ldots, \psi_n, \psi\} \), \( \psi \in G(T_2^*) \), is linearly independent mod \( G(T_1^*) \), this contradicts the fact that \( \psi - \sum_{i=1}^n d_i \psi_i \in G(T_1^*) \) where \( d_i = \psi(\nu_i) \). \( \square \)

**Lemma 4.** Suppose that \( X \) is a Banach space and \( X^* \) is its dual, and let \( T_1 : X^* \rightarrow X \) be an operator with domain \( D_1 \) and \( T_2 \) be a one-to-one restriction of \( T_1 \) with closed range having domain \( D_2 \subset D_1 \). If \( \dim(N(T_1)) < \infty \) and \( \dim^\perp R(T_2) \subseteq X < \infty \), then
\[ \dim \left( \frac{D_1}{D_2} \right) = \dim N(T_1) + \dim^\perp R(T_2). \]

**Proof.** In Lemma 2 we identify \( M \) with \( \perp R(T_2) \). Since \( R(T_2) \) is closed, \( M^\perp = R(T_2) \). We can conclude that
\[ \dim \left( \frac{R(T_1)}{R(T_2)} \right) = \dim^\perp R(T_2). \]

If \( k = \dim^\perp R(T_2) \), we can find nonzero elements \( f_1, \ldots, f_k \) in \( R(T_1) \) that are linearly independent mod \( R(T_2) \). Suppose \( y_i \in D_1 \) and \( M[y_i] = f_i, i = 1, \ldots, k. \)
Then it is easily verified that $S_1 = \{y_1, \ldots, y_k\}$ is a linearly independent set mod $D_2$ (for otherwise a linear combination of the $f_i$ would be in $R(T_2)$). Let $S_2 = \{z_1, \ldots, z_m\}$ be a basis of $N(T_1)$. We claim that $S_1 \cup S_2$ is linearly independent mod $D_2$. For assume that $\eta = \eta_1 + \eta_2 \in D_2$ where

$$\eta_1 = \sum_{i=1}^{k} c_i y_i, \quad \eta_2 = \sum_{i=1}^{m} d_i z_i$$

and not all the coefficients vanish. In particular, under this assumption, since $N(T_1) \cap D_2 = \{0\}$, not all the $c_i = 0$ vanish, for otherwise $\eta_2 \neq 0$ and $\eta_2 \in D_2 \cap N(T_1)$. But then

$$T_1(\eta) = T_1(\eta_1) = \sum_{i=1}^{k} c_i f_i \in R(T_2),$$

contradicting the linear independence of the $f_i$ mod $R(T_2)$. It follows that

$$\dim \left( \frac{D_2}{D_2} \right) \geq \dim N(T_1) + \dim \frac{1}{2} R(T_2) = m + k.$$ 

Suppose that we can adjoin an element $u \in D_1$ to $S_1 \cup S_2$ so that $S_3 = (S_1 \cup S_2) \cup \{u\}$ is linearly independent mod $D_2$. Since $N(T_1)$ is finite dimensional, it is complemented in $X$, and so $u$ can be written uniquely as a sum $u_1 + u_2$ where $u_1 \in N(T_1)$ and $u_2 \in (D_1 \setminus N(T_1)) \cup \{0\}$. If $\eta_3$ is an arbitrary linear combination of elements of $S_3$ with a nonzero coefficient $d$ of $u$, then

$$T_1(\eta_3) = \sum_{i=1}^{k} c_i f_i + d T_1(u_2) \notin R(T_2),$$

implying that $\dim(R(T_1)/R(T_2)) = k + 1$, which is false. $\square$

**Lemma 5.** For $p \in [1, \infty]$,

$$\dim N(T_p) = \dim \left( \frac{R(T_p)}{R(T_{0,p})} \right) = 1,$$

$$\dim \left( \frac{D_p}{D_{0,p}} \right) = 2.$$

**Proof.** Since $M$ is disconjugate on $I_a$ and since $q > 0$, it follows by Corollary 6.4 and Theorem 6.4 of Hartman [6] that there is a fundamental set of positive linearly independent solutions $y_1$ and $y_2$ of $M[y] = 0$, called respectively the principal and nonprincipal solutions, such that $y'_1 \leq 0$ and $y'_2 > 0$ on $I_a$. Additionally, $\lim_{t \to \infty} y_1 / y_2 = 0$. Suppose $y_p \in N(T_p)$ for $p \in [1, \infty]$. We claim that $y_p$ must be a multiple of $y_1$. For if $y_p = c_1 y_1 + c_2 y_2$ with $c_2 \neq 0$, then

$$\lim_{t \to \infty} y_p(t) = \lim_{t \to \infty} \left| y_2(c_1 y_1 / y_2 + c_2) \right| = |c_2| y_2.$$ 

(2.1)

Since $|y_p(t)|$ becomes arbitrarily close to a nondecreasing positive function, it cannot be in $L^p(I_a)$. This shows that $\dim N(T_p) \leq 1$, for $p \in [1, \infty]$, and if nontrivial, $N(T_p)$ is spanned by $\{y_1\}$. 

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We next show that \( N(T_p) \neq \{ 0 \} \). If \( r \in (1, \infty) \), then \( R(T_{0,r}) \) is closed by Lemma \( \mathbf{4} \) and from Theorem B (iv) and the fact that \( L^r(I_a) = L^r(I_b) \),
\begin{align}
(2.2) \quad & R(T_{0,r}) = \overline{\dim N(T_r)} = N(T_r), \\
(2.3) \quad & \overline{\dim R(T_{0,r})} = N(T_r).
\end{align}
Also, by Lemma \( \mathbf{1} \) (again), \( R(T_r) = L^r(I_a) \). We now identify \( M \) with \( N(T_r) \), \( X \) with \( L^r(I_a) \), and \( X^* \) with \( L^r(I_b) \). (2.2) and Lemma \( \mathbf{2} \) then give that
\begin{equation}
(2.4) \quad \dim \left( \frac{R(T_r)}{R(T_{0,r})} \right) = \dim N(T_r) \leq 1.
\end{equation}
Hence if \( \dim(R(T_r)/R(T_{0,r})) = 0 \), then \( N(T_r) = \{ 0 \} \), and by (2.3) and Lemma \( \mathbf{4} \), \( \dim(D_r/D_{0,r}) = \dim N(T_r) \). However, since we can find \( C_0^\infty \) linearly independent functions \( \phi_1, \phi_2 \) with support in \( I_a \) such that \( \phi_1(a) = 1 = \phi_2(b) \) and \( \phi_1'(a) = 0 = \phi_2'(b) \) it must be the case that \( \dim(D_r/D_{0,r}) \geq 2 \). So \( \dim N(T_r) = 2 \), which as we have seen from (2.1) is not possible given the properties of \( y_1 \) and \( y_2 \). This contradiction shows that \( \dim N(T_r) = 1 \) for \( r \in [1, \infty) \). If \( p = r \in (1, \infty) \) by (2.4), then \( \dim(R(T_p)/R(T_{0,p})) = 1 \) and if we choose \( r' = p \), then \( \dim N(T_p) = 1 \) for \( p \in [1, \infty) \). If \( p = \infty \), then \( y_1 \in N(T_\infty) \) since \( y_1 > 0 \) and \( y' \leq 0 \), and so \( \dim N(T_\infty) = 1 \). Since we have now established that \( \dim N(T_p) = \dim N(T_\infty) \) for \( p \in (1, \infty) \) by Lemma \( \mathbf{4} \) and (2.3), \( \dim(D_p/D_{0,p}) = 2 \). By Lemma \( \mathbf{3} \) \( \dim(D_1/D_{0,1}) = 2 \). Since \( T_1 \) is one-to-one on \( D_1 \cap N(T_1) \) we conclude that \( \dim(R(T_1)/R(T_{0,1})) = 1 \). The lemma is now established in all cases.

\textbf{Proof of Theorem 2} (i) and (ii) is the assertion of Lemma \( \mathbf{5} \). It follows that \( D_p = D_{0,p} \oplus \text{span} \{ \phi_1, \phi_2 \} \). Since \( \phi_1, \phi_2 \) vanish at \( \infty \), \( \lim_{t \to \infty} \{ y, z \}(t) = 0 \), which proves (iii).

\textbf{Proof of Theorem 2} This is also obvious from (2.1) and the proof of Lemma \( \mathbf{5} \). \( \square \)

\section*{References}


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