

SKIEW EXACTNESS PERTURBATION

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ABSTRACT. We offer a perturbation theory for finite ascent and descent properties of bounded operators.

There are various degrees of “skew exactness” ([11]; [8], (10.9.0.1), (10.9.0.2)) between compatible pairs of operators, bounded and linear between normed spaces:

1. Definition. Suppose $T : X \rightarrow Y$ and $S : Y \rightarrow Z$ are bounded and linear between normed spaces. Then we may classify the pair (S, T) as left skew exact if there is inclusion

$$(1.1) \quad S^{-1}(0) \cap T(X) = \{0\} ,$$

strongly left skew exact if there is $k > 0$ for which

$$(1.2) \quad \|T(\cdot)\| \leq k\|ST(\cdot)\| ,$$

and splitting left skew exact if there is $R \in BL(Z, Y)$ for which

$$(1.3) \quad T = RST .$$

Also we may classify the pair (S, T) as right skew exact if there is inclusion

$$(1.4) \quad S^{-1}(0) + T(X) = Y ,$$

strongly right skew exact if there is $k > 0$ for which: for every $y \in Y$ there is $x \in X$ for which

$$(1.5) \quad Sy = STx \text{ with } \|x\| \leq k\|y\| ,$$

and splitting right skew exact if there is $R \in BL(Y, X)$ for which

$$(1.6) \quad S = STR .$$

It is easy to see that

2. Theorem. *In the notation of Definition 1, there exist implications*

$$(2.1) \quad (1.3) \implies (1.2) \implies (1.1)$$

and

$$(2.2) \quad (1.6) \implies (1.5) \implies (1.4) .$$

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Proof. Most of this holds slightly more generally ([8], Theorems 10.1.2, 10.1.4), with a general operator $R' : X \rightarrow Z$ in place of the product ST . Note (cf. [4], (6.1)) that (1.1) holds iff

$$(2.3) \quad (ST)^{-1}(0) \subseteq T^{-1}(0) ,$$

and that (1.4) holds iff

$$(2.4) \quad S(Y) \subseteq ST(X). \quad \square$$

For Hilbert spaces X, Y, Z there exist ([8], Theorem 10.8.1) implications (1.2) \implies (1.3) and (1.5) \implies (1.6).

A slightly stronger version of condition (1.1) asks that

$$(2.5) \quad S^{-1}(0) \cap \text{cl } T(X) = \{0\} ,$$

which says that the operator $K_M J_N$ is one-to-one, where (cf. Yang [17]; [9]) $K_M : Y \rightarrow Y/M$ and $J_N : N \rightarrow Y$ are the natural quotient and injection induced by the subspaces $M = \text{cl } TX$ and $N = S^{-1}(0)$. Stronger again is the condition that there be $k > 0$ for which there is implication

$$(2.6) \quad y \in S^{-1}(0) \implies \|y\| \leq k \text{ dist}(y, T(X)) ,$$

which says that the same operator $K_M J_N$ is bounded below. Evidently

$$(2.7) \quad (1.2) \implies (2.6) \implies (2.5) \implies (1.1) :$$

if $k > 0$ satisfies (1.2) and if $Sy = 0$, then

$$\|y\| \leq \|y - Tx\| + \|Tx\| \leq \|y - Tx\| + k\|S(Tx - y)\| \leq (1 + k\|S\|)\|y - Tx\|. \quad \square$$

Condition (2.6), with $k = 1$, has been noticed by Anderson [1], who describes it by calling $T(X)$ *orthogonal to* $S^{-1}(0)$; we will prefer to say that $S^{-1}(0)$ is orthogonal to $T(X)$. Turnsek [16] has observed that it holds for certain operators on Banach algebras:

3. Theorem. *If $S \in BL(Y, Y)$, then (2.6) holds with $k = 1$ for (S, S) provided*

$$(3.1) \quad \|I - S\| \leq 1 .$$

Proof. Following the argument of Turnsek ([16], Theorem 1.1), write

$$S = I - U \text{ and } V_n = I + U + \dots + U^n ,$$

so that

$$(3.2) \quad SV_n = I - U^{n+1} = V_n S.$$

We have

$$Sy = 0 \implies (n+1)y = V_n y = (I - U^{n+1})x + V_n(y - Sx),$$

and hence

$$\|y\| \leq \frac{2}{n+1} \|x\| + \|y - Sx\| ;$$

now let $n \rightarrow \infty$. □

Alternatively, Theorem 3 is an application of Sinclair's Theorem ([15], Proposition 1; [3], Corollary 1), since here 0 is not in the interior of the "numerical range" of S . The argument of Theorem 3 suggests - wrongly - that we are using a weakened version of condition (1.3): we call the pair (S, T) *almost splitting left skew exact* if there are (R_n) in $BL(Z, Y)$ with

$$(3.3) \quad \|T - R_n S T\| \rightarrow 0 \text{ and } \sup_n \|R_n\| < \infty ,$$

and *almost splitting right skew exact* if instead (R_n) in $BL(Y, X)$ with

$$(3.4) \quad \|S - S T R_n\| \rightarrow 0 \text{ and } \sup_n \|R_n\| < \infty .$$

Also call (S, T) *almost strongly right skew exact* if there is $k > 0$ for which: for every $y \in Y$ there is (x_n) in X for which

$$(3.5) \quad \|S y - S T x_n\| \rightarrow 0 \text{ with } \sup_n \|x_n\| \leq k \|y\| .$$

Evidently (cf. [11], Theorem 10.1.2)

$$(3.6) \quad (1.3) \implies (3.3) \implies (1.2)$$

and

$$(3.7) \quad (1.6) \implies (3.4) \implies (3.5) ;$$

thus (3.3) implies (2.6). We do not, however, derive (3.3) for (S, S) from condition (3.1). We also remark that, whenever the space Z is complete, there is the implication

$$(3.8) \quad (1.4) \implies (3.5).$$

This ([2]; [5], Theorem 1.1; [8], Theorem 10.5.5) uses Baire's theorem.

Under certain circumstances the "left" and "right" skew exactnesses are equivalent. We begin (cf. [4], Lemma 6.2) by extending the finite ascent/descent characterizations:

4. Theorem. *Suppose, under the conditions of Definition 1, that $W \subseteq X$ with $T(W) \subseteq S^{-1}(0)$, and that $V \subseteq Y$ with $T(X) \subseteq S^{-1}(V)$. Then each of the following conditions is equivalent to (1.1):*

$$(4.1) \quad T^\vee : X/T^{-1}(0) \rightarrow Y/S^{-1}(0) \text{ one-to-one} ;$$

$$(4.2) \quad S^\wedge : T(X) \rightarrow V \text{ is one-to-one} .$$

Also each of the following conditions is equivalent to condition (1.4):

$$(4.3) \quad S^\wedge : T(X) \rightarrow S(Y) \text{ onto} ;$$

$$(4.4) \quad T^\vee : X/W \rightarrow Y/S^{-1}(0) \text{ is onto} .$$

Proof. The equivalences (1.1) \iff (4.1) and (1.4) \iff (4.3) are clear. We claim that (1.1) is equivalent to (4.2) with $V = Z$, and that this in turn is equivalent to (4.2) for arbitrary V for which $T(X) \subseteq S^{-1}V$. The second equivalence is clear; for the first note that for arbitrary $x \in X$ there is the implication

$$T x \in S^{-1}(0) \iff S T x = 0 .$$

We also claim that (1.4) is equivalent to (4.4) with $W = \{0\}$, and that this in turn is equivalent to (4.4) for arbitrary W for which $T(W) \subseteq S^{-1}(0)$. The second equivalence is clear; for the first note that for arbitrary $y \in Y$ there is the implication

$$y \in S^{-1}(0) + T(X) \iff S y \in S(TX) . \quad \square$$

If in particular $X = Y = Z$ and $ST = TS$, then (4.2) applies with $V = T(X)$, and (4.4) applies with $W = S^{-1}(0)$. We apply this in particular with $S = T^k$ for some $k \in \mathbf{N}$:

5. Theorem. *If $X = Y = Z$ and $S = T^k : Y \rightarrow Y$, with T in the “commutative closure” of the invertibles, in the sense that there are (R_n) in $BL(X, X)$ with*

$$(5.1) \quad R_n \in BL^{-1}(X, X) , R_n T = T R_n , \|R_n - T\| \rightarrow 0 ,$$

then the following are equivalent:

$$(5.2) \quad (TS)^{-1}(0) \subseteq S^{-1}(0) \text{ and } T(X) = \text{cl } T(X) ,$$

$$(5.3) \quad S(Y) \subseteq ST(X) \text{ and } T(X) = \text{cl } T(X) .$$

Proof. We recall ([6]; [8], Theorem 3.5.1) that for bounded linear operators $T : X \rightarrow Y$ between (possibly incomplete) normed spaces

$$(5.4) \quad T \text{ bounded below and a limit of dense range} \implies T \text{ almost open} ,$$

and hence ([6]; [8], Theorem 5.5.6) by duality

$$(5.5) \quad T \text{ almost open and a limit of bounded below} \implies T \text{ bounded below} .$$

Now if R_n commutes with T , then it leaves both $T(X)$ and $S^{-1}(0)$ invariant, and if R_n is invertible, then (cf. [8], Theorem 3.11.1) its restriction R_n^\wedge to $T(X)$ will be bounded below and its quotient on $Y/S^{-1}(0)$ will be onto. Thus if we assume (5.2), then by (4.1) and closed range, T^\vee will be bounded below and the limit of onto R_n^\vee , therefore onto, giving (5.3). If instead we assume (5.3), then by (4.3) S^\wedge will be onto and by closed range almost open, and the limit of bounded below $(R_n^k)^\wedge$, therefore bounded below, giving (5.2). \square

(5.2) and (5.3) are together equivalent to the condition that $T \in BL(X, X)$ is *polar* ([8], Definition 7.5.2) in the sense that $0 \in \mathbf{C}$ is at worst a pole of the resolvent function $(zI - T)^{-1}$. If we relax the closed range condition we can still [14] get one of the implications, provided we further tighten the approximation by commuting invertible operators:

6. Theorem. *Suppose that $S = T^k$ and that $0 \notin \text{int } \sigma(T)$. If the finite descent condition (1.4) holds, then so also does the finite ascent condition (5.2), including closed range.*

Proof. This is shown on Hilbert space ([14], Lemma 2.5) by Herrero, Larson and Wogen. Alternatively, since we are assuming that 0 is at worst on the boundary of the spectrum, then we can take the approximating invertible operators $R_n = T - \lambda_n I$ to be scalar perturbations of the operator T . Now if (1.4) holds, then the quotient operator T^\vee on $X/S^{-1}(0)$ is (4.5) onto, and the limit of operators $(T - \lambda_n I)^\vee$, which we claim are invertible. As in Theorem 5 it is clear that the quotient $(T - \lambda_n I)^\vee$ is onto: we claim it is also one-to-one. To see this recall that the operator $T - \lambda_n I$ is one-to-one and the restriction $(T - \lambda_n I)^\wedge = (-\lambda_n I)^\wedge$ to the subspace $T^{-1}(0)$ is onto, so that ([5], Theorem 3.11.2) the induced quotient is also one-to-one. For the closed range note that $T(X)$ now has a closed complement, and appeal to the “Lemma of Neuberger” ([8], Theorem 4.8.2). \square

Theorem 6 does not reverse:

7. Example. If

$$(7.1) \quad S = I - \lambda U \text{ or } S = I - \lambda V \text{ or } S = \lambda W ,$$

where $|\lambda| = 1$, U and V are the forward and backward shifts on ℓ_2 , and W the standard weight,

$$(7.2) \quad (Ux)_1 = 0 , (Ux)_{n+1} = x_n ; (Vx)_n = x_{n+1} ; (Wx)_n = (1/n)x_n ,$$

then S is one-to-one and not onto, therefore of finite descent and not of finite ascent, while

$$(7.3) \quad \|I - S\| = 1 \text{ so that } 0 \notin \text{int } \sigma(S) .$$

Proof. This is easily checked: note that, extended to all sequences, there is equivalence, for arbitrary $x \in X^{\mathbb{N}}$,

$$(7.4) \quad x \in (I - \lambda U)^{-1} \iff x \in (I - \bar{\lambda} V)^{-1} \iff x = x_1(1, \lambda, \lambda^2, \dots) . \quad \square$$

We need some auxiliary subspaces:

8. Definition. If $T \in BL(X, X)$ write

$$(8.1) \quad T^{-\infty}(0) = \bigcup_{n=1}^{\infty} T^{-n}(0) \text{ and } T^{\infty}(X) = \bigcap_{n=1}^{\infty} T^n(X)$$

for the hyperkernel and the hyperrange of T , and

$$(8.2) \quad E_X(T) = \sum_{\lambda \in \mathbb{C}} (T - \lambda I)^{-\infty}(0) \text{ and } F_X(T) = \bigcap_{\lambda \in \mathbb{C}} (T - \lambda I)^{\infty}(X) .$$

Each of the subspaces in Definition 8 is linear, not necessarily closed, and *hyperinvariant* under T . We recall that $T \in BL(X, X)$ is called *algebraic* if there is a nontrivial *polynomial* $0 \neq p \in \text{Poly}$ for which

$$(8.3) \quad p(T) = 0 ;$$

more generally T is said to be *locally algebraic* if

$$(8.4) \quad X = \bigcup \{p(T)^{-1}(0) : 0 \neq p \in \text{Poly}\} .$$

For the record

9. Theorem. If $T \in BL(X, X)$ for a Banach space X , then

$$(9.1) \quad T \text{ locally algebraic} \implies T \text{ algebraic} .$$

Necessary and sufficient for T to have finite descent is that

$$(9.2) \quad E_X(T) + T(X) = X .$$

Proof. The first part of this is known as *Kaplansky's Lemma*; the proof [10], [12] is a combination of Baire's theorem and the Euclidean algorithm for polynomials. The Euclidean algorithm also gives equality

$$(9.3) \quad E_X(T) = \bigcup \{p(T)^{-1}(0) : 0 \neq p \in \text{Poly}\} = \{x \in X : \dim \text{Poly}(T)x < \infty\} ,$$

and dually

$$(9.4) \quad F_X(T) = \bigcap \{p(T)(X) : 0 \neq p \in \text{Poly}\} .$$

Then again with a combination of Baire's theorem and the Euclidean algorithm, if $T \in BL(X, X)$ there is ([14], Lemma 2.4) $k \in \mathbf{N}$ for which

$$(9.5) \quad E_X(T) + T(X) = T^{-\infty}(0) + T(X) = T^{-k}(0) + T(X) . \quad \square$$

Dually, using the Euclidean algorithm, we get half way:

$$(9.6) \quad F_X(T) \cap T^{-1}(0) = T^\infty(X) \cap T^{-1}(0) .$$

For the essence of a possible spectral mapping theorem (cf. [11]), we have

10. Theorem. *If $S, T \in BL(X, X)$ satisfy $ST = TS$ and either*

$$(10.1) \quad S \in \{T^k : k \in \mathbf{N}\}$$

or

$$(10.2) \quad VS - TU = I \text{ with } \{U, V\} \subseteq \text{comm}(S, T) ,$$

then there is the equivalence

$$(10.3) \quad ST \text{ of finite ascent} \iff S, T \text{ of finite ascent} ,$$

and the equivalence

$$(10.4) \quad ST \text{ of finite descent} \iff S, T \text{ of finite descent} .$$

Proof. The backward implications are easy ([8], Theorem 7.9.2): if S and T commute and satisfy $S^{-k}(0) = S^{-k-1}(0)$ and $T^{-k}(0) = T^{-k-1}(0)$, then

$$\begin{aligned} (ST)^{-k}(0) &= S^{-k}T^{-k}(0) = S^{-k}T^{-k-1}(0) = T^{-k-1}S^{-k}(0) \\ &= T^{-k-1}S^{-k-1}(0) = (ST)^{-k-1}(0) . \end{aligned}$$

If instead $ST = TS$ with $S^k X = S^{k+1} X$ and $T^k X = T^{k+1} X$, then

$$(ST)^k X = S^k T^k X = S^k T^{k+1} X = T^{k+1} S^k X = T^{k+1} S^{k+1} X = (ST)^{k+1} X .$$

Also the forward implications are clear when (10.1) $S = T^k$ is a power of T . If instead we assume (10.2), then we argue

$$(10.5) \quad (ST)^{-1}(0) \subseteq T^{-1}(0) + T(X) \text{ and } (ST)X \supseteq T^{-1}(0) \cap T(X) ,$$

while if (U, V) satisfies (10.2), then for arbitrary $k \in \mathbf{N}$

$$(10.6) \quad V_k S^k - T^k U_k = I \text{ with } \{U_k, V_k\} \subseteq \text{comm}(S^k, T^k) .$$

To verify (10.5) argue

$$\begin{aligned} STx = 0 &\implies x + TUx = VSx \in T^{-1}(0) ; \\ T(Tx) = 0 &\implies Tx = TVSx - TUTx = (ST)(Vx) . \end{aligned}$$

For (10.6) note that for arbitrary $k \in \mathbf{N}$

$$VS - TU = I \implies V^{k+1}S^{k+1} - TU(I + VS + \dots + V^k S^k) = I . \quad \square$$

For an induced "spectrum" to be a closed set we have

11. Theorem. *$T \in BL(X, X)$ is of finite descent; then so is $T - \lambda I$ for sufficiently small $\lambda \in \mathbf{C}$.*

Proof. This has been shown on Hilbert space by Han, Larson, and Pan ([13], Lemma 2.2, Theorem 2.4). It is clear from the open mapping theorem (applied to condition (4.4) with $W = \{0\}$) that if condition (1.4) holds, then also

$$S^{-1}(0) + (T - U)(X) = Y$$

whenever $T - U \in BL(X, Y)$ is sufficiently close to $T \in BL(X, Y)$. The problem is that we must also perturb S . However if $S = T^k$ and $U = \lambda I$, so that $E_X(T - U) = E_X(T)$, then we can argue

$$E_X(T - U) + (T - U)(X) = E_X(T) + (T - U)(X) \supseteq S^{-1}(0) + (T - U)(X) = X.$$

The subspaces of Definition 8 lead to certain special kinds of operators:

12. Definition. We shall call $T \in BL(X, X)$ triangular if the subspace $E_X(T)$ is dense:

$$(12.1) \quad \text{cl } E_X(T) = X .$$

Dually $T \in BL(X, X)$ is co-triangular if the subspace $F_X(T)$ is trivial:

$$(12.2) \quad F_X(T) = \{0\} .$$

The shifts of Example 7 are either triangular or co-triangular:

13. Example. On each of the spaces c_0 and ℓ_p ($1 \leq p < \infty$), the forward shift U is triangular, the backward shift V is co-triangular and the standard weight W is both triangular and co-triangular.

Proof. The hyperkernel of the backward shift is dense, since it includes all the “terminating” sequences:

$$(13.1) \quad V^{-\infty}(0) \supseteq c_{00} .$$

Thus

$$(13.2) \quad E(V) \supseteq V^{-\infty}(0) \text{ is dense}$$

and also

$$(13.3) \quad F(V) = \bigcap_{|\lambda|=1} (V - \lambda I)^\infty(X) \supseteq \sum_{|\lambda|<1} (V - \lambda I)^{-\infty}(0) \supseteq V^{-\infty}(0) \text{ is dense} .$$

Since $U - \lambda I$ is one-to-one for every $\lambda \in \mathbf{C}$ we have

$$(13.4) \quad E(U) = \{0\} \text{ is trivial}$$

and also

$$(13.5) \quad F(U) \subseteq U^\infty(X) = \{0\} \text{ is trivial} .$$

Finally we notice that the weight W commutes with the projection UV , and more generally

$$(13.6) \quad WU^nV^n = U^nV^nW \quad (n \in \mathbf{N}) ;$$

also for each $n \in \mathbf{N}$

$$(13.7) \quad \begin{aligned} \left(\frac{1}{n}I - W\right)^{-1}(0) &= U^{n-1}(I - UV)V^{n-1}(X), \\ \left(\frac{1}{n}I - W\right)(X) &= (U^{n-1}(I - UV)V^{n-1})^{-1}(0) , \end{aligned}$$

so that $E(W)$ is dense and $F(W)$ is trivial. □

Triangularity and Fredholmness co-operate to generate finite ascent or descent:

14. Theorem. *If $T \in BL(X, X)$, then*

$$(14.1) \quad T \text{ upper semi-Fredholm and co-triangular} \implies T \text{ of finite ascent}$$

and

$$(14.2) \quad T \text{ lower semi-Fredholm and triangular} \implies T \text{ of finite descent} .$$

Proof. If $T \in BL(X, X)$ is upper semi-Fredholm, then the finite ascent condition can be written in the form

$$(14.3) \quad F_X(T) \cap T^{-1}(0) = \{0\} .$$

Indeed since ([8], Theorem 6.12.2) each power T^m is also upper semi-Fredholm, then $T^{-m}(0)$ is finite dimensional for each $m \in \mathbf{N}$ and $T^m(X)$ is closed; thus if for each $m \in \mathbf{N}$ we have $T^m(X) \cap T^{-1}(0) \neq \{0\}$, then there is (x_m) in X for which

$$\|T^m(x_m)\| = 1 \text{ and } T^{m+1}x_m = 0 .$$

By local compactness there is a subsequence

$$(y_m) = T^{\phi(m)}(x_{\phi(m)}) \rightarrow y_\infty \in T^\infty(X) ,$$

using the closedness of all the ranges, so that $\|y_\infty\| = 1$ and $y_\infty \in F_X(T) \cap T^{-1}(0)$. This proves (14.1); towards (14.2) we claim that for subspaces $Y, Z \subseteq X$

$$(14.4) \quad Y \text{ closed of finite codimension and } Z \text{ dense} \implies Y + Z = X ,$$

because if $\dim(X/Y) = n$ find successively e_1, e_2, \dots, e_n with

$$e_{j+1} \in Z \setminus (Y + \mathbf{C}e_1 + \mathbf{C}e_2 + \dots + \mathbf{C}e_j).$$

Applying this with $Y = T(X)$ and $Z = E_X(T)$ gives (14.2). \square

It is clear that in (14.1) we can replace the ‘‘co-triangular’’ condition (12.2) by the weaker condition (14.3); dually in (14.2) we can replace the triangular condition (12.1) by the weaker condition

$$(14.5) \quad \text{cl } E_X(T) + T(X) = X .$$

We cannot however remove the semi-Fredholm conditions: for example [12] if $T = U \otimes V$ is the tensor product of the shifts, then (14.3) holds but we never get $T^k(X) \cap T^{-1}(0) = \{0\}$. For operators which are both upper semi-Fredholm and of finite ascent, or lower semi-Fredholm of finite descent (‘‘semi Browder’’ in the sense of [8], Definition 7.9.1) the conditions of Theorem 10 can be replaced by simple commutivity ([8], Theorem 7.9.2).

REFERENCES

- [1] J. Anderson, *On normal derivations*, Proc. Amer. Math. Soc. **38** (1973) 136-140. MR **47**:875
- [2] M.R. Embry, *Factorization of operators on Banach spaces*, Proc. Amer. Math. Soc. **38** (1973) 587-590. MR **47**:849
- [3] C.-K. Fong, *Normal operators on Banach spaces*, Glasgow Math. Jour. **20** (1979) 163-168. MR **80e**:47020
- [4] S. Grabiner, *Generalizations of Fredholm operators*, Banach algebras '97 pp 170-187, de Gruyter 1998. MR **99i**:47022
- [5] R.E. Harte, *Berberian-Quigley and the ghost of a spectral mapping theorem*, Proc. Royal Irish Acad. **78A** (1978) 63-68. MR **80b**:47004
- [6] R.E. Harte, *Almost open mappings between normed spaces*, Proc. Amer. Math. Soc. **90** (1984) 243-249. MR **85b**:47024

- [7] R.E. Harte, *Almost exactness between normed spaces*, Proc. Amer. Math. Soc. **100** (1987) 257-265. MR **88g**:47004
- [8] R.E. Harte, *Invertibility and singularity*, Dekker, New York 1988.
- [9] R.E. Harte, *Eine Kleine Gapmusik*, PanAmerican Jour. **2** (1992) 101-102. MR **92k**:47003
- [10] R.E. Harte, *Taylor exactness and Kaplansky's lemma*, Jour. Op. Th. **25** (1991) 399-416. MR **94d**:47002
- [11] R.E. Harte, *Exactness plus skew exactness equals invertibility*, Proc. Royal Irish Acad. **97A** (1997) 15-18. MR **99h**:46085
- [12] R.E. Harte, *Variations on a theme of Kaplansky*, Matematicki Vesnik (to appear).
- [13] D. Han, D.R. Larson and Z. Pan, *The triangular extension spectrum and algebraic extensions of operators*, preprint.
- [14] D.A. Herrero, D.R. Larson and W.R. Wogen, *Semitriangular operators*, Houston Jour. Math. **17** (1991) 477-499. MR **92m**:47037
- [15] A.M. Sinclair, *Eigenvalues on the boundary of the numerical range*, Pacific Jour. Math. **35** (1970) 231-234. MR **42**:5086
- [16] A. Turnsek, *Elementary operators and orthogonality*, Linear Alg. Anal. **317** (2000) 207-216. MR **2002c**:47081
- [17] K.W. Yang, *Index of Fredholm operators*, Proc. Amer. Math. Soc. **41** (1973) 329-330. MR **47**:7492

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