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SKEW EXACTNESS PERTURBATION

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Abstract. We offer a perturbation theory for finite ascent and descent properties of bounded operators.

There are various degrees of "skew exactness" ([11]; [8], (10.9.0.1), (10.9.0.2)) between compatible pairs of operators, bounded and linear between normed spaces:

1. Definition. Suppose $T : X \to Y$ and $S : Y \to Z$ are bounded and linear between normed spaces. Then we may classify the pair (S, T) as left skew exact if there is inclusion

$$S^{-1}(0) \cap T(X) = \{0\} \ , \tag{1.1}$$

strongly left skew exact if there is $k > 0$ for which

$$\|T(\cdot)\| \leq k\|ST(\cdot)\| \ , \tag{1.2}$$

and splitting left skew exact if there is $R \in BL(Z, Y)$ for which

$$T = RST \ . \tag{1.3}$$

Also we may classify the pair (S, T) as right skew exact if there is inclusion

$$S^{-1}(0) + T(X) = Y \ , \tag{1.4}$$

strongly right skew exact if there is $k > 0$ for which: for every $y \in Y$ there is $x \in X$ for which

$$Sy = STx \ \textit{with} \ \|x\| \leq k\|y\| \ , \tag{1.5}$$

and splitting right skew exact if there is $R \in BL(Y, X)$ for which

$$S = STR \ . \tag{1.6}$$

It is easy to see that

2. Theorem. *In the notation of Definition* 1*, there exist implications*

$$(1.3) \Longrightarrow (1.2) \Longrightarrow (1.1) \tag{2.1}$$

and

$$(1.6) \Longrightarrow (1.5) \Longrightarrow (1.4) \ . \tag{2.2}$$

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Proof. Most of this holds slightly more generally ([8], Theorems 10.1.2, 10.1.4), with a general operator $R' : X \to Z$ in place of the product ST. Note (cf. [4], (6.1)) that (1.1) holds iff

$$(ST)^{-1}(0) \subseteq T^{-1}(0) \ , \tag{2.3}$$

and that (1.4) holds iff

$$S(Y) \subseteq ST(X). \quad \square \tag{2.4}$$

For Hilbert spaces X, Y, Z there exist ([8], Theorem 10.8.1) implications (1.2)$\Longrightarrow$(1.3) and (1.5)$\Longrightarrow$(1.6).

A slightly stronger version of condition (1.1) asks that

$$S^{-1}(0) \cap \text{cl } T(X) = \{0\} \ , \tag{2.5}$$

which says that the operator $K_M J_N$ is one-to-one, where (cf. Yang [17]; [9]) $K_M : Y \to Y/M$ and $J_N : N \to Y$ are the natural quotient and injection induced by the subspaces $M = \text{cl } TX$ and $N = S^{-1}(0)$. Stronger again is the condition that there be $k > 0$ for which there is implication

$$y \in S^{-1}(0) \Longrightarrow \|y\| \leq k \text{ dist}(y, T(X)) \ , \tag{2.6}$$

which says that the same operator $K_M J_N$ is bounded below. Evidently

$$(1.2) \Longrightarrow (2.6) \Longrightarrow (2.5) \Longrightarrow (1.1) \ : \tag{2.7}$$

if $k > 0$ satisfies (1.2) and if $Sy = 0$, then

$$\|y\| \leq \|y - Tx\| + \|Tx\| \leq \|y - Tx\| + k\|S(Tx - y)\| \leq (1 + k\|S\|)\|y - Tx\|. \quad \square$$

Condition (2.6), with $k = 1$, has been noticed by Anderson [1], who describes it by calling $T(X)$ *orthogonal to* $S^{-1}(0)$; we will prefer to say that $S^{-1}(0)$ is orthogonal to $T(X)$. Turnsek [16] has observed that it holds for certain operators on Banach algebras:

3. Theorem. *If $S \in BL(Y, Y)$, then* (2.6) *holds with $k = 1$ for (S, S) provided*

$$\|I - S\| \leq 1 \ . \tag{3.1}$$

Proof. Following the argument of Turnsek ([16], Theorem 1.1), write

$$S = I - U \text{ and } V_n = I + U + \ldots + U^n \ ,$$

so that

$$SV_n = I - U^{n+1} = V_n S. \tag{3.2}$$

We have

$$Sy = 0 \Longrightarrow (n+1)y = V_n y = (I - U^{n+1})x + V_n(y - Sx),$$

and hence

$$\|y\| \leq \frac{2}{n+1}\|x\| + \|y - Sx\| \ ;$$

now let $n \to \infty$. $\square$

Alternatively, Theorem 3 is an application of Sinclair's Theorem ([15], Proposition 1; [3], Corollary 1), since here 0 is not in the interior of the "numerical range" of S. The argument of Theorem 3 suggests - wrongly - that we are using a weakened version of condition (1.3): we call the pair (S,T) *almost splitting left skew exact* if there are (R_n) in $BL(Z,Y)$ with

$$\|T - R_nST\| \to 0 \text{ and } \sup_n\|R_n\| < \infty \ , \tag{3.3}$$

and *almost splitting right skew exact* if instead (R_n) in $BL(Y,X)$ with

$$\|S - STR_n\| \to 0 \text{ and } \sup_n\|R_n\| < \infty \ . \tag{3.4}$$

Also call (S,T) *almost strongly right skew exact* if there is $k > 0$ for which: for every $y \in Y$ there is (x_n) in X for which

$$\|Sy - STx_n\| \to 0 \text{ with } \sup_n\|x_n\| \le k\|y\| \ . \tag{3.5}$$

Evidently (cf. [11], Theorem 10.1.2)

$$(1.3) \Longrightarrow (3.3) \Longrightarrow (1.2) \tag{3.6}$$

and

$$(1.6) \Longrightarrow (3.4) \Longrightarrow (3.5) \ ; \tag{3.7}$$

thus (3.3) implies (2.6). We do not, however, derive (3.3) for (S,S) from condition (3.1). We also remark that, whenever the space Z is complete, there is the implication

$$(1.4) \Longrightarrow (3.5). \tag{3.8}$$

This ([2]; [5], Theorem 1.1; [8], Theorem 10.5.5) uses Baire's theorem.

Under certain circumstances the "left" and "right" skew exactnesses are equivalent. We begin (cf. [4], Lemma 6.2) by extending the finite ascent/descent characterizations:

4. Theorem. *Suppose, under the conditions of Definition* 1*, that* $W \subseteq X$ *with* $T(W) \subseteq S^{-1}(0)$*, and that* $V \subseteq Y$ *with* $T(X) \subseteq S^{-1}(V)$*. Then each of the following conditions is equivalent to* (1.1)*:*

$$T^\vee : X/T^{-1}(0) \to Y/S^{-1}(0) \ \textit{one-to-one} \ ; \tag{4.1}$$

$$S^\wedge : T(X) \to V \ \textit{is one-to-one} \ . \tag{4.2}$$

Also each of the following conditions is equivalent to condition (1.4)*:*

$$S^\wedge : T(X) \to S(Y) \ \textit{onto} \ ; \tag{4.3}$$

$$T^\vee : X/W \to Y/S^{-1}(0) \ \textit{is onto} \ . \tag{4.4}$$

Proof. The equivalences (1.1)$\Longleftrightarrow$(4.1) and (1.4)$\Longleftrightarrow$(4.3) are clear. We claim that (1.1) is equivalent to (4.2) with $V = Z$, and that this in turn is equivalent to (4.2) for arbitrary V for which $T(X) \subseteq S^{-1}V$. The second equivalence is clear; for the first note that for arbitrary $x \in X$ there is the implication

$$Tx \in S^{-1}(0) \Longleftrightarrow STx = 0 \ .$$

We also claim that (1.4) is equivalent to (4.4) with $W = \{0\}$, and that this in turn is equivalent to (4.4) for arbitrary W for which $T(W) \subseteq S^{-1}(0)$. The second equivalence is clear; for the first note that for arbitrary $y \in Y$ there is the implication

$$y \in S^{-1}(0) + T(X) \Longleftrightarrow Sy \in S(TX). \qquad \square$$

If in particular $X = Y = Z$ and $ST = TS$, then (4.2) applies with $V = T(X)$, and (4.4) applies with $W = S^{-1}(0)$. We apply this in particular with $S = T^k$ for some $k \in \mathbf{N}$:

5. Theorem. *If $X = Y = Z$ and $S = T^k : Y \to Y$, with T in the "commutative closure" of the invertibles, in the sense that there are (R_n) in $BL(X, X)$ with*

$$R_n \in BL^{-1}(X, X) \ , \ R_n T = T R_n \ , \ \|R_n - T\| \to 0 \ , \tag{5.1}$$

then the following are equivalent:

$$(TS)^{-1}(0) \subseteq S^{-1}(0) \text{ and } T(X) = \text{cl } T(X) \ , \tag{5.2}$$

$$S(Y) \subseteq ST(X) \text{ and } T(X) = \text{cl } T(X) \ . \tag{5.3}$$

Proof. We recall ([6]; [8], Theorem 3.5.1) that for bounded linear operators $T : X \to Y$ between (possibly incomplete) normed spaces

$$T \text{ bounded below and a limit of dense range } \implies T \text{ almost open} \ , \tag{5.4}$$

and hence ([6]; [8], Theorem 5.5.6) by duality

$$T \text{ almost open and a limit of bounded below } \implies T \text{ bounded below} \ . \tag{5.5}$$

Now if R_n commutes with T, then it leaves both $T(X)$ and $S^{-1}(0)$ invariant, and if R_n is invertible, then (cf. [8], Theorem 3.11.1) its restriction $R_n^\wedge$ to $T(X)$ will be bounded below and its quotient on $Y/S^{-1}(0)$ will be onto. Thus if we assume (5.2), then by (4.1) and closed range, $T^\vee$ will be bounded below and the limit of onto $R_n^\vee$, therefore onto, giving (5.3). If instead we assume (5.3), then by (4.3) $S^\wedge$ will be onto and by closed range almost open, and the limit of bounded below $(R_n^k)^\wedge$, therefore bounded below, giving (5.2). □

(5.2) and (5.3) are together equivalent to the condition that $T \in BL(X, X)$ is *polar* ([8], Definition 7.5.2) in the sense that $0 \in \mathbf{C}$ is at worst a pole of the resolvent function $(zI - T)^{-1}$. If we relax the closed range condition we can still [14] get one of the implications, provided we further tighten the approximation by commuting invertible operators:

6. Theorem. *Suppose that $S = T^k$ and that $0 \notin \text{int } \sigma(T)$. If the finite descent condition* (1.4) *holds, then so also does the finite ascent condition* (5.2)*, including closed range.*

Proof. This is shown on Hilbert space ([14], Lemma 2.5) by Herrero, Larson and Wogen. Alternatively, since we are assuming that 0 is at worst on the boundary of the spectrum, then we can take the approximating invertible operators $R_n = T - \lambda_n I$ to be scalar perturbations of the operator T. Now if (1.4) holds, then the quotient operator $T^\vee$ on $X/S^{-1}(0)$ is (4.5) onto, and the limit of operators $(T - \lambda_n I)^\vee$, which we claim are invertible. As in Theorem 5 it is clear that the quotient $(T - \lambda_n I)^\vee$ is onto: we claim it is also one-to-one. To see this recall that the operator $T - \lambda_n I$ is one-to-one and the restriction $(T - \lambda_n I)^\wedge = (-\lambda_n I)^\wedge$ to the subspace $T^{-1}(0)$ is onto, so that ([5], Theorem 3.11.2) the induced quotient is also one-to-one. For the closed range note that $T(X)$ now has a closed complement, and appeal to the "Lemma of Neuberger" ([8], Theorem 4.8.2). □

Theorem 6 does not reverse:

7. Example. If

$$S = I - \lambda U \text{ or } S = I - \lambda V \text{ or } S = \lambda W \ , \tag{7.1}$$

where $|\lambda| = 1$, U and V are the forward and backward shifts on ℓ_2, and W the standard weight,

$$(Ux)_1 = 0 \ , \ (Ux)_{n+1} = x_n \ ; \ (Vx)_n = x_{n+1} \ ; \ (Wx)_n = (1/n)x_n \ , \tag{7.2}$$

then S is one-to-one and not onto, therefore of finite descent and not of finite ascent, while

$$\|I - S\| = 1 \text{ so that } 0 \notin \operatorname{int} \sigma(S) \ . \tag{7.3}$$

Proof. This is easily checked: note that, extended to all sequences, there is equivalence, for arbitrary $x \in X^{\mathbf{N}}$,

$$x \in (I - \lambda U)^{-1} \Longleftrightarrow x \in (I - \overline{\lambda} V)^{-1} \Longleftrightarrow x = x_1(1, \lambda, \lambda^2, \ldots) \ . \quad \square \tag{7.4}$$

We need some auxiliary subspaces:

8. Definition. If $T \in BL(X, X)$ write

$$T^{-\infty}(0) = \bigcup_{n=1}^{\infty} T^{-n}(0) \ \text{ and } T^{\infty}(X) = \bigcap_{n=1}^{\infty} T^n(X) \tag{8.1}$$

for the hyperkernel and the hyperrange of T, and

$$E_X(T) = \sum_{\lambda \in \mathbf{C}} (T - \lambda I)^{-\infty}(0) \text{ and } F_X(T) = \bigcap_{\lambda \in \mathbf{C}} (T - \lambda I)^{\infty}(X) \ . \tag{8.2}$$

Each of the subspaces in Definition 8 is linear, not necessarily closed, and *hyperinvariant* under T. We recall that $T \in BL(X, X)$ is called *algebraic* if there is a nontrivial *polynomial* $0 \neq p \in \text{Poly}$ for which

$$p(T) = 0 \ ; \tag{8.3}$$

more generally T is said to be *locally algebraic* if

$$X = \bigcup \{p(T)^{-1}(0) : 0 \neq p \in \text{Poly}\} \ . \tag{8.4}$$

For the record

9. Theorem. *If $T \in BL(X, X)$ for a Banach space X, then*

$$T \textit{ locally algebraic} \implies T \textit{ algebraic} \ . \tag{9.1}$$

Necessary and sufficient for T to have finite descent is that

$$E_X(T) + T(X) = X \ . \tag{9.2}$$

Proof. The first part of this is known as *Kaplansky's Lemma*; the proof [10], [12] is a combination of Baire's theorem and the Euclidean algorithm for polynomials. The Euclidean algorithm also gives equality

$$E_X(T) = \bigcup \{p(T)^{-1}(0) : 0 \neq p \in \text{Poly}\} = \{x \in X : \dim \text{Poly}(T)x < \infty\} \ , \tag{9.3}$$

and dually

$$F_X(T) = \bigcap \{p(T)(X) : 0 \neq p \in \text{Poly}\} \ . \tag{9.4}$$

Then again with a combination of Baire's theorem and the Euclidean algorithm, if $T \in BL(X,X)$ there is ([14], Lemma 2.4) $k \in \mathbf{N}$ for which

$$E_X(T) + T(X) = T^{-\infty}(0) + T(X) = T^{-k}(0) + T(X) \ . \quad \square \tag{9.5}$$

Dually, using the Euclidean algorithm, we get half way:

$$F_X(T) \cap T^{-1}(0) = T^{\infty}(X) \cap T^{-1}(0) \ . \tag{9.6}$$

For the essence of a possible spectral mapping theorem (cf. [11]), we have

10. Theorem. *If $S, T \in BL(X,X)$ satisfy $ST = TS$ and either*

$$S \in \{T^k : k \in \mathbf{N}\} \tag{10.1}$$

or

$$VS - TU = I \text{ with } \{U, V\} \subseteq \mathrm{comm}(S,T) \ , \tag{10.2}$$

then there is the equivalence

$$ST \text{ of finite ascent} \iff S \ , \ T \text{ of finite ascent} \ , \tag{10.3}$$

and the equivalence

$$ST \text{ of finite descent} \iff S \ , \ T \text{ of finite descent} \ . \tag{10.4}$$

Proof. The backward implications are easy ([8], Theorem 7.9.2): if S and T commute and satisfy $S^{-k}(0) = S^{-k-1}(0)$ and $T^{-k}(0) = T^{-k-1}(0)$, then

$$\begin{aligned}(ST)^{-k}(0) &= S^{-k}T^{-k}(0) = S^{-k}T^{-k-1}(0) = T^{-k-1}S^{-k}(0) \\ &= T^{-k-1}S^{-k-1}(0) = (ST)^{-k-1}(0) \ .\end{aligned}$$

If instead $ST = TS$ with $S^kX = S^{k+1}X$ and $T^kX = T^{k+1}X$, then

$$(ST)^kX = S^kT^k(X) = S^kT^{k+1}X = T^{k+1}S^kX = T^{k+1}S^{k+1}X = (ST)^{k+1}X \ .$$

Also the forward implications are clear when (10.1) $S = T^k$ is a power of T. If instead we assume (10.2), then we argue

$$(ST)^{-1}(0) \subseteq T^{-1}(0) + T(X) \text{ and } (ST)X \supseteq T^{-1}(0) \cap T(X) \ , \tag{10.5}$$

while if (U,V) satisfies (10.2), then for arbitrary $k \in \mathbf{N}$

$$V_kS^k - T^kU_k = I \text{ with } \{U_k, V_k\} \subseteq \mathrm{comm}(S^k, T^k) \ . \tag{10.6}$$

To verify (10.5) argue

$$\begin{aligned}STx = 0 &\Longrightarrow x + TUx = VSx \in T^{-1}(0) \ ; \\ T(Tx) = 0 &\Longrightarrow Tx = TVSx - TUTx = (ST)(Vx) \ .\end{aligned}$$

For (10.6) note that for arbitrary $k \in \mathbf{N}$

$$VS - TU = I \Longrightarrow V^{k+1}S^{k+1} - TU(I + VS + \ldots + V^kS^k) = I. \qquad \square$$

For an induced "spectrum" to be a closed set we have

11. Theorem. *$T \in BL(X,X)$ is of finite descent; then so is $T - \lambda I$ for sufficiently small $\lambda \in \mathbf{C}$.*

Proof. This has been shown on Hilbert space by Han, Larson, and Pan ([13], Lemma 2.2, Theorem 2.4). It is clear from the open mapping theorem (applied to condition (4.4) with $W = \{0\}$) that if condition (1.4) holds, then also

$$S^{-1}(0) + (T - U)(X) = Y$$

whenever $T - U \in BL(X, Y)$ is sufficiently close to $T \in BL(X, Y)$. The problem is that we must also perturb S. However if $S = T^k$ and $U = \lambda I$, so that $E_X(T-U) = E_X(T)$, then we can argue

$$E_X(T - U) + (T - U)(X) = E_X(T) + (T - U)(X) \supseteq S^{-1}(0) + (T - U)(X) = X.$$

The subspaces of Definition 8 lead to certain special kinds of operators:

12. Definition. We shall call $T \in BL(X, X)$ triangular if the subspace $E_X(T)$ is dense:

$$\text{cl } E_X(T) = X \ . \tag{12.1}$$

Dually $T \in BL(X, X)$ is co-triangular if the subspace $F_X(T)$ is trivial:

$$F_X(T) = \{0\} \ . \tag{12.2}$$

The shifts of Example 7 are either triangular or co-triangular:

13. Example. On each of the spaces c_0 and ℓ_p $(1 \le p < \infty)$, the forward shift U is triangular, the backward shift V is co-triangular and the standard weight W is both triangular and co-triangular.

Proof. The hyperkernel of the backward shift is dense, since it includes all the "terminating" sequences:

$$V^{-\infty}(0) \supseteq c_{00} \ . \tag{13.1}$$

Thus

$$E(V) \supseteq V^{-\infty}(0) \text{ is dense} \tag{13.2}$$

and also

$$F(V) = \bigcap_{|\lambda|=1} (V - \lambda I)^{\infty}(X) \supseteq \sum_{|\lambda|<1} (V - \lambda I)^{-\infty}(0) \supseteq V^{-\infty}(0) \text{ is dense} \ . \tag{13.3}$$

Since $U - \lambda I$ is one-to-one for every $\lambda \in \mathbf{C}$ we have

$$E(U) = \{0\} \text{ is trivial} \tag{13.4}$$

and also

$$F(U) \subseteq U^{\infty}(X) = \{0\} \text{ is trivial} \ . \tag{13.5}$$

Finally we notice that the weight W commutes with the projection UV, and more generally

$$WU^nV^n = U^nV^nW \ (n \in \mathbf{N}) \ ; \tag{13.6}$$

also for each $n \in \mathbf{N}$

$$\begin{aligned} (\frac{1}{n}I - W)^{-1}(0) &= U^{n-1}(I - UV)V^{n-1}(X), \\ (\frac{1}{n}I - W)(X) &= (U^{n-1}(I - UV)V^{n-1})^{-1}(0) \ , \end{aligned} \tag{13.7}$$

so that $E(W)$ is dense and $F(W)$ is trivial. □

Triangularity and Fredholmness co-operate to generate finite ascent or descent:

14. Theorem. *If $T \in BL(X,X)$, then*

$$T \text{ upper semi-Fredholm and co-triangular} \implies T \text{ of finite ascent} \tag{14.1}$$

and

$$T \text{ lower semi-Fredholm and triangular} \implies T \text{ of finite descent}\ . \tag{14.2}$$

Proof. If $T \in BL(X,X)$ is upper semi-Fredholm, then the finite ascent condition can be written in the form

$$F_X(T) \cap T^{-1}(0) = \{0\}\ . \tag{14.3}$$

Indeed since ([8], Theorem 6.12.2) each power T^m is also upper semi-Fredholm, then $T^{-m}(0)$ is finite dimensional for each $m \in \mathbf{N}$ and $T^m(X)$ is closed; thus if for each $m \in \mathbf{N}$ we have $T^m(X) \cap T^{-1}(0) \neq \{0\}$, then there is (x_m) in X for which

$$\|T^m(x_m)\| = 1 \text{ and } T^{m+1}x_m = 0\ .$$

By local compactness there is a subsequence

$$(y_m) = T^{\phi(m)}(x_{\phi(m)}) \to y_\infty \in T^\infty(X)\ ,$$

using the closedness of all the ranges, so that $\|y_\infty\| = 1$ and $y_\infty \in F_X(T) \cap T^{-1}(0)$. This proves (14.1); towards (14.2) we claim that for subspaces $Y, Z \subseteq X$

$$Y \text{ closed of finite codimension and } Z \text{ dense} \implies Y + Z = X\ , \tag{14.4}$$

because if $\dim(X/Y) = n$ find successively $e_1, e_2, \ldots, e_n$ with

$$e_{j+1} \in Z \setminus (Y + \mathbf{C}e_1 + \mathbf{C}e_2 + \ldots + \mathbf{C}e_j).$$

Applying this with $Y = T(X)$ and $Z = E_X(T)$ gives (14.2). □

It is clear that in (14.1) we can replace the "co-triangular" condition (12.2) by the weaker condition (14.3); dually in (14.2) we can replace the triangular condition (12.1) by the weaker condition

$$\text{cl } E_X(T) + T(X) = X\ . \tag{14.5}$$

We cannot however remove the semi-Fredholm conditions: for example [12] if $T = U \otimes V$ is the tensor product of the shifts, then (14.3) holds but we never get $T^k(X) \cap T^{-1}(0) = \{0\}$. For operators which are both upper semi-Fredholm and of finite ascent, or lower semi-Fredholm of finite descent ("semi Browder" in the sense of [8], Definition 7.9.1) the conditions of Theorem 10 can be replaced by simple commutivity ([8], Theorem 7.9.2).

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