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THE EXCHANGE PROPERTY FOR PURELY INFINITE SIMPLE RINGS

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Abstract. It is proven that every purely infinite simple ring is an exchange ring. This result is applied to determine those Leavitt algebras that are exchange rings.

Introduction

Purely infinite simple C^*-algebras, introduced by J. Cuntz in 1981 [11], have played a central role in the development of the theory of C^*-algebras in the last two decades. One of the most important advances in the program of classifying separable C^*-algebras through K-theory, proposed by Elliott in the early seventies, was obtained in this context by Kirchberg [14] and Phillips [17], who showed that nuclear separable unital purely infinite simple C^*-algebras are classified by K-theoretic invariants. See [19] for a survey on the classification of nuclear C^*-algebras.

Recall that a C^*-algebra A has *real rank zero* if every selfadjoint element in A is a norm-limit of a sequence of selfadjoint invertible elements, see [8]. A fundamental result is that every purely infinite simple C^*-algebra has real rank zero. This was shown by Zhang in [21] and also by Brown and Pedersen in [8].

A suitable notion of purely infinite simple ring was introduced by Goodearl, Pardo and the author in [4]. A simple unital ring R is *purely infinite* in case it is not a division ring and for each nonzero element x in R, there are elements z, t in R such that $1 = zxt$. (The rings satisfying the latter property have been termed 1-simple rings by P. M. Cohn [10].) Since the C^*-algebras of real rank zero are exactly the ones that are exchange rings [2, Theorem 7.2], the fundamental result established by Zhang and Brown and Pedersen is equivalent to the statement that every purely infinite simple C^*-algebra is an exchange ring. (The definition of exchange ring is given at the beginning of Section 1.) This motivated the following question, posed in [4, 1.8(a)]: *Is every purely infinite simple ring an exchange ring?* In this paper, we answer this question in the affirmative. In fact, the proof we

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have found is quite simple, thus providing an elegant algebraic generalization of the fundamental C^*-algebra result mentioned above. The existing proofs of the fact that every purely infinite simple C^*-algebra has real rank zero are of a topological nature, and cannot be adapted to cover the result presented here.

We will also show a corresponding result for non-unital rings. The exchange property for a purely infinite simple ideal of a unital ring was a technical hypothesis that needed to be added to get some of the results on QB-rings obtained in [6]; see for example [6, Theorem 1.12, Proposition 3.2]. It follows from Corollary 1.2 that this hypothesis is automatically satisfied. The reader is referred to [5] for background material on QB-rings.

We will use our result to determine which Leavitt algebras are exchange rings. These algebras were defined and studied by Leavitt in [15] and [16], and their ring-theoretical properties have been investigated by several authors, among them Cohn, Skornyakov and Bergman, see e.g. [9, **2.11**] and [7].

1. Purely infinite rings

Following Warfield [20], we say that a unital ring R is an *exchange ring* in case the regular left R-module ${}_RR$ satisfies the (finite) exchange property, that is, for every R-module A and any decompositions

$$A = M' \oplus N = \bigoplus_{i=1}^{n} A_i$$

with $M' \cong {}_RR$, there exist submodules $A_i' \subseteq A_i$ such that

$$A = M' \oplus \big(\bigoplus_{i=1}^{n} A_i' \big).$$

By a result obtained independently by Goodearl [13] and Nicholson [18], R is an exchange ring if and only if for every element a in R there is an idempotent e in R such that $e \in Ra$ and $1 - e \in R(1-a)$.

If R is a ring and $e, f \in R$ are idempotents, we say that e and f are (Murray–von Neumann) *equivalent* (denoted $e \sim f$) provided that there exist elements $x \in eRf$ and $y \in fRe$ such that $xy = e$ and $yx = f$. This is equivalent to demanding that $eR \cong fR$ as right R-modules.

An idempotent e in a ring R is *infinite* if there exist orthogonal idempotents $f, g \in R$ such that $e = f + g$ while $e \sim f$ and $g \neq 0$. A simple unital ring R is said to be *purely infinite* in case every nonzero left ideal contains an infinite idempotent. By [4, Theorem 1.6], a simple ring R is purely infinite if and only if R is not a division ring and for every nonzero element $a \in R$ there are $x, y \in R$ such that $xay = 1$. See [4, Section 1] for the elementary properties and basic examples of purely infinite simple rings.

Theorem 1.1. *Every purely infinite simple unital ring is an exchange ring.*

Proof. Let R be a purely infinite simple ring. Let a be an element in R and set $b = 1-a$. By [18, Proposition 1.1], it is enough to check that there is an idempotent e such that $e \in Ra$ and $R = Re + Rb$. If $a = a^2$ there is nothing to prove. So we can assume that $Ra \cap Rb = R(ab) \neq 0$. By definition of purely infinite simple,

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there is a nonzero idempotent g such that $g \in Ra \cap Rb$. By [4, Theorem 1.6], there are $x, y \in R$ such that $xgy = 1$. Now consider the element

$$e = g + gya(1 - g).$$

Clearly, e is an idempotent, and since $g \in Ra$ we have $e \in Ra$. In order to prove that $R = Re + Rb$, it is enough to show that $a \in Re + Rb$. From the identity

$$xe = xg + (xgy)a(1 - g) = xg + a - ag$$

we get $a = xe + (a - x)g \in Re + Rg \subseteq Re + Rb$. Therefore $R = Re + Rb$, as desired. This completes the proof that R is an exchange ring. □

We now extend the above result to non-unital rings. The definition of purely infinite simple ring is the same as above: a simple ring R is *purely infinite* in case every nonzero left ideal contains an infinite idempotent. If x is a nonzero element in a purely infinite simple ring and $e = (ey)x$ is an infinite idempotent in the left ideal generated by x, then $f = x(ey)$ is an infinite idempotent in the right ideal generated by x, showing that the notion is left-right symmetric.

The concept of exchange ring was generalized to the non-unital case in [1]. Namely, a ring R is an *exchange ring* when for every element a in R there are an idempotent e in R and elements r, s in R such that $e = ra = a + s - sa$. Note that this reduces to the Goodearl–Nicholson characterization of exchange rings in case R is a unital ring.

We can now extend Theorem 1.1 to the non-unital case.

Corollary 1.2. *Let R be a purely infinite simple non-unital ring. Then R is an exchange ring.*

Proof. Take a nonzero idempotent p in R, and consider the corner ring $S = pRp$. It is easy to check that S is a purely infinite simple unital ring. It follows from Theorem 1.1 that S is an exchange ring.

To show that R is an exchange ring, we use the theory of Morita equivalence for rings without identity; see, for example, [12].

Let us denote by R^2 the set of all finite sums of products of two elements in R. Clearly R^2 is a two-sided ideal of R, and $R^2 \neq 0$ because R contains nonzero idempotents. Since R is a simple ring, we get $R = R^2$. Therefore R is an idempotent ring, and there is a Morita context (R, S, Rp, pR), showing that R and S are Morita equivalent. Since S is an exchange ring, it follows from [3, Theorem 2.3] that R is an exchange ring. □

2. An application

In this section, we apply the main result of this note to determine the Leavitt algebras that are exchange rings.

Leavitt algebras are universal examples of non-IBN algebras. For any field k, and for any two natural numbers m, n, we denote by $V_{m,n}$ the k-algebra with a universal isomorphism $i : nV_{m,n} \longrightarrow mV_{m,n}$, and we denote by $U_{m,n}$ the k-algebra with a universal pair of morphisms $i : nU_{m,n} \longrightarrow mU_{m,n}$ and $j : mU_{m,n} \longrightarrow nU_{m,n}$ such that $ji = 1_{nU_{m,n}}$. Leavitt, Cohn, Skornyakov and Bergman proved some fundamental properties of these algebras. In particular, the monoid of isomorphism classes of finitely generated projective modules of these examples was computed by Bergman in [7].

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Theorem 2.1. *Let k be a field, let m,n be positive integers with $m \leq n$, and let $U_{m,n}$ and $V_{m,n}$ be the Leavitt k-algebras defined above. Then the following conditions are equivalent:*

(a) *$U_{m,n}$ is an exchange ring;*
(b) *$V_{m,n}$ is an exchange ring;*
(c) *$m = 1$ and $n > 1$.*

Proof. The k-algebra $U_{m,n}$ has a system of $2nm$ generators x_{ij}, y_{ji}, $i = 1, \dots, n$, $j = 1, \dots, m$, with defining relations which in matrix form can be written

$$XY = I_n, \qquad X = (x_{ij}), \qquad Y = (y_{ji}).$$

The algebra $V_{m,n}$ is the factor ring of $U_{m,n}$ modulo the ideal generated by the relations

$$YX = I_m.$$

(a) $\Longrightarrow$ (b). This follows from the fact that a factor ring of an exchange ring is also an exchange ring.

(b) $\Longrightarrow$ (c). It is well known (and easy to prove) that an integral domain R is an exchange ring if and only if it is a local ring. Here we are using the standard definition for noncommutative local rings as the rings for which the non-units form a proper two-sided ideal.

If $m = n = 1$, then $V_{1,1} = k[t, t^{-1}]$, a Laurent polynomial ring. So $V_{1,1}$ is an integral domain, but it is not a local ring. It follows that $V_{1,1}$ is not an exchange ring.

If $m > 1$, then $V_{m,n}$ is an $(m-1)$-fir by [7, Theorem 6.1]. In particular $V_{m,n}$ is an integral domain. It is easy to see that the only invertible elements in $V_{m,n}$ are the nonzero elements of k; cf. [16, proof of Theorem 9]. Therefore, $V_{m,n}$ is an integral domain that is not a local ring, and so it is not an exchange ring.

(c) $\Longrightarrow$ (a). Assume $n > 1$. By [4, Theorem 4.2], the algebra $V_{1,n}$ is a purely infinite simple ring. It follows from Theorem 1.1 that $V_{1,n}$ is an exchange ring. Now $U_{1,n}$ has a unique maximal ideal M, which is the ideal generated by the idempotent $e := 1 - \sum_{i=1}^{n} y_i x_i$. The ideal M coincides with the socle of $U_{1,n}$, and it is a von Neumann regular ideal. In fact, it is easy to check that M is linearly spanned by an infinite family of matrix units; cf. [4, Section 4].

Thus we have proved that $U_{1,n}$ has a von Neumann regular ideal M such that the factor ring $U_{1,n}/M \cong V_{1,n}$ is an exchange ring. It follows from [1, Corollary 2.4] that $U_{1,n}$ is also an exchange ring. □

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