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ENTIRE INVARIANT SOLUTIONS TO MONGE-AMPÈRE EQUATIONS

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Abstract. We prove existence and regularity of entire solutions to Monge-Ampère equations invariant under an irreducible action of a compact Lie group.

We consider Monge-Ampère equations of the form

$$f(\nabla\phi)\det D_{ij}\phi = g(x), \tag{0.1}$$

where f and g are nonnegative measurable functions on $\mathbb{R}^n$. We recall first the concept of a weak solution of (0.1). Let ϕ be a convex function. Then $\nabla\phi$ is a well-defined multi-valued mapping: $(\nabla\phi)(x)$ is the set of slopes of all supporting hyperplanes to the graph of ϕ at $(x, \phi(x))$. If B is a subset of $\mathbb{R}^n$, let $\nabla\phi(B)$ be its image in the multi-valued sense. Then ϕ is a weak solution of (0.1) if

$$\int_B g(x)dx = \int_{\nabla\phi(B)} f(y)dy \tag{0.2}$$

for every Borel set B. Let us denote the right-hand side by $\omega(B, \phi, f)$. It can be shown that it is a Borel measure on $\mathbb{R}^n$. A basic result [1] is that if $u_k \to u$ compactly and $f_k \to f$ uniformly, then $\omega(\cdot, u_k, f_k)$ converges to $\omega(\cdot, u, f)$ weakly, i.e., as functionals on the space of compactly supported continuous functions.

Existence of weak solutions to (0.1) defined on all of $\mathbb{R}^n$ has been shown in [1] under the assumption that $g \in L^1(\mathbb{R}^n)$, and in [9] in the case when $f = 1$ and $g, 1/g$ are bounded. In this note we wish to give a simple proof of existence and regularity of entire solutions to (0.1) under a different type of assumption: f and g are invariant under an irreducible action of a compact (Lie) group. More precisely, we shall prove:

Theorem 0.1. *Let $K \subset O(n)$ be a compact subgroup acting irreducibly on $\mathbb{R}^n$, and let f, g be two nonnegative K-invariant measurable functions on $\mathbb{R}^n$. Furthermore, assume that f and g are locally bounded and that*

$$\int_{\mathbb{R}^n} f = +\infty.$$

Then there exists a (weak) convex K-invariant solution $\phi : \mathbb{R}^n \to \mathbb{R}$ of the Monge-Ampère equation (0.1).

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Theorem 0.2. *In the situation of Theorem 0.1 suppose, in addition, that f and g are strictly positive and of class $C^{p,\alpha}$, $p \geq 0$, $\alpha > 0$. Then any entire K-invariant convex weak solution ϕ of* (0.1) *is $C^{p+2,\alpha}$.*

Well-known examples show that this regularity result does not hold without the assumption of K-invariance, even when $f = g = 1$.

The above theorems have been proved before in [2] in the case when K is a reflection group. They can be applied to show existence of Kähler metrics with prescribed Ricci curvature on complexified symmetric spaces.

1. Proofs

We begin with a simple lemma:

Lemma 1.1. *Let $K \subset O(n)$ be a compact subgroup acting irreducibly on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, let C_x be the convex hull of the orbit $K \cdot x$. There exists an $\epsilon > 0$ such that the ball $B(0, \epsilon)$ is contained in C_x for any x with $|x| = 1$.*

Proof. Suppose that the conclusion does not hold. Then there exists a sequence of points x_k with $|x_k| = 1$ such that the boundary of C_{x_k} contains a point y_k with $y_k \to 0$. By choosing a convergent subsequence of x_k we find a point x_∞ on the unit sphere such that 0 lies on the boundary of C_{x_∞}. Since C_{x_∞} is convex, it lies to one side of any of its supporting hyperplanes at 0. Hence the orbit $K \cdot x_\infty$ lies to one side of a hyperplane H passing through 0. Therefore the center of mass of the orbit

$$m = \frac{1}{|K|} \int_K g \cdot x_\infty dg$$

cannot be the origin unless the whole orbit lies on H. On the other hand, m has to be the origin, since it is a K-invariant point and the representation is irreducible. It follows that the orbit $K \cdot x_\infty$ is contained in a hyperplane, and therefore the subspace generated linearly by $K \cdot x_\infty$ is a proper K-invariant subspace of $\mathbb{R}^n$, a contradiction. □

We now prove Theorem 0.1. Put $f_k = f + 1/k$ and $g_k = g + 1/k$, $k \in \mathbb{N}_+$. Let B_r denote the ball of radius r centered at the origin. Let R_k be a number defined by

$$\int_{B_{R_k}} f_k = \int_{B_k} g_k.$$

According to [3, 7] there exists a unique (up to a constant) strictly convex solution ϕ_k of

$$f_k(\nabla \phi_k) \det D_{ij} \phi_k = g_k(x) \tag{1.1}$$

which is of class $C^{1,\beta}$, for some $\beta > 0$, and such that $\nabla \phi_k$ maps B_k onto B_{R_k}. Moreover, since $K \subset O(n)$ and all the data are K-invariant, ϕ_k is K-invariant (this follows from uniqueness, since $\phi_k \circ g$, $g \in K$, is also a solution of (1.1)). To prove the existence of a weak solution to (0.1), defined on all of $\mathbb{R}^n$, it is enough to show that the functions ϕ_k are uniformly (in k) bounded on any ball B_R, $k \geq R$. Indeed, a bounded sequence of convex functions on a bounded open convex domain has a convergent subsequence. This follows from the elementary fact, which we will use repeatedly, that the slopes of supporting hyperplanes of a convex function, bounded by R on a domain G, are bounded by $2R/\delta$ on any subdomain G' such that $\operatorname{dist}(G', \partial G) \geq \delta$.

Let us show that the functions ϕ_k, $k \geq R$, $\phi_k(0) = 0$, are bounded on B_R uniformly in k. It is enough to show that $\nabla\phi_k$ are bounded uniformly on B_R. Suppose that there exists a sequence of points x_{k_j} in B_R such that $\left|\nabla\phi_{k_j}(x_{k_j})\right| \geq j$. Let $y_{k_j} = \nabla\phi_{k_j}(x_{k_j})$. Consider the unique solution ψ_k, $\psi_k(0) = 0$, to the equation

$$g_k(\nabla\psi_k)\det D_{ij}\psi_k = f_k(y)$$

mapping $B_{R_{k_j}}$ onto B_{k_j}. According to [3], $\nabla\psi_k$ is the inverse of $\nabla\phi_k$. Thus $\nabla\psi_{k_j}(y_{k_j}) = x_{k_j}$, and hence $\left|\nabla\psi_{k_j}(y_{k_j})\right| \leq R$. Observe that $\nabla\psi_k(0) = 0$, since $\nabla\psi_k$ is K-equivariant and the representation is irreducible. Let v_{k_j} be the unit vector in the direction y_{k_j}. Since ψ_k is convex, $\langle\nabla\psi_{k_j}(y), v_{k_j}\rangle \leq R$ for all $y \in [0, y_{k_j}]$, and therefore $\left|\psi_{k_j}(y)\right| \leq R|y|$ for such y. Now using Lemma 1.1, we conclude that for any $r > 0$, ψ_{k_j} is bounded by $\frac{R}{\epsilon}r$ on B_r for large j. It follows that there is a subsequence of ψ_{k_j} convergent to a convex solution $\psi : \mathbb{R}^n \to \mathbb{R}$ of

$$g(\nabla\psi)\det D_{ij}\psi = f.$$

The function ψ is bounded by Rr/ϵ on any B_r. Since ψ is convex, $\nabla\psi$ is bounded by $4R/\epsilon$ on $B_{r/2}$. Thus $U = \nabla\psi\big(\mathbb{R}^n\big)$ is contained in $B_{4R/\epsilon}$. However,

$$+\infty = \int_{\mathbb{R}^n} f = \int_U g,$$

which leads to a contradiction. Theorem 0.1 is proved.

Proposition 1.2. *Suppose, in addition, that* $\int_{\mathbb{R}^n} g > 0$. *Then any entire K-invariant convex solution ϕ of* (0.1) *is a proper function.*

Proof. Suppose that there exists an unbounded sequence of points x_k such that $\phi(x_k)$ is bounded. Using convexity of ϕ and Lemma 1.1, we conclude that ϕ is a bounded function. This, however, implies that ϕ, being convex and defined on all of $\mathbb{R}^n$, is constant, which contradicts the assumption on g. □

This proposition allows us to prove the regularity results of Theorem 0.2. The proof is essentially the same as the one given by Caffarelli and Viaclovsky [8]; we give it here for completeness.

Consider the convex (and bounded by the last proposition) sets $\Omega_c = \{x; \phi(x) \leq c\}$, $c \in \mathbb{R}$. For every c, we can find $R(c) > c$ such that $\operatorname{dist}\big(\Omega_c, \partial\Omega_{R(c)}\big) \geq 1$. By the argument used before, the slopes $\nabla\phi$ of ϕ are bounded by $2R(c)$ on Ω_c. If f and g are strictly positive and continuous, then equation (0.2) implies that there are constants λ_1, λ_2, depending on c, such that

$$\lambda_1|B| \leq |\nabla\phi(B)| \leq \lambda_2|B| \tag{1.2}$$

for any Borel subset $B \subset \Omega_c$. We can apply Corollary 2 in [4] to Ω_{c+1} and conclude that ϕ is strictly convex at any point of Ω_c. Since c was arbitrary, ϕ is strictly convex everywhere. Now the main result of [6] implies that ϕ is locally of class $C^{1,\beta}$, where "locally" means that β can be chosen uniformly only on bounded subsets of $\mathbb{R}^n$. We observe that the only additional properties of f and g we have used so far are the boundedness of $1/f$ and $1/g$. Therefore we have proved:

Proposition 1.3. *In the situation of Theorem 0.1 suppose, in addition, that $1/f$ and $1/g$ are locally bounded functions, and let ϕ be an entire K-invariant convex* *weak solution of* (0.1). *Then* ϕ *is of class* $C^{1,\beta}$ *on any compact subset* P, *where* β *may depend on* P. □

To finish the proof of Theorem 0.2, assume that f and g are of class $C^{0,\alpha}$. Now, ϕ is a solution of $\det D_{ij}\phi = \tilde{g}$, where $\tilde{g} = g/f(\nabla\phi)$. Since, by the last proposition, $\nabla\phi$ is Hölder continuous, $\tilde{g}$ is (locally) of class $C^{0,\gamma}$. Using (1.2), we can apply Theorem 2 in [5] to $u(x) = \phi(x) - c$, $c \in \mathbb{R}$, and the convex and bounded set $\Omega_c = \{x; u(x) \le 0\}$ to conclude that ϕ is locally of class $C^{2,\gamma}$ and hence globally C^2. Therefore $\tilde{g} = g/f(\nabla\phi)$ is of class $C^{0,\alpha}$ everywhere and, repeating the argument, we conclude that ϕ is $C^{2,\alpha}$. Higher regularity is standard.

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