PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 9, Pages 2671–2678
S 0002-9939(04)07375-7
Article electronically published on March 25, 2004

ASYMPTOTICALLY SYMMETRIC EMBEDDINGS AND SYMMETRIC QUASICIRCLES

ABDELKRIM BRANIA AND SHANSHUANG YANG

(Communicated by Juha M. Heinonen)

Abstract. A well-known characterization of quasicircles is the following: A Jordan curve J in the complex plane is a quasicircle if and only if it is the image of the unit circle under a quasisymmetric embedding. In this paper we try to characterize a subclass of quasicircles, namely, symmetric quasicircles, by introducing the concept of asymptotically symmetric embeddings. We show that a Jordan curve J in the complex plane is a symmetric quasicircle if and only if it is the image of the unit circle under an asymptotically symmetric embedding.

1. Introduction

This paper is largely motivated by the following well-known characterization of quasicircles, which is a combination of several important results in the theory of quasiconformal mappings in the plane (see, for example, [Po, Proposition 5.10 and Theorem 5.11]).

Theorem A. *A Jordan curve J in the complex plane is a quasicircle if and only if it is the image of the unit circle under a quasisymmetric embedding.*

A Jordan curve J in the complex plane $\mathbb{C}$ is a *quasicircle* if it is the image of the unit circle S^1 under a quasiconformal map of the plane. This has been a much studied subject in analysis and geometry since Ahlfors introduced this concept in the 1960s (see the survey article [Ge]). The idea of quasisymmetric maps was introduced by Ahlfors and Beurling in their study of boundary correspondence of quasiconformal maps of the upper half plane [Ah], [BA]. This concept was later promoted by Tukia and Väisälä, who introduced and studied quasisymmetric maps between arbitrary metric spaces [TV]. Following their definition, an embedding $f : X \to Y$ (in metric spaces) is called *quasisymmetric*, abbreviated QS, if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\text{(QS)} \qquad \frac{|a-x|}{|b-x|} \le t \Longrightarrow \frac{|f(a)-f(x)|}{|f(b)-f(x)|} \le \eta(t)$$

for all $a, b, x \in X$ with $b \neq x$ and for all $t \ge 0$. In this case we also say that f is η-QS. Here and in what follows we use the notation $|a-b|$ for the distance

Received by the editors September 18, 2002 and, in revised form, June 12, 2003.
2000 *Mathematics Subject Classification.* Primary 30C62.
Key words and phrases. Quasisymmetric maps, quasicircles, asymptotically symmetric maps, symmetric quasicircles.

between a and b in any metric space. Quasisymmetry turns out to be a very useful tool in the study of analysis and geometry in general metric spaces with no smooth structure (see [HK], [He]).

Our goal here is to obtain a characterization parallel to Theorem A for a subclass of quasicircles, namely, symmetric quasicircles. A Jordan curve J in the plane $\mathbb{C}$ is said to be a *symmetric quasicircle* if

$$\max_{w\in J(a,b)} \frac{|a-w|+|w-b|}{|a-b|} \to 1 \tag{1.1}$$

as $a, b \in J$ and $|a-b| \to 0$, where $J(a,b)$ denotes the subarc of J between a and b of smaller diameter. This class of quasicircles was introduced by Becker and Pommerenke [BP] as *asymptotically conformal curves* in connection with the study of boundary behavior of conformal mappings. They gave a number of analytical characterizations for symmetric quasicircles in terms of conformal mappings (see [Po, Theorem 11.1]). The term symmetric quasicircle is due to Gardiner and Sullivan, who further studied this class of quasicircles from a topological point of view [GS]. Several other characterizations are given in [GS], [WY], [Ka]. It follows from (1.1) and Ahlfors' three-point condition that symmetric quasicircles are quasicircles. It is also easy to see that smooth Jordan curves are symmetric quasicircles and symmetric quasicircles do not allow corners. On the other hand, a symmetric quasicircle can also be very wild in the sense that it may possess a tangent only on a set of zero linear measure (see [Po, p. 249] for such an example).

We achieve our goal by introducing a natural subclass of QS embeddings, called asymptotically symmetric embeddings. The main results of this paper are Theorems 3.1 and 3.2, which can be summarized as saying that a Jordan curve $J \subset \mathbb{C}$ is a symmetric quasicircle if and only if it is the image of the unit circle under an asymptotically symmetric embedding.

This paper is organized as follows. In Section 2 we give the definition and some elementary properties of asymptotically symmetric embeddings. The main results, a characterization of symmetric quasicircles using asymptotically symmetric maps, are given in Section 3. In Section 4 we investigate the relation between the asymptotically symmetric maps introduced here and the symmetric homeomorphisms of the unit circle introduced by Gardiner and Sullivan [GS].

The authors would like to thank the referee for valuable suggestions and comments.

2. Asymptotically symmetric maps

In this section we introduce the concept of asymptotically symmetric embeddings in general metric spaces and establish some elementary properties which are of independent interest and will be needed later.

2.1. Definition. An embedding $f : X \to Y$ (in metric spaces) is called *asymptotically symmetric*, abbreviated AS, if for any $\epsilon > 0$ and $t > 0$ there is a $\delta > 0$ such that

$$\frac{|a-x|}{|b-x|} \le t \Longrightarrow \frac{|f(a)-f(x)|}{|f(b)-f(x)|} \le (1+\epsilon)t \tag{AS}$$

for all $a, b, x \in X$ (with $x \neq b$) contained in a ball of radius δ.

One could say that condition (AS) is a local, but strengthened, version of condition (QS) with a general homeomorphism $\eta(t)$ being replaced by $(1+\epsilon)t$ in infinitesimal scales. The following result shows that the class of AS maps is truly a subclass of QS maps in compact metric spaces.

2.2. Proposition. *Let X be a compact metric space. If $f : X \to Y$ is AS, then it is QS.*

Proof. For $t > 0$, let

$$\eta_1(t) = \sup\left\{\frac{|f(a)-f(x)|}{|f(b)-f(x)|} : \frac{|a-x|}{|b-x|} \le t; a,b,x \in X\right\}.$$

According to [TV, 2.1], we only need to show that $\eta_1(t) < \infty$ for $t > 0$ and that $\eta_1(t) \to 0$ as $t \to 0$. This follows routinely from condition (AS) and the compactness of X by contrapositive arguments. □

It should be pointed out that the conclusion in the above proposition is no longer true if one does not assume the compactness of X. Consider the map $f : \mathbb{R} \to \mathbb{R}$ of the real line determined by

$$f(x) = \begin{cases} x, & x \le 1, \\ x^2, & x \ge 2, \\ 4 - 8x + 6x^2 - x^3, & 1 \le x \le 2. \end{cases}$$

One can verify that f is AS, but not QS. The next result shows that, like QS maps, the class of AS maps has a group structure in the compact case.

2.3. Proposition. *Let X, Y, Z be compact metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are AS, so are the inverse f^{-1} and the composition $g \circ f$.*

Proof. Let $f : X \to Y$ be AS with X and $Y = f(X)$ being compact. For simplicity of notation, we let x' denote $f(x)$ for any point $x \in X$. To show that the inverse f^{-1} is also AS, for any $\epsilon' > 0$ and $t' > 0$ we need to find $\delta' > 0$ such that

$$\frac{|a'-x'|}{|b'-x'|} \le t' \Longrightarrow \frac{|a-x|}{|b-x|} \le (1+\epsilon')t' \tag{2.1}$$

for all $a', b', x' \in Y$ contained in a ball of radius δ'. Since f is AS, for $\epsilon = \epsilon'/2$ and $t = 1/[(1+\epsilon')t']$ there exists $\delta > 0$ such that

$$\frac{|b-x|}{|a-x|} \le t \Longrightarrow \frac{|b'-x'|}{|a'-x'|} \le (1+\epsilon)t < \frac{1}{t'} \tag{2.2}$$

for all $a, b, x \in X$ contained in a ball of radius δ. Next, since Y is compact, it follows that f^{-1} is uniformly continuous. Choose $\delta' > 0$ such that balls of radius δ' in Y are mapped into balls of radius δ in X by f^{-1}. With such a choice of δ', (2.2) yields (2.1), as desired.

Finally, it is a routine exercise to verify that the composition $g \circ f$ is also AS, and we leave the details to the reader. □

3. A characterization of symmetric quasicircles

In this section we characterize symmetric quasicircles using the asymptotically symmetric maps introduced above and obtain a result corresponding to Theorem A for quasicircles.

3.1. Theorem. *Let J be a Jordan curve in the plane and f a conformal map of the unit disk $\mathbb{D}$ onto the interior domain Ω. If J is a symmetric quasicircle, then the boundary extension $f : S^1 \to J$ is AS.*

Before proving Theorem 3.1, we recall the following facts about the Teichmüller ring domain and the Teichmüller function, which will be needed later. For a curve family Γ in $\mathbb{C}$, let $M(\Gamma)$ denote the modulus of Γ. The Teichmüller ring domain is the domain whose complement consists of the intervals $E = [-1, 0]$ and $F = [t, \infty]$, $t > 0$. It is well known that the modulus of the curve family joining E and F can be expressed as

$$M(\Gamma) = \lambda(t),$$

where $\lambda(t)$ is the Teichmüller function, which is continuous and strictly decreasing with

$$\lambda(0) = \lim_{t\to 0} \lambda(t) = \infty \quad \text{and} \quad \lambda(\infty) = \lim_{t\to\infty} \lambda(t) = 0.$$

Another fact we need is the following comparison principle. Let E and F be disjoint continua with $a, b \in E$ and $c, d \in F$. Then the modulus of the curve family $\Delta(E, F; \mathbb{C})$ joining E and F in $\mathbb{C}$ has the following lower bound:

$$M(\Delta(E, F; \mathbb{C})) \geq \lambda(t),$$

where t is the cross-ratio of a, b, c, d:

$$t = [a, b, c, d] = \frac{|b - c||a - d|}{|a - b||c - d|}.$$

For more details about the modulus and the Teichmüller ring domain, we refer the reader to [Ah] and [AVV].

Proof of Theorem 3.1. Suppose $f : S^1 \to J$ is not AS. Then there exist $\epsilon > 0$ and $t > 0$ such that for each $\delta_n = \frac{1}{n}$ $(n = 1, 2, 3, ...)$ there exist points $a_n, b_n, x_n \in S^1$ contained in a ball of radius δ_n with the property that

$$|a_n - x_n| \leq t|b_n - x_n| \quad \text{and} \quad |f(a_n) - f(x_n)| > (1 + \epsilon)t|f(b_n) - f(x_n)|.$$

By passing to subsequences, we may further assume that, as $n \to \infty$, $a_n, b_n, x_n \to x \in S^1$ for some x,

$$\frac{|a_n - x_n|}{|b_n - x_n|} \to t_0 \leq t \tag{3.1}$$

for some $t_0 \geq 0$ and

$$\frac{|f(a_n) - f(x_n)|}{|f(b_n) - f(x_n)|} \to t_0' \geq (1 + \epsilon)t \tag{3.2}$$

for some $t_0' \leq \infty$. Note that $t_0 < (1 + \epsilon)t \leq t_0'$.

We are going to derive a contradiction from (3.1) and (3.2) by means of modulus estimates. For this we consider three possible configurations, by passing to subsequences if necessary: Configuration AXB: x_n lies on the smaller component of $S^1 \backslash \{a_n, b_n\}$ for all $n \geq 1$ (namely, x_n is "between" a_n and b_n); Configuration XAB: a_n lies on the smaller component of $S^1 \backslash \{x_n, b_n\}$ for all $n \geq 1$; Configuration ABX: b_n lies on the smaller component of $S^1 \backslash \{a_n, x_n\}$ for all $n \geq 1$.

We deal with Configuration AXB first and reduce the other two cases to this case. For Configuration AXB, fix $d \in S^1$ with $d \neq x$ and let $d' = f(d)$. In what follows, for any point p in the closed unit disk we let p' denote the image $f(p)$. Next, for each n let E_n and F_n denote the disjoint circular arcs on S^1 from a_n to x_n and from

b_n to d, respectively, and let $\Gamma_n = \Delta(E_n, F_n; \mathbb{C})$ and $\Gamma'_n = \Delta(f(E_n), f(F_n); \mathbb{C})$. It follows from the conformal invariance of the modulus and the comparison principle that

$$M(\Gamma_n) = \lambda(\tau_n), \quad M(\Gamma'_n) \geq \lambda(\tau'_n), \tag{3.3}$$

where

$$\tau_n = \frac{|x_n - b_n||a_n - d|}{|a_n - x_n||b_n - d|} \to \frac{1}{t_0}, \quad \tau'_n = \frac{|x'_n - b'_n||a'_n - d'|}{|a'_n - x'_n||b'_n - d'|} \to \frac{1}{t'_0}$$

by (3.1) and (3.2). Let

$$\Lambda = \lim_{n\to\infty} \lambda(\tau_n) = \lambda(1/t_0) \ \text{ and } \ \Lambda' = \lim_{n\to\infty} \lambda(\tau'_n) = \lambda(1/t'_0).$$

Then it follows that $\Lambda < \Lambda'$, since the Teichmüller function $\lambda(\tau)$ is strictly decreasing.

We shall derive a contradiction by estimating the upper bound of $M(\Gamma'_n)$. Fix any $\epsilon' > 0$. Since J is a symmetric quasicircle, by [Po, Theorem 11.1], f has a $(1+\epsilon')$−QC extension to a disk $B(0, 1+\delta) = \{z : |z| < 1+\delta\}$ for some $\delta > 0$. Fix $R > 0$ such that $B(x', R) \subset f(B(0, 1+\delta))$ and fix $r > 0$ such that $\log(R/r) = 2\pi/\epsilon'$. For large n we split Γ'_n into two subfamilies: $\Gamma'_{n,1} = \{\gamma \in \Gamma'_n : \gamma \subset B(x', R)\}$ and $\Gamma'_{n,2} = \Gamma'_n \backslash \Gamma'_{n,1}$. Then, by monotonicity and quasi-invariance of the modulus, we obtain that

$$M(\Gamma'_{n,1}) \leq (1+\epsilon')M(f^{-1}(\Gamma'_{n,1})) \leq (1+\epsilon')M(\Gamma_n). \tag{3.4}$$

Furthermore, we note that for large n the family $\Gamma'_{n,2}$ is minorized by the family of curves joining the circles $|z - x'| = r$ and $|z - x'| = R$. Thus

$$M(\Gamma'_{n,2}) \leq \frac{2\pi}{\log(R/r)} = \epsilon'. \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), we obtain that

$$\lambda(\tau'_n) \leq M(\Gamma'_n) \leq M(\Gamma'_{n,1}) + M(\Gamma'_{n,2}) \leq (1+\epsilon')\lambda(\tau_n) + \epsilon'.$$

Thus, letting $n \to \infty$ yields

$$\Lambda' \leq (1+\epsilon')\Lambda + \epsilon' < \Lambda' \ \text{ when } \ \epsilon' < \frac{\Lambda' - \Lambda}{1+\Lambda};$$

a contradiction. This completes the proof for Configuration AXB.

Next we consider Configuration XAB. Since J is a symmetric quasicircle, by (3.2) and (1.1) we obtain that

$$\begin{aligned}\frac{1}{t'_0} &= \lim_{n\to\infty} \frac{|b'_n - x'_n|}{|a'_n - x'_n|} \\ &= \lim_{n\to\infty} \frac{|b'_n - a'_n| + |a'_n - x'_n|}{|a'_n - x'_n|} = 1 + \lim_{n\to\infty} \frac{|b'_n - a'_n|}{|a'_n - x'_n|}.\end{aligned}$$

Similarly, by (3.1) we have

$$\frac{1}{t_0} = \lim_{n\to\infty} \frac{|b_n - x_n|}{|a_n - x_n|} = 1 + \lim_{n\to\infty} \frac{|b_n - a_n|}{|a_n - x_n|}.$$

Furthermore, we observe in this case that $t'_0 \leq 1$. Thus if we interchange x_n with a_n, Configuration XAB reduces to Configuration AXB and the corresponding limits

(3.1) and (3.2) become

$$\lim_{n\to\infty} \frac{|a_n - x_n|}{|b_n - x_n|} = \frac{t_0}{1-t_0} \tag{3.1a}$$

and

$$\lim_{n\to\infty} \frac{|a'_n - x'_n|}{|b'_n - x'_n|} = \frac{t'_0}{1-t'_0}, \tag{3.2a}$$

respectively. Since the function $x/(1-x)$ is strictly increasing on $[0,1]$, we note that

$$0 \le \frac{t_0}{1-t_0} < \frac{t'_0}{1-t'_0} \le \infty.$$

Therefore, as shown in Configuration AXB, (3.1a) and (3.2a) lead to a contradiction.

Finally we consider Configuration ABX. As in Configuration XAB, it follows from (3.1), (3.2) and (1.1) that

$$t_0 = \lim_{n\to\infty} \frac{|a_n - x_n|}{|b_n - x_n|} = 1 + \lim_{n\to\infty} \frac{|a_n - b_n|}{|b_n - x_n|}$$

and

$$t'_0 = \lim_{n\to\infty} \frac{|a'_n - x'_n|}{|b'_n - x'_n|} = 1 + \lim_{n\to\infty} \frac{|a'_n - b'_n|}{|b'_n - x'_n|}.$$

Thus if we interchange x_n with b_n, Configuration ABX reduces to Configuration AXB and the corresponding limits (3.1) and (3.2) become

$$\lim_{n\to\infty} \frac{|a_n - x_n|}{|b_n - x_n|} = t_0 - 1 \tag{3.1b}$$

and

$$\lim_{n\to\infty} \frac{|a'_n - x'_n|}{|b'_n - x'_n|} = t'_0 - 1, \tag{3.2b}$$

respectively. Here we have $0 \le t_0 - 1 < t'_0 - 1 \le \infty$. Therefore, as shown in Configuration AXB, (3.1b) and (3.2b) lead to a contradiction. This completes the proof of Theorem 3.1. □

3.2. Theorem. *If $f : S^1 \to \mathbb{C}$ is AS, then $J = f(S^1)$ is a symmetric quasicircle.*

Proof. First we observe that f is quasisymmetric by Proposition 2.2. Thus it follows from Theorem A that J is a quasicircle. Next, assume J is not a symmetric quasicircle. By definition (1.1), there are sequences $a_n, b_n, c_n \in S^1$, with $b_n \in S^1(a_n, c_n)$, all converging to the same point a, such that

$$\lim_{n\to\infty} \frac{|a'_n - b'_n| + |b'_n - c'_n|}{|a'_n - c'_n|} = 1 + \epsilon_0 > 1,$$

where p' denotes the image $f(p)$ for any point $p \in S^1$. By passing to subsequences, we may further assume that the following limits exist:

$$\lim_{n\to\infty} \frac{|a'_n - b'_n|}{|a'_n - c'_n|} = \lambda_1, \quad \lim_{n\to\infty} \frac{|b'_n - c'_n|}{|a'_n - c'_n|} = \lambda_2.$$

Then $\lambda_1 + \lambda_2 = 1 + \epsilon_0 > 1$. On the other hand, since $f^{-1} : J \to S^1$ is also AS by Proposition 2.3, one can deduce that the same limits hold for the corresponding points on S^1:

$$\lim_{n\to\infty} \frac{|a_n - b_n|}{|a_n - c_n|} = \lambda_1, \quad \lim_{n\to\infty} \frac{|b_n - c_n|}{|a_n - c_n|} = \lambda_2.$$

Thus we have

$$\lim_{n\to\infty} \frac{|a_n - b_n| + |b_n - c_n|}{|a_n - c_n|} = \lambda_1 + \lambda_2 > 1,$$

which contradicts the fact that S^1 is a circle, and hence proves that J must be a symmetric quasicircle. □

One can easily see that if S^1 is replaced by a symmetric quasicircle, the proof of Theorem 3.2 is still valid. We record this fact as follows.

3.3. Corollary. *If $J \subset \mathbb{C}$ is a symmetric quasicircle and $f : J \to \mathbb{C}$ an AS embedding, then $f(J)$ is also a symmetric quasicircle.*

Note that Corollary 3.3 also follows from Theorem 3.1, Proposition 2.3, and Theorem 3.2 by precomposing f with the boundary map of a conformal map of the unit disk onto the interior domain of J. Combining Theorems 3.1 and 3.2, we obtain the following characterization for symmetric quasicircles.

3.4. Corollary. *A Jordan curve J in $\mathbb{C}$ is a symmetric quasicircle if and only if it is the image of the unit circle under an asymptotically symmetric embedding $f : S^1 \to \mathbb{C}$.*

4. Relation between AS maps and Gardiner-Sullivan symmetric maps

In the study of topological structures of the quasisymmetric group QS of all quasisymmetric homeomorphisms $h : S^1 \to S^1$, Gardiner and Sullivan introduced the concept of symmetric homeomorphisms of the unit circle [GS]. According to [GS, Definition 2.1 and Theorem 2.1], a homeomorphism $f : S^1 \to S^1$ is *symmetric* if and only if it satisfies condition (AS) for $t = 1$. For this reason, we call such a homeomorphism (or embedding) *weakly asymptotically symmetric*, or WAS for short. This is similar to the term weakly quasisymmetric maps, or WQS, coined by Tukia and Väisälä [TV], for maps satisfying condition (QS) for $t = 1$. This weaker version of the QS condition was first introduced by Beurling and Ahlfors, and called the M-condition with $M = \eta(1)$, for the boundary extensions of quasiconformal maps of the upper half-plane onto itself. Tukia and Väisälä showed that for a large class of spaces, conditions (QS) and (WQS) are equivalent [TV, 2.15, 2.16] (see also [He, 10.19]). In particular, for homeomorphisms of the unit circle they are equivalent. For AS maps we establish the following result.

4.1. Theorem. *A homeomorphism $h : S^1 \to S^1$ is AS if and only if it is WAS (or symmetric in the sense of Gardiner and Sullivan).*

Proof. Obviously, if $h : S^1 \to S^1$ is AS, then it is WAS. Now assume h is WAS. Then it is quasisymmetric. By the sewing property of quasisymmetric maps (see [Le, Lemma III.1.1]), h is induced by a Jordan curve J; that is, there are conformal maps f and g from the unit disk $\mathbb{D}$ onto the interior domain Ω of J and from $\mathbb{D}^*$ onto the exterior domain Ω^* of J, respectively, such that $h = f^{-1} \circ g|_{S^1}$. Here we use f and g to denote their boundary extensions as well. Since h is symmetric in the

sense of Gardiner and Sullivan, by [GS, Theorem 6.3] J is a symmetric quasicircle. Therefore, it follows from Theorem 3.1 and Proposition 2.3 that h is AS. □

As one can see, the above proof is very indirect and non-elementary. It would be interesting to obtain a more direct proof using the metric condition and relying less heavily on the conformal structure of the unit circle. Such a proof will likely lead to a more general result about the relation between condition (AS) and condition (WAS). By [He, Corollary 10.22], it is known that for an embedding f from a connected subset in a Euclidean space into a Euclidean space it is QS if and only if it is WQS. We close this paper with the following open question: for an embedding f from a Jordan curve in the plane into the plane, is it true that f is AS if and only if it is WAS? Or, more generally, under what assumption (on the set) are the conditions (AS) and (WAS) equivalent? We believe that results in this paper (Theorem 4.1 in particular) may be used to attack this problem when f is a homeomorphism between symmetric quasicircles. We will investigate this and more general relations between the condition (AS) and the seemingly weaker condition (WAS) in a future project.

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Department of Mathematics, Morehouse College, Atlanta, Georgia 30314
E-mail address: abrania@morehouse.edu

Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322
E-mail address: syang@mathcs.emory.edu