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MIXED-MEAN INEQUALITIES FOR SUBSETS

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ABSTRACT. For $A \subset X = \{x_1, ..., x_n \mid x_i \geq 0,\ i = 1, 2, ..., n\}$, let a_A and g_A denote the arithmetic mean and geometric mean of elements of A, respectively. It is proved that if k is an integer in $(\frac{n}{2}, n]$, then

$$\Big(\prod_{\substack{|A|=k\\ A\subset X}} a_A\Big)^{\frac{1}{C_n^k}} \geq \frac{1}{C_n^k}\Big(\sum_{\substack{|A|=k\\ A\subset X}} g_A\Big),$$

with equality if and only if $x_1 = ... = x_n$. Furthermore, as a generalization of this inequality, a mixed power-mean inequality for subsets is established.

1. Introduction

The classical arithmetic-geometric mean inequality is one of the most important analytic inequalities, and is used in almost every branch of mathematics. There is a huge amount of work on its generalization (see [1], [3]-[7]).

In [2] (also see [6, 3.9.69]), Carlson established the following elegant mixed-mean inequality:

Let the arithmetic and geometric means of the real nonnegative numbers $x_1, ..., x_n$ taken $n-1$ at a time be denoted by

$$a_i = \frac{x_1 + ... + x_n - x_i}{n-1}, \qquad g_i = \Big(\frac{x_1 \cdots x_n}{x_i}\Big)^{\frac{1}{n-1}}.$$

Then for $n \geq 3$,

$$\Big(a_1 \cdots a_n\Big)^{\frac{1}{n}} \geq \frac{g_1 + ... + g_n}{n}.$$

In this note, we establish a new mixed arithmetic-geometric mean inequality for subsets, which is an extension of the Carlson inequality and also an extension of the arithmetic-geometric mean inequality. Furthermore, we generalize our mixed arithmetic-geometric mean inequality for subsets to mixed power mean.

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Our main results are the following two theorems.

Theorem 1. *Let $X = \{x_1, ..., x_n \mid x_i \geq 0,\ i = 1, 2, ..., n\}$. For $A \subset X$, let a_A and g_A denote the arithmetic mean and geometric mean of all elements of A, respectively. If k is an integer in $(\frac{n}{2}, n]$, then*

$$(1) \qquad (\prod_{\substack{|A|=k \\ A\subset X}} a_A)^{\frac{1}{C_n^k}} \geq \frac{1}{C_n^k}(\sum_{\substack{|A|=k \\ A\subset X}} g_A),$$

with equality if and only if $x_1 = ... = x_n$.

Theorem 2. *Let $X = \{x_1, ..., x_n \mid x_i \geq 0,\ i = 1, 2, ..., n\}$. For $A \subseteq X$ and $p \in R^+$, let a_{pA} denote the p-th power mean of all elements of A, i.e.,*

$$a_{pA} = \Big(\frac{\sum_{x_i\in A} x_i^p}{|A|}\Big)^{\frac{1}{p}}.$$

If k is an integer in $(\frac{n}{2}, n]$ and $p > q > 0$, then

$$(2) \qquad \Big(\frac{\sum_{\substack{|A|=k \\ A\subset X}} (a_{pA})^q}{C_n^k}\Big)^{\frac{1}{q}} \geq \Big(\frac{\sum_{\substack{|A|=k \\ A\subset X}} (a_{qA})^p}{C_n^k}\Big)^{\frac{1}{p}},$$

with equality if and only if $x_1 = ... = x_n$.

Remark 1. Taking $k = n$ in Theorem 1, inequality (1) is just the arithmetic-geometric mean inequality. Taking $k = n$ in Theorem 2, inequality (2) is just the famous power-mean inequality. Taking $k = n - 1$ in Theorem 1, inequality (1) is just the Carlson inequality. For $p = 1$ and passing to the limit as $q \to 0$ in Theorem 2, we see that (2) implies (1). Hence, inequality (2) is a generalization of (1).

Remark 2. If $k \leq [\frac{n}{2}]$, taking $x_1 = x_2 = ... = x_k = l,\ x_{k+1} = ... = x_n = 0$ in (1), then the left-hand side of (1) equals l/C_n^k, but the right-hand side is zero. Hence the statement of Theorem 1 fails for $k \leq [\frac{n}{2}]$.

2. Proof of main results

Proof of Theorem 1. Put $I = \{1, 2, ..., C_n^k\}$ and $\mathcal{X}_k = \{A \subset X \mid |A| = k\}$, and note that there is an injective mapping $f : I \longrightarrow \mathcal{X}_k$.

We will prove the following equality:

$$(3) \qquad a_{f(i)} = \frac{1}{C_n^k}(a_{f(i)\cap f(1)} + a_{f(i)\cap f(2)} + ... + a_{f(i)\cap f(C_n^k)}).$$

For this, let $f(i) = \{x_{i_1}, x_{i_1}, ..., x_{i_k}\}$, so that the left-hand side of (3) is equal to

$$\frac{1}{k}(x_{i_1} + x_{i_2} + ... + x_{i_k}).$$

We may assume that the right-hand side of (3) can be written as

$$b_1x_{i_1} + b_2x_{i_2} + ... + b_kx_{i_k},$$

where $b_i \in R^+$ $(i = 1, 2, ..., k)$.

Obviously, x_{i_j} and x_{i_l} $(j, l \in \{1, ..., k\}, j \neq l)$ have the same coefficient. So it follows that

$$(4) \qquad b_1 = b_2 = ... = b_k.$$

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Since $k > \frac{n}{2}$, we have

$$f(i) \cap f(j) \neq \Phi, \qquad j = 1, 2, ..., C_n^k.$$

Thus the sum of the coefficients of all elements of $a_{f(i)\cap f(j)}$ equals 1; hence

$$b_1 + b_2 + ... + b_k = \frac{1}{C_n^k}(\underbrace{1 + 1 + ... + 1}_{C_n^k}) = 1. \tag{5}$$

From (4) and (5), we obtain

$$b_1 = b_2 = ... = b_k = \frac{1}{k}.$$

Thus (3) is proved.

Next, we will prove the following analogue to (3):

$$g_{f(i)} = \Big(\prod_{j=1}^{C_n^k} g_{f(i)\cap f(j)}\Big)^{\frac{1}{C_n^k}}. \tag{6}$$

By applying the arithmetic-geometric inequality, we get

$$a_{f(i)\cap f(j)} \geq g_{f(i)\cap f(j)}. \tag{7}$$

From (3) and (7), we infer that

$$a_{f(i)} \geq \frac{1}{C_n^k} \sum_{j=1}^{C_n^k} g_{f(i)\cap f(j)}.$$

Therefore,

$$\begin{aligned} \Big(\prod_{\substack{|A|=k \\ A\subset X}} a_A\Big)^{\frac{1}{C_n^k}} &= \Big(\prod_{i=1}^{C_n^k} a_{f(i)}\Big)^{\frac{1}{C_n^k}} \\ &\geq \frac{1}{C_n^k}\Big(\prod_{i=1}^{C_n^k}(\sum_{j=1}^{C_n^k} g_{f(i)\cap f(j)})\Big)^{\frac{1}{C_n^k}}. \end{aligned} \tag{8}$$

On the other hand, using the discrete case of Hölder's inequality in the form

$$\sum_{k=1}^{n}\Big(\prod_{j=1}^{m} x_{jk}\Big)^{\frac{1}{m}} \leq \left(\prod_{j=1}^{n}\Big(\sum_{k=1}^{m} x_{jk}\Big)\right)^{\frac{1}{m}}, \tag{9}$$

where $n, m \in N^+$ and $x_{jk} \geq 0$ $(j, k = 1, 2, ..., m)$ (see [6, 2.14.2]), we obtain

$$\Big(\prod_{i=1}^{C_n^k}\Big(\sum_{j=1}^{C_n^k} g_{f(i)\cap f(j)}\Big)\Big)^{\frac{1}{C_n^k}} \geq \sum_{i=1}^{C_n^k}\Big(\prod_{j=1}^{C_n^k} g_{f(i)\cap f(j)}\Big)^{\frac{1}{C_n^k}}. \tag{10}$$

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Combining (6), (8) and (10), we see that

$$\Big(\prod_{\substack{|A|=k\\ A\subset X}} a_A\Big)^{\frac{1}{C_n^k}} \geq \frac{1}{C_n^k}\sum_{i=1}^{C_n^k}\Big(\prod_{j=1}^{C_n^k} g_{f(i)\cap f(j)}\Big)^{\frac{1}{C_n^k}}$$

$$= \frac{1}{C_n^k}\sum_{i=1}^{C_n^k} g_{f(i)} = \frac{1}{C_n^k}\sum_{\substack{|A|=k\\ A\subset X}} g_A,$$

which is just the inequality (1). □

Proof of Theorem 2.

Lemma 1. *Suppose that* (c_{ij}) *is an* $m\times n$ *matrix with nonnegative entries. If* $q<1$*, then the q-th power mean of all the arithmetic means of the rows is greater than or equal to the arithmetic mean of all the q-th power means of the columns.*

Proof. It suffices to prove the inequality

$$\Big(\frac{\sum_{i=1}^{m}\Big(\frac{\sum_{j=1}^{n} c_{ij}}{n}\Big)^q}{m}\Big)^{\frac{1}{q}} \geq \frac{\sum_{j=1}^{n}\Big(\frac{\sum_{i=1}^{m}(c_{ij})^q}{m}\Big)^{\frac{1}{q}}}{n},$$

or equivalently,

$$\Big(\sum_{i=1}^{m}\Big(\sum_{j=1}^{n} c_{ij}\Big)^q\Big)^{\frac{1}{q}} \geq \sum_{j=1}^{n}\Big(\sum_{i=1}^{m}(c_{ij})^q\Big)^{\frac{1}{q}},$$

which is just Minkowski's inequality. □

We will prove the inequality (2) in two steps.

First, we prove that if (2) holds for $p=1$, then (2) holds for $p\in R^+$.

In fact, for $A=\{x_{i_1},...,x_{i_k}\}\subset X$, let $y_i={x_i}^p$, $\alpha=\frac{q}{p}$, $Y=\{y_1,...,y_n\}$, and $A'=\{y_{i_1},...,y_{i_k}\}\subset Y$. Then, the left-hand side of (2) equals

$$\Big(\frac{\sum_{\substack{|A|=k\\ A\subset X}}\Big(\frac{\sum_{x_i\in A} x_i^p}{k}\Big)^{\frac{q}{p}}}{C_n^k}\Big)^{\frac{1}{q}} = \Big(\frac{\sum_{\substack{|A'|=k\\ A'\subset Y}}\Big(\frac{\sum_{y\in A'} y}{k}\Big)^{\alpha}}{C_n^k}\Big)^{\frac{1}{q}},$$

and the right-hand side of (2) is

$$\Big(\frac{\sum_{\substack{|A|=k\\ A\subset X}}\Big(\frac{\sum_{x_i\in A} x_i^q}{k}\Big)^{\frac{p}{q}}}{C_n^k}\Big)^{\frac{1}{p}} = \Big(\frac{\sum_{\substack{|A'|=k\\ A'\subset Y}}\Big(\frac{\sum_{y\in A'} y^{\alpha}}{k}\Big)^{\frac{1}{\alpha}}}{C_n^k}\Big)^{\frac{1}{p}}.$$

Hence, (2) is equivalent to the following inequality:

$$\Big(\frac{\sum_{\substack{|A'|=k\\ A'\subset Y}}(a_{1A'})^{\alpha}}{C_n^k}\Big)^{\frac{1}{\alpha}} \geq \frac{\sum_{\substack{|A'|=k\\ A'\subset Y}} a_{\alpha A'}}{C_n^k},$$

which is just inequality (2) for $p=1$ $(q=\alpha<1)$.

Second, we prove that (2) holds when $p=1$.

In fact, let $C=(c_{ij})$ be a $C_n^k\times C_n^k$ matrix with rows indexed by $\mathcal{X}_k=\{f(i)|\ f(i)\subset X,\ |f(i)|=k\}$ and columns indexed also by $\mathcal{X}_k$, where

$$c_{ij}=a_{qf(i)\cap f(j)}.$$

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The q-th power mean of the j-th column equals

$$\begin{aligned}\Big(\frac{1}{C_n^k}\sum_{i=1}^{C_n^k}(a_{qf(i)\cap f(j)})^q\Big)^{\frac{1}{q}} &= \Big(\frac{1}{C_n^k}\sum_{i=1}^{C_n^k}\Big(\Big(\sum_{x\in f(i)\cap f(j)}\frac{x^q}{|f(i)\cap f(j)|}\Big)^{\frac{1}{q}}\Big)^q\Big)^{\frac{1}{q}}\\ &= \Big(\frac{1}{C_n^k}\sum_{i=1}^{C_n^k}\sum_{x\in f(i)\cap f(j)}\frac{x^q}{|f(i)\cap f(j)|}\Big)^{\frac{1}{q}}.\end{aligned}$$

Since the elements of $f(j)$ are in the same situation, the right-hand side of the above equality does not contain the term x_s^q (if $x_s \notin f(j)$) and the sum of the coefficients of all x_t^q (if $x_t \in f(j)$) equals 1. Thus we have

$$\Big(\frac{1}{C_n^k}\sum_{i=1}^{C_n^k}(a_{qf(i)\cap f(j)})^q\Big)^{\frac{1}{q}} = \Big(\frac{1}{k}\sum_{x\in g(j)}x^q\Big)^{\frac{1}{q}}.$$

Hence, the arithmetic mean of all the q-th power means of the columns is equal to

$$\frac{1}{C_n^k}\sum_{j=1}^{C_n^k}\Big(\frac{1}{k}\sum_{x\in g(j)}x^q\Big)^{\frac{1}{q}} = \frac{1}{C_n^k}\sum_{\substack{|A|=k\\ A\subset X}}a_{qA}. \tag{11}$$

On the other hand, according to the famous power mean inequality, and noting that $q < 1$, we have

$$c_{ij} = a_{qf(i)\cap f(j)} \le a_{1f(i)\cap f(j)}.$$

It follows that the arithmetic mean of the i-th row is less than or equal to

$$\begin{aligned}\frac{1}{C_n^k}\sum_{j=1}^{C_n^k}a_{1f(i)\cap f(j)} &= \frac{1}{C_n^k}\sum_{j=1}^{C_n^k}\sum_{x\in f(i)\cap f(j)}\frac{x}{|f(i)\cap f(j)|}\\ &= \frac{1}{k}\sum_{x\in f(j)}x.\end{aligned}$$

Hence, the q-th power mean of all the arithmetic means of the rows is less than or equal to

$$\Big(\frac{1}{C_n^k}\sum_{j=1}^{C_n^k}\Big(\frac{\sum_{x\in f(j)}x}{k}\Big)^q\Big)^{\frac{1}{q}} = \Big(\frac{1}{C_n^k}\sum_{\substack{|A|=k\\ A\subset X}}(a_{1A})^q\Big)^{\frac{1}{q}}. \tag{12}$$

Applying Lemma 1, it follows from (11) and (12) that

$$\Big(\frac{1}{C_n^k}\sum_{\substack{|A|=k\\ A\subset X}}(a_{1A})^q\Big)^{\frac{1}{q}} \ge \frac{1}{C_n^k}\sum_{\substack{|A|=k\\ A\subset X}}a_{qA}.$$

Thus, (2) is true for $p = 1$. □

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