MIXED-MEAN INEQUALITIES FOR SUBSETS

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(Communicated by Carmen C. Chicone)

Abstract. For $A \subseteq X = \{x_1, \ldots, x_n \mid x_i \geq 0, \ i = 1, 2, \ldots, n\}$, let $a_A$ and $g_A$ denote the arithmetic mean and geometric mean of elements of $A$, respectively. It is proved that if $k$ is an integer in $(n^2, n]$, then

$$\left( \prod_{\substack{|A|=k \\ A \subseteq X}} a_A \right)^{\frac{1}{n^k}} \geq \frac{1}{C_n} \left( \sum_{\substack{|A|=k \\ A \subseteq X}} g_A \right)^{\frac{1}{n^k}}$$

with equality if and only if $x_1 = \ldots = x_n$. Furthermore, as a generalization of this inequality, a mixed power-mean inequality for subsets is established.

1. Introduction

The classical arithmetic-geometric mean inequality is one of the most important analytic inequalities, and is used in almost every branch of mathematics. There is a huge amount of work on its generalization (see [1], [3]-[7]).

In [2] (also see [3, 3.9.69]), Carlson established the following elegant mixed-mean inequality:

Let the arithmetic and geometric means of the real nonnegative numbers $x_1, \ldots, x_n$ taken $n-1$ at a time be denoted by

$$a_i = \frac{x_1 + \ldots + x_n - x_i}{n-1}, \quad g_i = \left( \frac{x_1 \cdots x_n}{x_i} \right)^{\frac{1}{n-1}}.$$ 

Then for $n \geq 3$,

$$\left( a_1 \cdots a_n \right)^{\frac{1}{n}} \geq \frac{g_1 + \ldots + g_n}{n}.$$ 

In this note, we establish a new mixed arithmetic-geometric mean inequality for subsets, which is an extension of the Carlson inequality and also an extension of the arithmetic-geometric mean inequality. Furthermore, we generalize our mixed arithmetic-geometric mean inequality for subsets to mixed power mean.
Our main results are the following two theorems.

**Theorem 1.** Let $X = \{x_1, \ldots, x_n \mid x_i \geq 0, \ i = 1, 2, \ldots, n\}$. For $A \subseteq X$, let $a_A$ and $g_A$ denote the arithmetic mean and geometric mean of all elements of $A$, respectively. If $k$ is an integer in $(\frac{n}{2}, n]$, then

$$
\left( \prod_{|A| = k} a_A \right)^{\frac{1}{k}} \geq \frac{1}{C_n^k} \left( \sum_{|A| = k} g_A \right),
$$

with equality if and only if $x_1 = \ldots = x_n$.

**Theorem 2.** Let $X = \{x_1, \ldots, x_n \mid x_i \geq 0, \ i = 1, 2, \ldots, n\}$. For $A \subseteq X$ and $p \in R^+$, let $a_{pA}$ denote the $p$-th power mean of all elements of $A$, i.e.,

$$
a_{pA} = \left( \frac{\sum_{x \in A} x^p}{|A|} \right)^{\frac{1}{p}}.
$$

If $k$ is an integer in $(\frac{n}{2}, n]$ and $p > q > 0$, then

$$
\left( \prod_{|A| = k} a_{pA} \right)^{\frac{q}{k}} \geq \frac{1}{C_n^k} \left( \sum_{|A| = k} (a_{qA})^p \right)^{\frac{1}{p}},
$$

with equality if and only if $x_1 = \ldots = x_n$.

**Remark 1.** Taking $k = n$ in Theorem 1, inequality (1) is just the arithmetic-geometric mean inequality. Taking $k = n$ in Theorem 2, inequality (2) is just the famous power-mean inequality. Taking $k = n - 1$ in Theorem 1, inequality (1) is just the Carlson inequality. For $p = 1$ and passing to the limit as $q \to 0$ in Theorem 2, we see that (2) implies (1). Hence, inequality (2) is a generalization of (1).

**Remark 2.** If $k \leq \lfloor \frac{n}{2} \rfloor$, taking $x_1 = x_2 = \ldots = x_k = l$, $x_{k+1} = \ldots = x_n = 0$ in (1), then the left-hand side of (1) equals $l/C_n^k$, but the right-hand side is zero. Hence the statement of Theorem 1 fails for $k \leq \lfloor \frac{n}{2} \rfloor$.

2. Proof of Main Results

**Proof of Theorem 1.** Put $I = \{1, \ldots, C_n^k\}$ and $A_k = \{A \subseteq X \mid |A| = k\}$, and note that there is an injective mapping $f : I \rightarrow A_k$.

We will prove the following equality:

$$
a_{f(i)} = \frac{1}{C_n^k} \left( a_{f(i) \cap f(1)} + a_{f(i) \cap f(2)} + \ldots + a_{f(i) \cap f(C_n^k)} \right).
$$

For this, let $f(i) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$, so that the left-hand side of (3) is equal to

$$
\frac{1}{k} (x_{i_1} + x_{i_2} + \ldots + x_{i_k}).
$$

We may assume that the right-hand side of (3) can be written as

$$
b_1 x_{i_1} + b_2 x_{i_2} + \ldots + b_k x_{i_k},
$$

where $b_i \in R^+$ ($i = 1, 2, \ldots, k$).

Obviously, $x_{i_j}$ and $x_{i_l}$ ($j, l \in \{1, \ldots, k\}, j \neq l$) have the same coefficient. So it follows that

$$
b_1 = b_2 = \ldots = b_k.
$$
Since $k > \frac{n}{2}$, we have
\[ f(i) \cap f(j) \neq \Phi, \quad j = 1, 2, \ldots, C_n^k. \]
Thus the sum of the coefficients of all elements of $a_{f(i) \cap f(j)}$ equals 1; hence
\[ b_1 + b_2 + \ldots + b_k = \frac{1}{C_n^k} \left(1 + 1 + \ldots + 1\right) = 1. \]
From (4) and (5), we obtain
\[ b_1 = b_2 = \ldots = b_k = \frac{1}{k}. \]
Thus (3) is proved.

Next, we will prove the following analogue to (3):
\[ g_{f(i)} = \left(\prod_{j=1}^{C_n^k} g_{f(i) \cap f(j)}\right)^{\frac{1}{C_n^k}}. \]
By applying the arithmetic-geometric inequality, we get
\[ a_{f(i) \cap f(j)} \geq g_{f(i) \cap f(j)}. \]
From (3) and (7), we infer that
\[ a_{f(i)} \geq \frac{1}{C_n^k} \sum_{j=1}^{C_n^k} g_{f(i) \cap f(j)}. \]
Therefore,
\[ \left(\prod_{\|A\|=k} a_A\right)^{\frac{1}{C_n^k}} = \left(\prod_{i=1}^{C_n^k} a_{f(i)}\right)^{\frac{1}{C_n^k}} \geq \frac{1}{C_n^k} \left(\prod_{i=1}^{C_n^k} \left(\sum_{j=1}^{C_n^k} g_{f(i) \cap f(j)}\right)^{\frac{1}{C_n^k}}\right). \]
On the other hand, using the discrete case of Hölder’s inequality in the form
\[ \sum_{k=1}^{n} \left(\prod_{j=1}^{m} x_{jk}\right)^{\frac{1}{m}} \leq \left(\prod_{j=1}^{m} \left(\sum_{k=1}^{n} x_{jk}\right)^{\frac{1}{n}}\right)^{\frac{1}{m}}, \]
where $n, m \in \mathbb{N}^+$ and $x_{jk} \geq 0$ ($j, k = 1, 2, \ldots, m$) (see [3] 2.14.2), we obtain
\[ \left(\prod_{i=1}^{C_n^k} \left(\sum_{j=1}^{C_n^k} g_{f(i) \cap f(j)}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{C_n^k}} \geq \sum_{i=1}^{C_n^k} \left(\prod_{j=1}^{C_n^k} g_{f(i) \cap f(j)}\right)^{\frac{1}{C_n^k}}. \]
Combining (6), (8) and (10), we see that

\[
Y_j A_j = k A X a A_k C_{kn} \sum_{i=1}^{C_k n} g_{f(i) \cap f(j)} \frac{1}{C_{kn} n} = \frac{1}{C_{kn} n} \sum_{|A| = k} g_{f(i)} = \frac{1}{C_{kn} n} \sum_{A \subseteq X} g_A,
\]

which is just the inequality (1).

\[\square\]

**Proof of Theorem 2.**

**Lemma 1.** Suppose that \((c_{ij})\) is an \(m \times n\) matrix with nonnegative entries. If \(q < 1\), then the \(q\)-th power mean of all the arithmetic means of the rows is greater than or equal to the arithmetic mean of all the \(q\)-th power means of the columns.

**Proof.** It suffices to prove the inequality

\[
\sum_{i=1}^{m} \left( \frac{\sum_{j=1}^{n} c_{ij}}{n} \right)^{\frac{q}{n}} \geq \frac{\sum_{j=1}^{n} \left( \sum_{i=1}^{m} c_{ij} \right)^{\frac{q}{n}}}{m},
\]

or equivalently,

\[
\left( \sum_{i=1}^{m} \left( \sum_{j=1}^{n} c_{ij} \right)^{q} \right)^{\frac{1}{q}} \geq \sum_{j=1}^{n} \left( \sum_{i=1}^{m} c_{ij} \right)^{\frac{1}{q}},
\]

which is just Minkowski’s inequality. \[\square\]

We will prove the inequality (2) in two steps.

First, we prove that if (2) holds for \(p = 1\), then (2) holds for \(p \in \mathbb{R}^+\).

In fact, let \(A = \{x_1, \ldots, x_k\} \subset X\), let \(y_i = x_i^p\), \(\alpha = \frac{q}{p}\), \(Y = \{y_1, \ldots, y_n\}\), and \(A' = \{y_1, \ldots, y_k\} \subset Y\). Then, the left-hand side of (2) equals

\[
\left( \frac{\sum_{|A| = k} (\sum_{x_i \in A} x_i^p)^\alpha}{C_{kn} k} \right)^{\frac{1}{\alpha}} = \left( \frac{\sum_{|A'| = k} (\sum_{y_j \in A'} y_j^\alpha)^{\frac{1}{\alpha}}}{C_{kn} k} \right)^{\frac{1}{\alpha}},
\]

and the right-hand side of (2) is

\[
\left( \sum_{|A| = k} \left( \frac{\sum_{x_i \in A} x_i^p}{k} \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} = \left( \frac{\sum_{|A'| = k} (\sum_{y_j \in A'} y_j^\alpha)^{\frac{1}{\alpha}}}{C_{kn} k} \right)^{\frac{1}{\alpha}}.
\]

Hence, (2) is equivalent to the following inequality:

\[
\left( \frac{\sum_{|A'| = k} (a_{A'})^{\alpha}}{C_{kn} k} \right)^{\frac{1}{\alpha}} \geq \frac{\sum_{|A'| = k} a_{A'}}{C_{kn} k},
\]

which is just inequality (2) for \(p = 1\) \((q = \alpha < 1)\).

Second, we prove that (2) holds when \(p = 1\).

In fact, let \(C = (c_{ij})\) be a \(C_{kn} \times C_{kn}\) matrix with rows indexed by \(X_k = \{f(i) \mid f(i) \subset X, \ |f(i)| = k\}\) and columns indexed also by \(X_k\), where

\[
c_{ij} = a_q f(i) \cap f(j)\].

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The $q$-th power mean of the $j$-th column equals
\[
\left( \frac{1}{C_n} \sum_{i=1}^{C_n} (a_{qf(i)\cap f(j)})^q \right)^{\frac{1}{q}} = \left( \frac{1}{C_n} \sum_{i=1}^{C_n} \left( \left( \sum_{x \in f(i) \setminus f(j)} x^q \right) / |f(i) \cap f(j)| \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} = \left( \frac{1}{C_n} \sum_{i \in f(j)} x \right)^{\frac{1}{q}}.
\]
Since the elements of $f(j)$ are in the same situation, the right-hand side of the above equality does not contain the term $x^q$ (if $x \not\in f(j)$) and the sum of the coefficients of all $x^q$ (if $x \in f(j)$) equals 1. Thus we have
\[
\left( \frac{1}{C_n} \sum_{i=1}^{C_n} (a_{qf(i)\cap f(j)})^q \right)^{\frac{1}{q}} = \left( \frac{1}{k} \sum_{x \in g(j)} x^q \right)^{\frac{1}{q}}.
\]
Hence, the arithmetic mean of all the $q$-th power means of the columns is equal to
\[
\frac{1}{C_n} \sum_{j=1}^{C_n} \left( \frac{1}{k} \sum_{x \in g(j)} x^q \right)^{\frac{1}{q}} = \frac{1}{C_n} \sum_{|A|=k} a_{qA}.
\]
On the other hand, according to the famous power mean inequality, and noting that $q < 1$, we have
\[c_{ij} = a_{qf(i)\cap f(j)} \leq a_{1f(i)\cap f(j)}.
\]
It follows that the arithmetic mean of the $i$-th row is less than or equal to
\[
\frac{1}{C_n} \sum_{j=1}^{C_n} a_{1f(i)\cap f(j)} = \frac{1}{C_n} \sum_{j=1}^{C_n} \sum_{x \in f(i) \setminus f(j)} x / |f(i) \cap f(j)| = \frac{1}{k} \sum_{x \in f(j)} x.
\]
Hence, the $q$-th power mean of all the arithmetic means of the rows is less than or equal to
\[
\left( \frac{1}{C_n} \sum_{j=1}^{C_n} \left( \frac{\sum_{x \in f(j)} x}{k} \right)^q \right)^{\frac{1}{q}} = \left( \frac{1}{C_n} \sum_{|A|=k} (a_{1A})^q \right)^{\frac{1}{q}}.
\]
Applying Lemma 1, it follows from (11) and (12) that
\[
\left( \frac{1}{C_n} \sum_{|A|=k} (a_{1A})^q \right)^{\frac{1}{q}} \geq \frac{1}{C_n} \sum_{|A|=k} a_{qA}.
\]
Thus, (2) is true for $p = 1$. 
\[\square\]
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