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INDECOMPOSABLES OF MULTIPLICATIVE FIBRATIONS

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Abstract. Given a multiplicative fibration $F \xrightarrow{j} E \xrightarrow{\pi} B$ we study the module of indecomposables $QH^*(E;\mathbb{Z}_p)$ for p a prime.

A fibration $F \xrightarrow{j} E \xrightarrow{\pi} B$ is a multiplicative fibration if E and B are connected H-spaces and $\pi : E \longrightarrow B$ is an H-map. In this note we study the module of indecomposables $QH^*(E;\mathbb{Z}_p)$ for p a prime.

Theorem 0.1 (Main Theorem)**.** *Let $F \xrightarrow{j} E \xrightarrow{\pi} B$ be a multiplicative fibration.*

(a) *Let p be an odd prime. If $n \not\equiv 0 \bmod 2p$ and $n \not\equiv \pm 1 \bmod 2p$, then there are exact sequences*

$$PH_n(F;\mathbb{Z}_p) \xrightarrow{Pj_*} PH_n(E;\mathbb{Z}_p) \xrightarrow{P\pi_*} PH_n(B;\mathbb{Z}_p),$$

$$QH^n(F;\mathbb{Z}_p) \xleftarrow{Qj^*} QH^n(E;\mathbb{Z}_p) \xleftarrow{Q\pi^*} QH^n(B;\mathbb{Z}_p).$$

(b) *Let $n \equiv 0 \bmod 2p$ for p odd or n even for $p = 2$. Suppose*

$$QH^n(F;\mathbb{Z}_p) \oplus \sum_{k=1}^{\infty} Q^n(\xi^k H^*(F;\mathbb{Z}_p)) = 0$$

where $\xi : H^*(F;\mathbb{Z}_p) \longrightarrow H^*(F;\mathbb{Z}_p)$ is the pth power map. Then $QH^n(E;\mathbb{Z}_p) = Q\pi^* QH^n(B;\mathbb{Z}_p)$.

Theorems similar to the Main Theorem have been proved under more specialized assumptions. For example, if $H^*(E;\mathbb{Z}_p)$ is a $U(M)$ module, such multiplicative fibrations were studied by Massey and Peterson [6, 7]. In the case of loop fibrations, these sequences have been considered by Goerss, Lannes and Morel [2]. Moore and Smith also study multiplicative fibrations using the Eilenberg-Moore spectral sequence [10]. We will use this theorem in a future paper to investigate the cohomology of finite H-spaces with nontrivial Steenrod action on the even degree indecomposables. As a corollary of the Main Theorem, we make some observations about the ring structure of $H_*(E;\mathbb{Z}_p)$.

We assume all spaces have the homotopy type of path connected CW complexes with basepoint and all homology and cohomology modules are finitely generated.

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The ground field will be the field $\mathbb{Z}_p$ of p elements where p is a prime. Given a connected Hopf algebra $A, I(A)$ will denote the elements of A of positive degree. $P(A)$ and $Q(A)$ will denote the primitives and indecomposables respectively. $\alpha_A : I(A) \longrightarrow Q(A) = I(A)/I(A)^2$ will denote the projection map.

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Proposition 0.2. *Let A and B be connected commutative Hopf algebras over $\mathbb{Z}_p$ with B a sub-Hopf algebra of A.*

- (a) *Let $n \not\equiv 0 \bmod 2p$, for p odd or n odd for $p = 2$. Then the inclusion map $i : B \longrightarrow A$ induces a monomorphism $Qi : Q^nB \longrightarrow Q^nA$.*
- (b) *Let $n \equiv 0 \bmod 2p$ for p odd or n even for $p = 2$. Suppose $0 \neq \alpha_B(b) \in Q^nB \cap \ker Qi$. Then $\alpha_B(b)$ can be represented by a p^kth power of an algebra generator $a \in A$ for some $k \geq 1$.*

Proof. Suppose $0 \neq \alpha_B(b) \in Q^n(B) \cap \ker Qi$. Let $B(n-1)$ be the sub-Hopf algebra of B generated by elements of B of degree less than n. We have a commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\pi_B"] \arrow[d, "i"'] & B/\!/B(n-1) \arrow[d, "\theta"] \\
A \arrow[r, "\pi_A"] & A/\!/B(n-1)
\end{tikzcd}$$

where π_B, π_A are Hopf algebra epimorphisms and $0 \neq \pi_B(b) \in P^n(B/\!/B(n-1)) \cong Q^nB$. By [3, p. 8] there exist isomorphisms $B \cong B(n-1) \otimes B/\!/ B(n-1)$ and $A \cong B(n-1) \otimes A/\!/B(n-1)$ such that the following diagram commutes:

$$\begin{tikzcd}
B \arrow[r, phantom, "\cong"] \arrow[d, "i"'] & B(n-1) \otimes B/\!/B(n-1) \arrow[d, "1 \otimes \theta"] \\
A \arrow[r, phantom, "\cong"] & B(n-1) \otimes A/\!/B(n-1)
\end{tikzcd}$$

Therefore, θ is a monomorphism.

We have a commutative diagram:

$$\begin{tikzcd}
 & Q^nB \arrow[r, "Qi"] \arrow[d, equal, "\wr"] & Q^nA \arrow[r, "Q\pi_A"] & Q^n\Big(A/\!/B(n-1)\Big) \\
0 \arrow[r] & P^n\Big(B/\!/B(n-1)\Big) \arrow[rr, "P\theta"] & & P^n\Big(A/\!/B(n-1)\Big) \arrow[u] \\
 & & & P^n\Big(\xi\big(A/\!/B(n-1)\big)\Big) \arrow[u] \\
 & & & 0 \arrow[u]
\end{tikzcd} \tag{0.1}$$

The bottom row is exact since θ is monic. The right column is exact by [9, Proposition 4.2.21]. Now given $Qi\alpha_B(b) = 0$, by diagram (0.1), it follows that

$\pi_A(b) \in P^n(A/\!\!/B(n-1))$ is decomposable. By exactness of the right column of (0.1), $\pi_A(b)$ is a pth power. In case (a) $n \not\equiv 0 \mod 2p$; this is impossible and $Qi : Q^nB \to Q^nA$ is a monomorphism. In case (b) $n \equiv 0 \mod 2p$ for p odd or n is even for $p = 2$. For each $k \geq 1$ there are exact sequences [9, Proposition 4.2.21]

$$0 \longrightarrow P(\xi^{k+1}(A/\!\!/B(n-1))) \longrightarrow P(\xi^k(A/\!\!/B(n-1))) \longrightarrow Q(\xi^k(A/\!\!/B(n-1))).$$

By using these sequences we can show that $\alpha_B(b)$ can be represented by a p^kth power of a generator of A for some $k \geq 1$. This proves (b). □

Let $F \xrightarrow{j} E \xrightarrow{\pi} B$ be a multiplicative fibration. By [1, Lemma 5.1], $\pi_1(B)$ acts trivially on $H_*(F;\mathbb{Z}_p)$. Hence the Serre spectral sequence for the fibration satisfies

$$E_2^{r,s} = H^r(B;\mathbb{Z}_p) \otimes H^s(F;\mathbb{Z}_p) \tag{0.2}$$

and

$$E_\infty \cong \mathcal{G}_r H^*(E;\mathbb{Z}_p).$$

This is a first quadrant spectral sequence of Hopf algebras [1, p. 166]. Furthermore, by [1, Theorem 5.8],

$$E_k \cong B_k \otimes C_k \otimes M_k \otimes N_k \quad \text{for} \quad k \geq 2 \quad \text{as algebras} \tag{0.3}$$

where

$$B_k = E_k^{0,*} \text{ and } C_k = E_k^{*,0} \text{ are sub-Hopf algebras of } E_k.$$

$M_k = \Lambda(x_1, \ldots, x_\ell)$ and $N_k = \Lambda(w_1, \ldots, w_n)$ with $\dim x_i \equiv 1 \mod 2p$ and $\dim w_i \equiv -1 \mod 2p$. Equation (0.3) implies

$$QE_\infty^{r,s} = 0 \quad \text{for} \quad r > 0 \quad \text{and} \quad s > 0 \quad \text{and} \quad r + s \not\equiv \pm 1 \mod 2p. \tag{0.4}$$

Hence, if $n \not\equiv \pm 1 \mod 2p$,

$$Q^nE_\infty = QE_\infty^{0,n} \oplus QE_\infty^{n,0}. \tag{0.5}$$

By [8, Theorem 5.8], $j^* : H^n(E;\mathbb{Z}_p) \longrightarrow H^n(F;\mathbb{Z}_p)$ is the composition

$$H^n(E;\mathbb{Z}_p) \longrightarrow E_\infty^{0,n} = E_{n+1}^{0,n} \subset \cdots \subset E_2^{0,n} = H^n(F;\mathbb{Z}_p), \tag{0.6}$$

and $\pi^* : H^n(B;\mathbb{Z}_p) \longrightarrow H^n(E;\mathbb{Z}_p)$ is the composition

$$H^n(B;\mathbb{Z}_p) = E_2^{n,0} \twoheadrightarrow E_{n+1}^{n,0} = E_\infty^{n,0} \subseteq H^n(E;\mathbb{Z}_p). \tag{0.7}$$

Proof of the Main Theorem 0.1. The sequences of (a) are dual. So it suffices to prove

$$QH^n(B;\mathbb{Z}_p) \xrightarrow{Q\pi^*} QH^n(E;\mathbb{Z}_p) \xrightarrow{Qj_*} QH^n(F;\mathbb{Z}_p)$$

is exact if p is an odd prime, $n \not\equiv 0 \mod 2p$ and $n \not\equiv \pm 1 \mod 2p$. Note that πj is null homotopic. Therefore, $Qj^*Q\pi^* = 0$. So it suffices to prove $\ker Qj^* \subseteq \operatorname{im} Q\pi^*$.

Let $\alpha : IH^*(E;\mathbb{Z}_p) \longrightarrow QH^*(E;\mathbb{Z}_p)$ be the projection. Let $\alpha(w) \in \ker Qj^* \cap QH^n(E;\mathbb{Z}_p)$. The Serre spectral sequence is a spectral sequence of algebras. Hence, w must produce a nonzero element of Q^nE_∞. By (0.5),

$$Q^nE_\infty = QE_\infty^{0,n} \oplus QE_\infty^{n,0}.$$

Suppose $\alpha(w)$ has a nonzero component in $QE_\infty^{0,n}$. By (0.3) and (0.6), $E_\infty^{0,*}$ is a sub-Hopf algebra of $H^*(F;\mathbb{Z}_p)$. By Proposition 0.2 (a), since $n \not\equiv 0 \mod 2p$, and p is odd, a nonzero element of $QE_\infty^{0,n}$ produces a nonzero element of $QH^n(F;\mathbb{Z}_p)$.

By (0.6), this would imply that $Qj^*(\alpha(w)) \neq 0$. But $\alpha(w) \in \ker Qj^*$. Therefore, $\alpha(w)$ has zero component in $QE_\infty^{0,n}$ and $\alpha(w)$ lies in $QE_\infty^{n,0}$. By (0.7),

$$H^n(B;\mathbb{Z}_p) \longrightarrow E_\infty^{n,0} \quad \text{is onto.} \tag{0.8}$$

Hence $\alpha(w)$ lies in the image of $Q\pi^*$. This proves (a). To prove (b) let $\alpha(w) \in QH^n(E;\mathbb{Z}_p)$ with $n \equiv 0 \bmod 2p$ for p odd or n even for $p = 2$. Suppose $\alpha(w)$ has nonzero component x in $QE_\infty^{0,n}$. Since $QH^n(F;\mathbb{Z}_p) = 0$, the inclusion map induces the trivial map

$$Qi : QE_\infty^{0,n} \longrightarrow QH^n(F;\mathbb{Z}_p).$$

By Proposition 0.2 (b), x can be represented by a p^kth power of an algebra generator of $H^*(F;\mathbb{Z}_p)$. But $Q^n(\xi^k H^*(F;\mathbb{Z}_p)) = 0$. Hence $\alpha(w)$ has no component in $QE_\infty^{0,n}$ and $\alpha(w)$ lies in $QE_\infty^{n,0}$. By (0.8), $\alpha(w)$ lies in the image of $Q\pi^*$. □

Let $f : B \to K(\mathbb{Z}_p, \ell + 1)$ be an H-map, and let E be the fibre of f. We have the following multiplicative fibration:

$$\begin{tikzcd} K(\mathbb{Z}_p,\ell) \arrow[d, "j"] \\ E \arrow[d, "\pi"] \\ B \end{tikzcd}$$

In general, the algebra structure of $H_*(E;\mathbb{Z}_p)$ is difficult to compute. We can, however, make the following observations.

Corollary 0.3. *Let p be an odd prime. Let $s, t \in PH_*(E;\mathbb{Z}_p)$ with $\pi_*(s) \neq 0, \pi_*(t) \neq 0$, and $[\pi_*(s), \pi_*(t)] = 0$. Suppose $\deg[s,t] = n$ and $n \not\equiv \pm 1 \mod 2p$ and $n \not\equiv 0 \mod 2p$. If $QH^n(K(\mathbb{Z}_p,\ell);\mathbb{Z}_p) = 0$, then $[s,t] = 0$.*

Proof. By the Main Theorem 0.1 (a), since $PH_n(K(\mathbb{Z}_p,\ell);\mathbb{Z}_p) = 0$, if $[s,t] \neq 0$, then $\pi_*[s,t] = [\pi_*(s), \pi_*(t)] \neq 0$. We conclude $[s,t] = 0$. □

Corollary 0.4. *Let $t \in PH_{2m}(E;\mathbb{Z}_p)$ with $\pi_*(t) \neq 0$, and $(\pi_*(t))^p = 0$. If $QH^{2mp}(K(\mathbb{Z}_p,\ell);\mathbb{Z}_p) \oplus \sum_{k=1}^{\infty} Q^{2mp}(\xi^k H^*(K(\mathbb{Z}_p,\ell);\mathbb{Z}_p)) = 0$, then $t^p = 0$.*

Proof. If $t^p \neq 0$, there exists an indecomposable $\gamma \in QH^{2mp}(E;\mathbb{Z}_p)$ with $\langle t^p, \gamma\rangle = 1$. By the Main Theorem 0.1 (b), $\gamma = \pi^*(\gamma_1) + d$ where γ_1 is indecomposable and d is decomposable. Then $1 = \langle t^p, \gamma\rangle = \langle t^p, \pi^*(\gamma_1) + d\rangle = \langle t^p, \pi^*(\gamma_1)\rangle = \langle(\pi_*(t))^p, \gamma_1\rangle = 0$. This is a contradiction. Therefore, $t^p = 0$. □

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