INDECOMPOSABLES OF MULTIPlicative Fibrations

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Abstract. Given a multiplicative fibration \( F \rightarrow E \rightarrow B \) we study the module of indecomposables \( \mathbb{Q}H^n(E; \mathbb{Z}_p) \) for \( p \) a prime.

A fibration \( F \rightarrow E \rightarrow B \) is a multiplicative fibration if \( E \) and \( B \) are connected \( H \)-spaces and \( \pi : E \rightarrow B \) is an \( H \)-map. In this note we study the module of indecomposables \( \mathbb{Q}H^n(E; \mathbb{Z}_p) \) for \( p \) a prime.

Theorem 0.1 (Main Theorem). Let \( F \rightarrow E \rightarrow B \) be a multiplicative fibration.

(a) Let \( p \) be an odd prime. If \( n \equiv 0 \mod 2p \) and \( n \equiv \pm 1 \mod 2p \), then there are exact sequences

\[
\begin{align*}
PH_n(F; \mathbb{Z}_p) & \xrightarrow{P^*} PH_n(E; \mathbb{Z}_p) \xrightarrow{\pi^*} PH_n(B; \mathbb{Z}_p), \\
\mathbb{Q}H^n(F; \mathbb{Z}_p) & \xrightarrow{Q^*} \mathbb{Q}H^n(E; \mathbb{Z}_p) \xrightarrow{\pi^*} \mathbb{Q}H^n(B; \mathbb{Z}_p).
\end{align*}
\]

(b) Let \( n \equiv 0 \mod 2p \) for \( p \) odd or \( n \) even for \( p = 2 \). Suppose

\[\mathbb{Q}H^n(F; \mathbb{Z}_p) \oplus \sum_{k=1}^{\infty} \mathbb{Q}^n(\xi^k H^*(F; \mathbb{Z}_p)) = 0\]

where \( \xi : H^*(F; \mathbb{Z}_p) \rightarrow H^*(F; \mathbb{Z}_p) \) is the \( p \)th power map. Then \( \mathbb{Q}H^n(E; \mathbb{Z}_p) = \mathbb{Q}^n \pi^* \mathbb{Q}H^n(B; \mathbb{Z}_p) \).

Theorems similar to the Main Theorem have been proved under more specialized assumptions. For example, if \( H^*(E; \mathbb{Z}_p) \) is a \( U(M) \) module, such multiplicative fibrations were studied by Massey and Peterson [6, 7]. In the case of loop fibrations, these sequences have been considered by Goerss, Lannes and Morel [2]. Moore and Smith also study multiplicative fibrations using the Eilenberg-Moore spectral sequence [10]. We will use this theorem in a future paper to investigate the cohomology of finite \( H \)-spaces with nontrivial Steenrod action on the even degree indecomposables. As a corollary of the Main Theorem, we make some observations about the ring structure of \( H_*(E; \mathbb{Z}_p) \).

We assume all spaces have the homotopy type of path connected CW complexes with basepoint and all homology and cohomology modules are finitely generated.

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The ground field will be the field \( \mathbb{Z}_p \) of \( p \) elements where \( p \) is a prime. Given a connected Hopf algebra \( A, I(A) \) will denote the elements of \( A \) of positive degree. \( P(A) \) and \( Q(A) \) will denote the primitives and indecomposables respectively. \( \alpha_A : I(A) \longrightarrow Q(A) = I(A)/I(A)^2 \) will denote the projection map.

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**Proposition 0.2.** Let \( A \) and \( B \) be connected commutative Hopf algebras over \( \mathbb{Z}_p \) with \( B \) a sub-Hopf algebra of \( A \).

(a) Let \( n \neq 0 \mod 2p \), for \( p \) odd or \( n \) odd for \( p = 2 \). Then the inclusion map \( i : B \longrightarrow A \) induces a monomorphism \( \varphi : Q^n B \longrightarrow Q^n A \).

(b) Let \( n \equiv 0 \mod 2p \) for \( p \) odd or \( n \) even for \( p = 2 \). Suppose \( 0 \neq \alpha_B(b) \in Q^n B \cap \ker \varphi \). Then \( \alpha_B(b) \) can be represented by a \( p^k \)th power of an algebra generator \( a \in A \) for some \( k \geq 1 \).

**Proof.** Suppose \( 0 \neq \alpha_B(b) \in Q^n B \cap \ker \varphi \). Let \( B(n - 1) \) be the sub-Hopf algebra of \( B \) generated by elements of \( B \) of degree less than \( n \). We have a commutative diagram:

\[
\begin{array}{cccccc}
B & \xrightarrow{\pi_B} & B/B(n-1) & \xrightarrow{i} & A & \xrightarrow{\pi_A} & \text{A}/B(n-1) \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Q}^n B & \xrightarrow{Q\varphi} & \text{Q}^n A & \xrightarrow{Q\pi_A} & \text{Q}^n (\text{A}/B(n-1)) \\
\end{array}
\]

where \( \pi_B, \pi_A \) are Hopf algebra epimorphisms and \( 0 \neq \pi_B(b) \in P^n(B/B(n-1)) \cong Q^n B \). By [3, p. 8] there exist isomorphisms \( B \cong B(n-1) \otimes B/B(n-1) \) and \( A \cong B(n-1) \otimes \text{A}/B(n-1) \) such that the following diagram commutes:

\[
\begin{array}{cccccc}
B & \xrightarrow{i} & B(n-1) \otimes B/B(n-1) & \xrightarrow{1 \otimes \theta} & \text{A}/B(n-1) \\
\downarrow & & \downarrow & & \\
B(n-1) \otimes \text{A}/B(n-1) \\
\end{array}
\]

Therefore, \( \theta \) is a monomorphism.

We have a commutative diagram:

\[
\begin{array}{ccccc}
\text{Q}^n B & \xrightarrow{Q\varphi} & \text{Q}^n A & \xrightarrow{Q\pi_A} & \text{Q}^n (\text{A}/B(n-1)) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{P^n(B/B(n-1))} & P^n(\text{A}/B(n-1)) & \xrightarrow{P\theta} & P^n(\text{A}/B(n-1)) \\
\downarrow & & \downarrow & & \\
0 & & \downarrow & & \\
\end{array}
\]

(0.1)

The bottom row is exact since \( \theta \) is monic. The right column is exact by [9, Proposition 4.2.21]. Now given \( Q\varphi \alpha_B(b) = 0 \), by diagram (0.1), it follows that
Let $F \xrightarrow{j} E \xrightarrow{\pi} B$ be a multiplicative fibration. By [1, Lemma 5.1], $\pi_1(B)$ acts trivially on $H_*(F; \mathbb{Z}_p)$. Hence, the Serre spectral sequence for the fibration satisfies

$$E_2^{r,s} = H^r(B; \mathbb{Z}_p) \otimes H^s(F; \mathbb{Z}_p)$$

and

$$E_\infty \cong G_*H^*(E; \mathbb{Z}_p).$$

This is a first quadrant spectral sequence of Hopf algebras [1, p. 166]. Furthermore, by [1, Theorem 5.8],

$$E_k \cong B_k \otimes C_k \otimes M_k \otimes N_k \quad \text{for} \quad k \geq 2$$

as algebras

where

$$B_k = E^{0,*}_k \quad \text{and} \quad C_k = E^{*,0}_k$$

are sub-Hopf algebras of $E_k$. $M_k = \Lambda(x_1, \ldots, x_\ell)$ and $N_k = \Lambda(w_1, \ldots, w_n)$ with $\dim x_i \equiv 1 \mod 2p$ and $\dim w_i \equiv -1 \mod 2p$. Equation (0.3) implies

$$Q^rE^{s,*}_\infty = 0 \quad \text{for} \quad r > 0 \quad \text{and} \quad s > 0 \quad \text{and} \quad r + s \not\equiv \pm 1 \mod 2p.$$

Hence, if $n \not\equiv \pm 1 \mod 2p$,

$$Q^rE^{s,*}_\infty = QE^{0,n}_\infty \oplus QE^{n,0}_\infty.$$  

By [8, Theorem 5.8], $j^*: H^*(E; \mathbb{Z}_p) \longrightarrow H^*(F; \mathbb{Z}_p)$ is the composition

$$H^*(E; \mathbb{Z}_p) \longrightarrow E^{0,n}_\infty \subseteq E^{2,n}_\infty \subseteq \cdots \subseteq E^{2,n}_\infty = H^*(F; \mathbb{Z}_p),$$

and \(\pi^*: H^*(B; \mathbb{Z}_p) \longrightarrow H^*(E; \mathbb{Z}_p)\) is the composition

$$H^*(B; \mathbb{Z}_p) = E^{2,0}_\infty \longrightarrow E^{2,0}_{n+1} = E^{2,0}_\infty \subseteq H^*(E; \mathbb{Z}_p).$$

**Proof of the Main Theorem** [7,1] The sequences of (a) are dual. So it suffices to prove

$$QH^*(B; \mathbb{Z}_p) \xrightarrow{Q\pi^*} QH^*(E; \mathbb{Z}_p) \xrightarrow{Qj^*} QH^*(F; \mathbb{Z}_p)$$

is exact if $p$ is an odd prime, $n \not\equiv 0 \mod 2p$ and $n \not\equiv \pm 1 \mod 2p$. Note that $\pi j$ is null homotopic. Therefore, $Qj^*Q\pi^* = 0$. So it suffices to prove $\ker Qj^* \subseteq \im Q\pi^*$.

Let $\alpha: H^*(E; \mathbb{Z}_p) \longrightarrow QH^*(E; \mathbb{Z}_p)$ be the projection. Let $\alpha(w) \in \ker Qj^* \cap QH^*(E; \mathbb{Z}_p)$. The Serre spectral sequence is a spectral sequence of algebras. Hence, $w$ must produce a nonzero element of $Q^rE^*_\infty$. By (0.4),

$$Q^rE^*_\infty = QE^{0,n}_\infty \oplus QE^{n,0}_\infty.$$  

Suppose $\alpha(w)$ has a nonzero component in $QE^{0,n}_\infty$. By (0.3) and (1.10), $E^{0,n}_\infty$ is a sub-Hopf algebra of $H^*(F; \mathbb{Z}_p)$. By Proposition (0.2), since $n \not\equiv 0 \mod 2p$, and $p$ is odd, a nonzero element of $QE^{0,n}_\infty$ produces a nonzero element of $QH^*(F; \mathbb{Z}_p)$. 


By (0.7), this would imply that \( Qj^*(\alpha(w)) \neq 0 \). But \( \alpha(w) \in \ker Qj^* \). Therefore, \( \alpha(w) \) has zero component in \( QE^{0,n}_\infty \) and \( \alpha(w) \) lies in \( QE^{n,0}_\infty \). By (0.7),

\[(0.8) \quad H^n(B; \mathbb{Z}_p) \to E^{n,0}_\infty \text{ is onto.}\]

Hence \( \alpha(w) \) lies in the image of \( Q\pi^* \). This proves (a). To prove (b) let \( \alpha(w) \in QH^n(E; \mathbb{Z}_p) \) with \( n \equiv 0 \mod 2p \) for \( p \) odd or \( n \) even for \( p = 2 \). Suppose \( \alpha(w) \) has nonzero component \( x \) in \( QE^{0,n}_\infty \). Since \( QH^n(F; \mathbb{Z}_p) = 0 \), the inclusion map induces the trivial map

\[Q_i : QE^{0,n}_\infty \to QH^n(F; \mathbb{Z}_p).\]

By Proposition (0.2) (b), \( x \) can be represented by a \( p^k \)-th power of an algebra generator of \( H^*(F; \mathbb{Z}_p) \). But \( Q^n(\xi^k H^*(F; \mathbb{Z}_p)) = 0 \). Hence \( \alpha(w) \) has no component in \( QE^{0,n}_\infty \) and \( \alpha(w) \) lies in \( QE^{n,0}_\infty \). By (0.8), \( \alpha(w) \) lies in the image of \( Q\pi^* \). \( \square \)

Let \( f : B \to K(\mathbb{Z}_p, \ell + 1) \) be an \( H \)-map, and let \( E \) be the fibre of \( f \). We have the following multiplicative fibration:

\[
\begin{array}{c}
K(\mathbb{Z}_p, \ell) \\
\downarrow j \\
E \\
\downarrow \pi \\
B
\end{array}
\]

In general, the algebra structure of \( H_*(E; \mathbb{Z}_p) \) is difficult to compute. We can, however, make the following observations.

**Corollary 0.3.** Let \( p \) be an odd prime. Let \( s, t \in PH_*(E; \mathbb{Z}_p) \) with \( \pi_*(s) \neq 0, \pi_*(t) \neq 0 \), and \( [\pi_*(s), \pi_*(t)] = 0 \). Suppose \( \deg[s, t] = n \) and \( n \not\equiv \pm 1 \mod 2p \) and \( n \not\equiv 0 \mod 2p \). If \( QH^n(K(\mathbb{Z}_p, \ell); \mathbb{Z}_p) = 0 \), then \( [s, t] = 0 \).

**Proof.** By the Main Theorem (0.1) (a), since \( PH_*(K(\mathbb{Z}_p, \ell); \mathbb{Z}_p) = 0 \), if \( [s, t] \neq 0 \), then \( \pi_*[s, t] = [\pi_*(s), \pi_*(t)] \neq 0 \). We conclude \( [s, t] = 0 \). \( \square \)

**Corollary 0.4.** Let \( t \in PH_{2m}(E; \mathbb{Z}_p) \) with \( \pi_*(t) \neq 0 \), and \( (\pi_*(t))^p = 0 \). If \( QH^{2mp}(K(\mathbb{Z}_p, \ell); \mathbb{Z}_p) \oplus \sum_{k=1}^{\infty} Q^{2mp}(\xi^k H^*(K(\mathbb{Z}_p, \ell); \mathbb{Z}_p)) = 0 \), then \( t^p = 0 \).

**Proof.** If \( t^p \neq 0 \), there exists an indecomposable \( \gamma \in QH^{2mp}(E; \mathbb{Z}_p) \) with \( \langle t^p, \gamma \rangle = 1 \). By the Main Theorem (0.1) (b), \( \gamma = \pi^*(\gamma_1) + d \) where \( \gamma_1 \) is indecomposable and \( d \) is decomposable. Then \( 1 = \langle t^p, \gamma \rangle = \langle t^p, \pi^*(\gamma_1) + d \rangle = \langle t^p, \pi^*(\gamma_1) \rangle = \langle (\pi_*(t))^p, \gamma_1 \rangle = 0 \). This is a contradiction. Therefore, \( t^p = 0 \). \( \square \)

**References**


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